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GEOMETRIC CHARACTERIZATION OF LINE GEOMETRY OVER QUADRATICALLY COMPLETE FIELD

Introduction

Usually, 3-dimensional projective geometry PG_3 is considered as a two-sorted theory of incidence relation between two types of objects: points and planes (cf. [3]). Consequently, lines of PG_3 are objects defined within the framework of this theory, and theory of lines (usually called the line geometry) constitutes as (proper) part of PG_3 (cf. [6]). The set of lines has a well-known interpretation in 5-dimensional geometry (so-called the geometry on Plucker' quadric). It would be interesting to define the line geometry as an axiomatic theory of certain relations between lines.

This paper constitutes a continuation of [4], where we proposed an axiom system for line geometry. The theory presented in [4] can be considered as a formalization of (3-dimensional, not necessarily Pappian) projective geometry in one-sorted language with one primitive notion. The variables of this language stand for *lines* and denoted by capital Latin letters. The *predicate* is denoted by the symbol "-". The atomic formula $A - B$ is expressed as "A intersects B".

In this paper, similarly as in [4], we shall use the following abbreviations: we shall write $\neq (A_1A_2\dots A_n)$ instead of $\bigwedge_{i=1}^n \bigwedge_{j=1}^n (i \neq j => A_i \neq A_j)$,
$-(A_1A_2\dots A_n)$ instead of $\bigwedge_{i=1}^n \bigwedge_{j=1}^n (i \neq j => A_i - A_j)$,
$B - (A_1A_2\dots A_n)$ instead of $\bigwedge_{i=1}^n (B - A_i)$,
$(B_1B_2\dots B_m) - (A_1A_2\dots A_n)$ instead of $\bigwedge_{i=1}^n \bigwedge_{j=1}^n (B_i - A_j)$,
$A \div B$ instead of $\neg(A - B)$ (A does not intersect B),
$\div(A_1A_2\dots A_n)$ instead of $\bigwedge_{i=1}^n \bigwedge_{j=1}^n (i \neq j => A_i - A_j)$,
where $\bigwedge_{i=1}^n \sigma_i$ denotes the conjuction $\sigma_1 \wedge \sigma_2 \wedge \dots \wedge \sigma_n$.

Let us recall the axioms of the theory presented in [4], describing structures $\langle \mathsf{L}, - \rangle$. For every axiom we give its (intended) meaning, expressed in terms of traditional projective geometry.

A1. $\exists AB(A \neq B \wedge A - B)$ (nonempty set L and nonempty relation "-").

A2. $A - B \Rightarrow B - A$ (symmetry of intersecting).

A3. $A - A$ (reflexivity of intersecting).

A4. $A - B \Rightarrow \exists CD(-(ABC) \wedge (ABD) \wedge (C \div D))$ (existence of "triangle" and "tripod", every two intersecting lines induce at least two objects: "point" and "plane", the dimension of a induced projective space is greater than 2).

A5. $-(ABCD) \wedge -(ABCE) \wedge \exists F(-(ABF) \wedge F \div C)) \Rightarrow D - E \wedge A = B$ (transitivity of "triangle" and "tripod" dependence, i.e. Pash axiom in terms of line geometry).

A6. $-(ABC) \wedge D - E \Rightarrow \exists F(F - (ABCDE))$ (every two "points" can be connected with a line, and every two "planes" have a common line).

A7. $-(ABCD) \Rightarrow \exists E(-(ABCDE) \wedge \forall FG(F \div A \vee F \div B \vee G \div C \vee G \div D \vee E - (FG) \vee A = B \vee C = D))$ (every two pencils in a "star" or in a "ruled plane" have a common line – the dimension of a induced projective space is less than 4). If the predicate $\mathbf{P}$ is such that

$$\mathbf{P}(ABC) \Leftrightarrow (A \neq B \wedge -(ABC) \wedge \forall D(-(ABD) \Rightarrow D - C)),$$

then the axiom A7 can be expressed in a shorter form:

A7′. $-(ABCD) \Rightarrow \exists E(\mathbf{P}(ABE) \wedge \mathbf{P}(CDE)) \vee A = B \vee C = D)$.

A8. $-(ABC) \wedge -(ABD) \wedge D \div C \wedge E - (AB) \Rightarrow E - C \vee E - D \vee A = B$ (every two intersecting lines induce at most two objects: "point" and "plane").

A9. $\exists D(A \div D \wedge B \div D \wedge C \div D)$ (there are at least three "points" on a line, a line belongs at most to three "planes", existence of a "quadrangle" on a "plane").

The subject of our considerations is theory $Cn(\{A1, A1, \ldots, A9\})$ the set of consequences of the axioms A1, A2, ..., A9. In this theory we introduced the $\mathbf{T}$ ($\mathbf{T}$ – triangle/tripod) relation defined in the following way:

$$\mathbf{T}(ABC) \Leftrightarrow (\neq (ABC \wedge -(ABC) \wedge \exists D(-(ABD) \wedge C \div D)).$$

The above relation was used to introduce the sets $[ABC] = \{X \in \mathsf{L} : -(ABCX) \wedge \mathbf{T}(ABC)\}$ so-called *varieties*. In the set V of all varieties $a, b, c \ldots$ we were defined the relation $\equiv_V$ in the following manner: $a \equiv_V b \Leftrightarrow |a \cap b| = 1 \vee a = b$ for all $a, b \in V$. This relation, as an equivalence relation, define the division set $V/_{\equiv_V}$ containing two elements V_1, V_3. The elements of V_1 we called points (denoted by small Latin letters) and the elements of V_3 we called planes (denoted by small Greek letters). For $a \in V_1$ and $\alpha \in V_3$ we set $a|\alpha$ iff $a \cap \alpha \neq \emptyset$ and then $\langle V_1, V_3, |\rangle$ is a 3-dimensional projective space, where lines can be identified with the elements of L. Thus, as it was already said, Desarguesian projective geometry can be represented in the geometry of lines defined in [4].

1. Pappus–Gallucci' axiom

The interesting question is: "What axiom must we add to obtain such theory in which Pappian projective geometry can be represented?" and: "Is it simply expressed in the terms of "lines" and "intersects relation"?". Notice, that the well known forms of Pappus' theorem are not simple. In classical geometry we have Gallucci' theorem concerning eighth lines: *If all three skew lines intersect other three skew lines, then any transversal of the first set intersects any transversal line of the second set* (cf. [2]). One may show that Gallucci' property is equivalent to Pappus' theorem. We will prove it (the proof of this fact can be found in [1], but here we give another proof where we use Veblen–Young's quadranglian 6-tuple of the points). First, we notice that Pappus' theorem is equivalent to the commutativity of Veblen–Young's quadranglian 6-tuple of the points (cf. [3]). Precisely, Pappus' theorem holds if and only if

$$a_1 \neq a_4 \wedge a_2 \neq a_5 \wedge \begin{bmatrix} a_1 & a_4 \\ a_2 & a_5 \\ a_3 & a_6 \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 & a_4 \\ a_2 & a_5 \\ a_6 & a_3 \end{bmatrix},$$

where matrices denote Veblen–Young's quadranglian 6-tuples.

Now, as an additional axiom of the theory of lines, let us take the following phrase (Gallucci' property):

A10. $\div(ABC) \wedge \div(EFG) \wedge (ABC) - (EFG) \wedge D - (EFG) \wedge H - (ABC) \Rightarrow D - H$.

For convenience, especially in this part of article, we will use the incidence symbol "|" in the following sense:

$$a|A \text{ (or } A|a \text{ iff } A \in a, \text{ for any point } a \text{ and line } A,$$

and similarly

$$A|\alpha \text{ (or } \alpha|A) \text{ iff } A \in \alpha, \text{ for any plane } \alpha \text{ and line } A.$$

We shall write

$$a_1, a_2, \ldots, a_n|A \text{ instead of } \bigwedge_{i=1}^{n} a_i|A, \text{ for points } a_1, a_2, \ldots, a_n \text{ and line } A.$$

Similar abbreviations will be used for remaining combinations of points, lines and planes. And other agreement yet: the unique line C belonging to set $a \cap b$ will be denoted by ab, and the unique line C belonging to set $\alpha \cap \beta$ will be denoted by $\alpha\beta$.

Now we are going to prove the Gallucci' property implies Pappus' theorem.

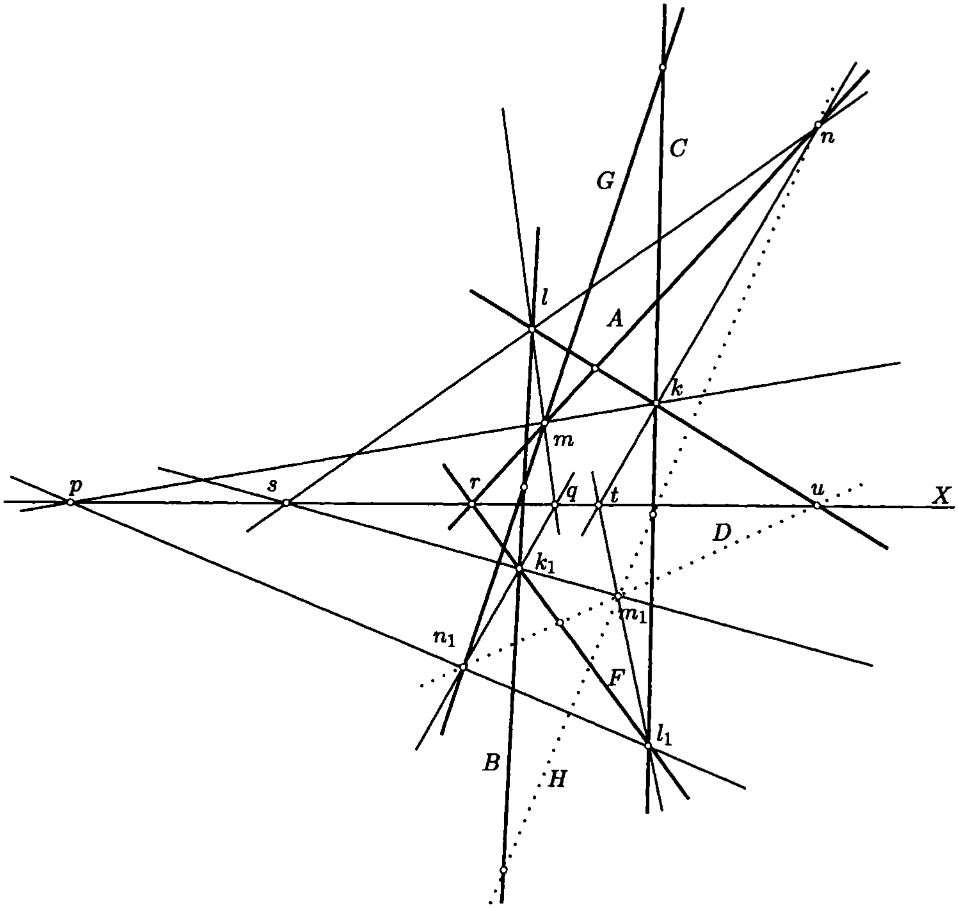

Fig. 1

Consider any Veblen–Young six-tuple $\begin{bmatrix} s & p \\ r & u \\ t & q \end{bmatrix}$ of the points s, r, t, p, u, q and line X such that $s, r, t, p, u, q | X$ and $s \neq p$ and $r \neq u$. This six-tuple is defined on a plane, which we denote by π, by certain quadrangle (in Fig. 1. we have a quadrangle $lmkn$), such that the vertices of this quadrangle are not incident with X. Naturally $X|\pi$. In 3-dimensional projective geometry an existence of such objects is obvious. Let us denote $E = lk$ and $A = mn$. Obviously $A - E$ and $A \neq E$ and $A, E|\pi$. Let k_1 be any point no incident with π. Then $k_1 \neq s$ and $k_1 \neq r$ and $k_1 \neq q$ we denote by π_1 a plane determined by the line X and point k_1. Let us consider three lines sk_1, rk_1, qk_1. Naturally $sk_1, rk_1, qk_1|\pi_1$. Set $F = rk_1$. Let n_1 be a point incident with the line qk_1 such that $n_1 \neq q$ and $n_1 \neq k_1$. Thus n_1 is not incident with π and hence

$n_1 \neq u$. Let us denote by D the line un_1. We have $F - D, F - A$, and $F \neq D$ because $r \neq u$. Notice that $D \neq qk_1$. Indeed, if $D = qk_1$ were fulfilled, then we would have $q = u$, hence we would have $D|n_1, k_1$ and $s = u$. We should obtain the equality $q = s = u$, which is impossible, because then the points l, m, k, n do not form a quadrangle. From $F \neq D$ it follows that n_1 is not incident with F and, next, the existence of a point $l_1 (l_1 \neq k)$ such that $l_1|pn_1$ and $l_1|rk_1$. Notice that the points k_1, n_1, l_1 form a triangle. Let us denote by B, G, C the lines $n_1 m, lk_1, kl_1$ respectively. We have $B - G, C - G$ and $B, C, G \neg | \pi, \pi_1$ (see Fig. 1.). Next notice that $A \div B$. Indeed, for if $A - B$, then we should have $l|A$. It is with conflict with a fact that the points l, m, k, n form the quadrangle. In the same manner we prove that $A \div C$. Further on we shall prove that $B \div C$. Indeed, for if $B - C$, then would must be $C|\alpha(ns, sk_1)$, hence we should have $l_1 = k_1$. It is impossible, because the points k_1, n_1, l_1 form a triangle. Finally, we have $\div(ABC)$. Below we shall prove that $\div(EFG)$. First notice that $E \div F$, because the condition $E - F$, together $E|\pi$ and $F|\pi_1$, implies $r = u$. This is contradictory with our assumption about points r, u. The facts $E \div G, G \div F$ we can prove in the same manner as it was done lines A, B, C. It is easy to see that $(ABC) - (EFG) \wedge D - (EFG)$. Now let us consider two planes: $\beta(n, B)$ and $\gamma(n, C)$. We have $\beta \neq \gamma$, because $B \div C$. Therefore it exists one and only one line incident with β, γ. Put $H = \beta\gamma$. Thus $H - (ABC)$ holds. Last of all the predecessor of implication in axiom A10 is satisfied. Hence we obtain $D - H$. Let us denote by m_1 the point determined by lines D, H and by δ – the plane determined by these lines. From $n, k|\delta$ we have $t|\delta$ and $t|m_1 l_1$. Let us consider the quadrangle $(l_1 m_1 k_1 n_1)$. From the above considerations it follows that this quadrangle induces five points r, t, p, u, q from the six-tuple $\begin{bmatrix} p & s' \\ u & r \\ q & l \end{bmatrix}$. The sixth point is uniquely determined. So $s' = s$. Thus $m_1|sk_1$.
From the described above configuration (Fig. 1.) we obtain the quadrangle $(m_1 l_1 n_1 k_1)$ and, realized by it, Veblen–Young's six-tuple $\begin{bmatrix} s & p \\ r & u \\ q & t \end{bmatrix}$. Thus the implication $\begin{bmatrix} s & p \\ r & u \\ t & q \end{bmatrix} \Rightarrow \begin{bmatrix} s & p \\ r & u \\ q & t \end{bmatrix}$ is true. And this implication, together with $s \neq p$ and $r \neq u$ (our assumption), denotes that Pappus' axiom holds (cf [3], p. 76). □

The opposite implication (from Pappus' theorem to Gallucci' proposition) we may prove in the following way: for given eight lines A, B, C, D, E, F, G, h satisfying the left side of Gallucci' implication we can construct the configuration, which ilustrates Fig. 1. The consequence of the property: $s \neq p \wedge r \neq u \wedge \begin{bmatrix} s & p \\ r & u \\ t & q \end{bmatrix} \Rightarrow \begin{bmatrix} s & p \\ r & u \\ q & t \end{bmatrix}$ is an incidence of the line D and the point m_1 determined by the line H and lines A, B, C, E, F, G. This gives $D - H$. The detailed proof will be omitted on acount of a similarity of the reasoning to considerations concerning the first part of proof of the equivalence of Gallucci' theorem and Pappus' proposition. Moreover the remark about it we can find in [2] (cf. [2], p. 277). □◇

Therefore the theory $Cn(\{A1, A2, \ldots, A10\})$ is Pappian geometry of lines.

Pappus' axiom allows us to introduce the notion of *quadric* in our theory. Let A, B, C be any lines such that $\div(ABC)$. Let us consider the set of all lines intersecting these lines.

DEFINITION 1. $]ABC[= \{X : X - (ABC)\}$.

This set will be called a *half-quadric* generated by lines A, B, C. Notice that $A \notin\,]ABC[$ and $B \notin]ABC[$ and $C \notin\,]ABC[$. We shall prove that

THEOREM 1. $|]ABC[| \geq 3$.

Proof. Basing on the theorems T16, T12, T4a of the work [4] let us consider three lines A, B, C such that $\div(ABC)$. From T16 there exist three points b, b', b'' such that $B \in b$ and $B \in b'$ and $B \in b''$. By T12 there exist A', C' such that $A' \in b$ and $C' \in b$ and $A' - A$ and $C' - C$. From T4a there exist two lines A'', C'' such that $[AA'A'']$ and $[CC'C'']$ are planes. Obviously the planes $[AA'A'']$ and $[CC'C'']$ have only one common line X. Notice that

$$X - A \text{ because } X \in [AA'A''],$$
$$X - C \text{ because } X \in [CC'C''],$$
$$X - B \text{ because } X \in b.$$

Repeating the construction above for points b' and b'' we obtain two lines Y, Z. Notice that $\neq (XYZ)$. Indeed, if $\neq (XYZ)$ were not fulfiled then $X = Y$ would hold. Consequently, $X \in b$ and $X \in b'$, on other hand $B \in b$ and $B \in b'$. Since $b \neq b'$, so $|b \cap b'| = 1$. Thus $B = X$. But $B \div A$ and $B \div C$. We obtain a contradiction. ◇

One may show that every two lines belonging to the half-quadric are skew, i.e.

THEOREM 2. $X, Y \in\,]ABC[\Rightarrow X \div Y$.

Proof. Suppose that $X - Y$. Simultaneously from D1 we have $X - (ABC)$ and $Y - (ABC)$. Thus we obtain $-(XYA)$ and $-(XYB)$ and

$-(XYC)$. By A8 we deduce that $C - A$ or $C - B$. We obtain a contradiction to $\div(ABC)$. $\diamond$

DEFINITION 2. A half-quadric $]MNO[$ will be called *associated* with a half-quadric $]ABC[$ iff $M, N, O \in\ ]ABC[$.

From Theorems 2 and 1 it follows that for every half-quadric it exists a half-quadric associated with it.

THEOREM 3. *If* $]ABC[$ *is associated with* $]DEF[$, *then* $]DEF[$ *is associated with* $]ABC[$.

Proof. Assume that $]ABC[$ is associated with $]DEF[$. This means that $A, B, C \in\ ]DEF[$ and thus we have $(ABC) - (DEF)$. Obviously we have $(DEF) - (ABC)$. On the other hand $\div(ABC)$ holds. Thus $D, E, F \in\ ]ABC[$. $\diamond$

Now we are going to prove that the definition of half-quadric does not depend on lines which generate it. We shall prove the following, so-called exchange theorem.

THEOREM 4. *Let* $]ABC[$ *and* $]DEF[$ *be any two half-quadrics mutually associated. If* $U \in\ ]DEF[$ *and* $U \neq A$ *and* $U \neq B$, *then* $]ABC] =]ABU[$.

Proof. Let $U \in\ ]DEF[$. Consider two cases. $\diamond$

P1: $U = C$. We immediately obtain the thesis.

P2. $U \neq C$. We prove two inclusions.

"$\subset$". Take $W \in\ ]ABC[$. Then $W - (ABC)$. Simultaneously $U - (DEF)$. By A10 we have $W - U$. Thus $W \in\ ]ABU[$. $\square$

"$\supset$". Take $W \in\ ]ABU[$. Then $W - (ABU)$. Thus we have $\div(ABU) \wedge \div(DEF) \wedge (ABU) - (DEF) \wedge C - (DEF) \wedge W - (ABU)$.

By A10 we obtain $W - C$. Thus $W - (ABC)$, i,e, $W \in\ ]ABC[$. $\square\diamond$

The axiom A10 implies the *pappian line geometry* and in consequence the geometry over commutative field. Now we are going to show a simple axiom, which implies the geometry over *quadratically complete field.*

2. An axiom inducing the geometry of lines over quadratically complete field

Let us take a phrase

A11. $\exists E(ABCD) - E)$, i.e. for any four lines it exists a line intersecting every line of them of four.

And next consider the following

DEFINITION 3. We say that $D-]ABC[$ (a line D intersects quadratic $]ABC[$) iff it exists a line $E \in\ ]ABC[$ with $D - E$.

We have following

LEMMA 1. *D–$]ABC[$ iff it exists a line E such that $E - (ABCD)$.*

Proof. "$\Rightarrow$": Let D–$]ABC[$. Then, by Definition 3, it exists E such that $E \in]ABC[$ and $D - E$. But $E \in]ABC[$ denotes $E - (ABC)$. In consequence we obtain $E - (ABCD)$.

"$\Leftarrow$": Let three skew lines A, B, C (which define the quadric $]ABC[$) and any line D be given. Since $E - (ABC)$ then $E \in]ABC[$. Then we have $E \in]ABC[$ and $D - E$. We obtain D–$]ABC[$. $\diamond$

Accepting the axiom A11 and using Lemma 1 we can formulate

THEOREM 5. *For any quadric $]ABC[$ and any line D the condition D–$]ABC[$ is satisfied, i.e. every line intersects every quadric.*

Consider the ruled quadric $x_1^2 + x_2^2 - x_3^2 - x_0^2 = 0$ and the line $\{x_1 = 1, x_2 = \lambda, x_3 = 0, x_4 = 0\}$. The existence of a common point of the above quadric and line is equivalent to the existence of a solution of equation $1 + \lambda^2 = 0$ (generally: the existence of a common point of every quadric and every lines is equivalent to the existence of a solution of arbitrary quadratic equation). This equation has a root in quadratically complete field. Hence the theory Cn(A1,A2,...,A10,A11) can be treated as the *line geometry over quadratically complete field.*

3. Final comments

A) Both axioms A10 and A11 are very simple. It appears, that in the line geometry such complicated properties as Pappus' theorem and property of "quadratic completeness of field" are expressed very simply and very naturally. One believes that the language accepted in the description of line geometry is straightforward as possible.

B) It is worthwhile to remind, that no finite field is algebraically closed. Then, the obtained above axiomatics excludes all finite geometries of lines.

C) The theory Cn({A1, A2, ..., A10}) admits the geometries over finite (so-called Galois) fields. Particularly in LG(2), i.e. in line geometry over GF(2) (GF(q) — Galois field of q elements) any half-quadric has exactly three elements. This is a consequence of the number of points incident with any line [5]. In (3-dimensional) projective geometry PG(3,q) over GF(q) every line is incident with $q + 1$ points. Indeed, from theorem 1 and 2 it follows that every point incident with a line belonging to a half-quadric Σ lies on one and only one line from the half-quadric Σ' associded with Σ. This reasoning shows that every half-quadric in LG(q) contains at most $q + 1$ lines.

D) The interesting role in the study of the structure of line geometry may be played by computer programs written in PROLOG language. By appropriate simple programs, constructed by means of PROLOG with use of

Plucker's coordinates, the structure LG(q) has been analysed. Particularly, in this way, it was checked for $q = 2, 3, 4$ (for not large numbers) that every quadric in LG(q) contains exactly $q + 1$ lines. Generally, PROLOG may be an interesting computer system in geometrical research, especially in study of geometric configurations in axiomatically defined theories, wherever the investigation concerns finite structures, or if the obtained, in whis way, result may be treated as the search problem concerning any structure (finite or infinite). The results obtained by computer programs most often allow us to formulate definitions and theorems, sometimes they may be treated as proofs.

References

[1] F. Bachmann: *Aufbau der Geometrie aus dem Spielungsbegriff.* Springer Verlag 1959.

[2] H. S. M. Coxeter: *Wstp do geometrii dawnej i nowej.* PWN Warszawa 1967 (*Introduction to Geometry.* New York 1961).

[3] M. Kordos: *Podstawy geometrii rzutowej i rzutowo-metrycznej.* Warszawa 1984 (*Foundations of Projective and Projective-Metric Geometry.* Warsaw 1984).

[4] E. Koźniewski: An axiom system for the line geometry, *Demonstratio Math.* 21 (1988) 1071–1087.

[5] F. Karteszi: *Introduction to finite geometries.* Akademiai Kiado, Budapest 1976.

[6] J. G. Semple, L. Roth: *Introduction to Algebraic Geometry.* Clarendon Press, Oxford 1986.

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