DEMONSTRATIO MATHEMATICA
Vol. XXIX No 3 1996

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LOCAL CHARACTERIZATION OF FUNCTIONS WITH CLOSED GRAPHS

Let X and Y be topological spaces. A function $f : X \to Y$ has a closed graph if the set $G(f) = \{(x, f(x) : x \in X\}$ is a closed subset of $X \times Y$. Functions with closed graphs play an important rule in functional analysis and they have been studied extensively. The purpose of the present paper is to introduce a suitable pointwise definition of that notion and to give a characterization of the set of all closedness graph points.

For a subset A of a topological space we denote by $\operatorname{Cl} A$ and $\operatorname{Int} A$ the closure and the interior of A, respectively. The letters $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ stand for the set of positive integers, rational and real numbers, respectively. For $x \in X$ denote by $\mathcal{U}_x$ the family of all neighbourhoods of x.

In [6] it is shown (for compact Hausdorff X and Hausdorff Y also in [13]) that a function $f : X \to Y$ has a closed graph if and only if $C(f, x) = \{f(x)\}$, where $C(f, x)$ is the cluster set of f at x defined by $C(f, x) = \bigcap_{U \in \mathcal{U}_x} \operatorname{Cl} f(U)$ $(= \{y \in Y$: there exists a net x_α in X with $\lim x_\alpha = x$ and $\lim f(x_\alpha) = y\})$. Hence the following definition seems to be reasonable.

DEFINITION 1. We say that a function $f : X \to Y$ has a closed graph at $x \in X$ if $C(f, x) = \{f(x)\}$.

Hence f has a closed graph if and only if it has a closed graph at each point. Denote by $H(f)$ the set of all closedness graph points of $f : X \to Y$. Further denote by $C(f)$ and $D(f)$ the set of all continuity and discontinuity points of f, respectively. Obviously, for a Hausdorff Y (but not for an arbitrary Y) we have $C(f) \subset H(f)$ (e.g. [7]).

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 54C08.
Key words and phrases. Closed graph, Continuity, c-continuity.
Supported by Grant GA-SAV 367.

A function $f : X \to Y$ is c-continuous at $x \in X$ if for each open neighbourhood V of $f(x)$ such that $Y \setminus V$ is compact there is an open neighbourhood U of x such that $f(U) \subset V$. A function is c-continuous if it is such at every point [5]. Denote by $T(f)$ the set of all c-continuity points of f. Evidently $C(f) \subset T(f)$.

It is known that a function $f : X \to Y$ with a closed graph is c-continuous [10]. If Y is locally a compact Hausdorff space, then these properties are equivalent [10]. The assumptions on Y cannot be omitted [12]. We shall show that these assertions are true also pointwisely.

PROPOSITION 1. *We have* $H(f) \subset T(f)$.

Proof. Fix $x \in H(f)$ and let V be an open neighbourhood of $f(x)$ such that $K = Y \setminus V$ is compact. Then $C(f,x) \subset V$ and hence $C(f,x) \cap K = \emptyset$. Since K is compact and $(\operatorname{Cl} f(U) \cap K)_{U \in \mathcal{U}_x}$ is a family of closed subsets of K with $\bigcap_{U \in \mathcal{U}_x} \operatorname{Cl} f(U) \cap K = \emptyset$ there are neighbourhoods $U_1, \dots U_n$ of x such that $\bigcap_{i=1}^n \operatorname{Cl} f(U_i) \cap K = \emptyset$. Now $U = \bigcap_{i=1}^n U_i$ is a neighbourhood of x and $f(U) \subset \bigcap_{i=1}^n f(U_i) \subset V$. □

PROPOSITION 2. *Let Y be a Hausdorff locally compact space. Then* $T(f) = H(f)$.

Proof. Fix $x \in T(f)$ and take $y \neq f(x)$. Then there is a closed compact neighbourhood K of y such that $f(x) \notin K$. The c-continuity of f at x implies that there is an open neighbourhood U of x such that $f(U) \subset Y \setminus K$. Therefore $y \notin \operatorname{Cl} f(U)$ and $x \in H(f)$. □

Denote by $B(f)$ the set of all local boundedness points of $f : X \to Y$, i.e. $B(f) = \{x \in X :$ there is a compact set K in Y such that $x \in \operatorname{Int} f^{-1}(K)\}$. Evidently $B(f)$ is an open set. In [3] it is shown (also in [8] for metrizable X and $Y = \mathbb{R}$) that for a function f with a closed graph we have $B(f) \subset C(f)$. Also this is true pointwise.

PROPOSITION 3. *We have* $B(f) \cap H(f) \subset C(f)$.

Proof. Fix $x \in B(f) \cap H(f)$ and let V be an open neighbourhood of $f(x)$. Then one can find a compact set K and an open neighbourhood G of x with $f(G) \subset K$. Now $K \setminus V$ is compact and $C(f,x) \subset V$. Hence $(\operatorname{Cl} f(U) \cap (K \setminus V))_{U \in \mathcal{U}_x}$ is a family of closed subsets of $K \setminus V$ with $\bigcap_{U \in \mathcal{U}_x} \operatorname{Cl} f(U) \cap (K \setminus V) = \emptyset$. Hence there are neighbourhoods $U_1, \dots, U_n$ of x with $\bigcap_{i=1}^n \operatorname{Cl} f(U_i) \cap (K \setminus V) = \emptyset$. Then $H = G \cap \bigcap_{i=1}^n U_i$ is a neighbourhood of x. If $y \in H$, then $f(y) \in K$ and $f(y) \notin K \setminus V$. Therefore $f(y) \in V$ and $x \in C(f)$. □

From Propositions 2 and 3 we obtain $B(f) \cap T(f) \subset C(f)$ for a locally compact Hausdorff space Y. We shall show that the local compactness can be omitted.

PROPOSITION 4. *Let Y be a Hausdorff space. Then $B(f) \cap T(f) \subset C(f)$.*

Proof. Fix $x \in B(f) \cap T(f)$ and let V be an open neighbourhood of $f(x)$. Then one can find a compact set K and an open neighbourhood G of x with $f(G) \subset K$. Since Y is Hausdorff space so K is closed. Hence $W = V \cup (Y \setminus K)$ is an open neighbourhood of $f(x)$ such that the set $Y \setminus W = K \setminus V$ is compact. Hence there is a neighbourhood U of x with $f(U) \subset W$. Now $f(U \cap G) \subset V$ and $x \in C(f)$. □

Remark 1. The assumption "Y is a Hausdorff space" in Proposition 4 cannot be omitted. Let $X = \mathbb{R}$ with the usual topology and let $Y = \mathbb{R}$ with the topology $\mathcal{T}$, where $A \in \mathcal{T}$ if $A = \emptyset$ or $A = Y$ or $A = (a, \infty)$ for some $a \in \mathbb{R}$. Let $f : X \to Y$ be definded by $f(x) = 1$ for irrational x and $f(x) = 0$ for rational x. Then $B(f) = T(f) = X$ and $C(f) = \emptyset$.

PROPOSITION 5. *Let Y be a Hausdorff locally compact space. Then $B(f) \cap H(f) = C(f) = B(f) \cap T(f)$.*

Proof. We have $C(f) \subset H(f)$. Further, if $x \in C(f)$ and K is a compact neighbourhood of $f(x)$, then there is a neighbourhood U of x with $f(U) \subset K$, i.e. $x \in B(f)$. □

It is known that if Y is compact [8], [6] or if X is first countable and Y is countably compact [9], [6] or if X is saturated and Y is regular countably compact [5], then functions with closed graphs are continuous. However, from their proofs it follows that under above assumptions on X and Y we have $H(f) \subset C(f)$ and if moreover Y is a Hausdorff space, then $H(f) = C(f)$. It is easy to see the following

PROPOSITION 6. *If A is a subset of X and $f : X \to Y$ has a closed graph at $x \in A$, then $f|A : A \to Y$ has a closed graph at x.*

Now we shall characterize the set $H(f)$. We recall that a metric space (Y, d) is called b-compact if every bounded subset of Y has the compact closure [6; p.29].

THEOREM 1. *Let X be a topological space and let Y be a b-compact metric space. Then $H(f)$ is a G_δ set.*

Proof. Let b be a point in Y.For $a \in Y$ and $n \in \mathbb{N}$ denote by

$$S_n^a = \{y \in Y : d(y, a) \geqq \tfrac{1}{n}\},$$
$$T_n^a = \{y \in Y : d(y, b) \leqq n\} \text{ and}$$
$$K_n^a = S_n^a \cap T_n^a.$$

Notice that K_n^a is a closed bounded subset of Y and therefore it is compact. Put

$$A_n^a = \{x \in X : \text{ there is } U \in \mathcal{U}_x \text{ with } f(U) \subset Y \setminus K_n^a\}$$

and observe that every A_n^a is an open set. Therefore

$$A = \bigcap_{n=1}^{\infty} \bigcup_{a \in Y} A_n^a$$

is a G_δ set. We shall show that $A = H(f)$.

Fix $x \in A$ and let $y \neq f(x)$. Then there is $n_1 \in \mathbb{N}$ such that $d(y, f(x)) > \frac{3}{n_1}$. Further, there are $n_2, n_3 \in \mathbb{N}$ such that $d(y,b) < \frac{n_2}{2}$ and $d(f(x), b) < n_3$. Put $n = \max\{2, n_1, n_2, n_3\}$. Since $x \in A$, there is $a \in Y$ such that $x \in A_n^a$. Then $d(f(x), a) < \frac{1}{n}$. (If namely $d(f(x), a) \geqq \frac{1}{n}$, then $f(x) \in K_n^a$ and hence $x \notin A_n^a$, a contradiction.) This yields to $\frac{3}{n} < d(y, f(x)) \leqq d(y,a) + d(a, f(x)) < \frac{1}{n} + d(y,a)$ and therefore $d(y,a) > \frac{2}{n}$. Let U be a neighbourhood of x with $f(U) \subset Y \setminus K_n^a$. Put $G = \{z \in Y : d(y,z) < \frac{1}{n}\}$.

Let $z \in G$. Then $\frac{2}{n} < d(y,a) \leqq d(y,z) + d(z,a) < d(z,a) + \frac{1}{n}$ and thus $\frac{1}{n} < d(z,a)$, i.e. $z \in S_n^a$. Further, $d(z,b) \leqq d(z,y) + d(y,b) < \frac{1}{n} + \frac{n}{2} < n$, thus $z \in T_n^a$. Therefore $z \in K_n^a$ and $G \subset K_n^a$. This yields to $G \cap f(U) = \emptyset$ and $y \notin \mathrm{Cl}\, f(U)$. Therefore $y \notin C(f,x)$ and $x \in H(f)$.

Now fix $x \in H(f)$ and $n \in \mathbb{N}$. Then, by Proposition 1, $x \in T(f)$ and since $K_n^{f(x)}$ is compact closed and $f(x) \notin K_n^{f(x)}$, so there is a neighbourhood U of x with $f(U) \subset Y \setminus K_n^{f(x)}$. Thus $x \in A_n^{f(x)}$. But $n \in \mathbb{N}$ is arbitrary and hence $x \in A$. □

Remark 2. Obviously every b-compact metric space is locally compact. Hence, by Proposition 2, also the set $T(f)$ is a G_δ set. However, Theorem 1 is not true if we replace "Y is b-compact metric" by "Y is locally compact metric". Let $X = \mathbb{R}$ with the ususal topology and let $Y = \mathbb{R}$ with the discrete metric (i.e. $d(a,b) = 1$ for $a \neq b$). Then Y is a locally compact metric space. If $f : X \to Y$ is defined by $f(x) = 0$ for rational x and $f(x) = 1$ for irrational x, then $H(f) = \mathbb{Q}$ is not a G_δ set.

We recall that a topological space is almost resolvable if it is a countable union of sets with empty interiors. Every first countable topological space without isolated points, locally compact Hausdorff topological space without isolated points, real linear topological space or separable topological space without isolated points is almost resolvable [2]. A topological space is perfect if every closed subset of this space is G_δ [4]. A space is perfect normal if it is normal (need not be T_1) and perfect.

THEOREM 2. *Let X be an almost resolvable topological space. Let H be a subset of X. Then H is a G_δ set if and only if $H = H(f)$ for some $f : X \to \mathbb{R}$.*

Proof. Sufficiency follows from Theorem 1. By [2] there is a function $f : X \to [0,1]$ with $C(f) = H$ and by Proposition 5 we have $H = H(f)$. □

LEMMA 1. *Let* $f, g : X \to \mathbb{R}$. *Then* $C(f) \cap H(g) \subset H(f+g)$.

Proof. Let $x \in C(f) \cap H(g)$. For every $n \in \mathbb{N}$ there is a neighbourhood V_n of x such that $f(V_n) \subset (f(x) - \frac{1}{n}, f(x) + \frac{1}{n})$. Further, by Proposition 1 there is a neighbourhood W_n of x such that $g(W_n) \subset (-\infty, -2n) \cup (g(x) - \frac{1}{n}, g(x) + \frac{1}{n}) \cup (2n, \infty)$. Now for $n > |f(x)| + 1$ we have $(f+g)(V_n \cap W_n) \subset (-\infty, -n) \cup (f(x) + g(x) - \frac{2}{n}, f(x) + g(x) + \frac{2}{n}) \cup (n, \infty)$. This yields to $C(f+g, x) \subset \bigcap_{n=1}^{\infty} \mathrm{Cl}(f+g)(V_n \cap W_n) \subset \{f(x) + g(x)\}$ and therefore $x \in H(f+g)$. □

THEOREM 3. *Let X be a Baire almost resolvable perfectly normal topological space. Let C, H be subsets of X. Then the following conditions (A) and (B) are equivalent*:

(A) *There exists a function $f : X \to \mathbb{R}$ such that $C = C(f)$ and $H = H(f)$*;

(B) (i) *C and H are G_δ sets*,

(ii) $C \subset H$,

(iii) *C is open in H*,

(iv) $\mathrm{Int}(H \setminus C) = \emptyset$.

Proof. (A) ⇒ (B):

(i) By Theorem 2.

(ii) Obvious.

(iii) The set $B(f)$ is open and by Proposition 5 we have $B(f) \cap H(f) = C(f)$.

(iv) Suppose that $G = \mathrm{Int}(H(f) \setminus C(f)) \neq \emptyset$. By Proposition 6 we have $G = G \cap H(f) \subset H(f|G)$. Therefore $f|G$ has a closed graph and since G is a Baire space so by [14] $D(f|G)$ is closed and nowhere dense in G. However, since G is open, we have $C(f|G) = C(f) \cap G = \emptyset$, a contradiction.

(B) ⇒ (A): Put $S = \bigcup\{E \subset X : E \text{ is open and } E \cap H = C\}$. Then S is an open set, $C \subset S$ and $S \cap H = C$. Since X is almost resolvable we have $X = \bigcup_{m=1}^{\infty} X_m$, where $\mathrm{Int}\, X_m = \emptyset$ and $X_m \cap X_n = \emptyset$ if $m \neq n$. Further, by (i), $C = \bigcap_{n=1}^{\infty} C_n$ and $H = \bigcap_{n=1}^{\infty} H_n$, where C_n and H_n are open. We can assume that $C_{n+1} \subset C_n \subset S$ and $H_{n+1} \subset H_n$ for each $n \in \mathbb{N}$. Put $C_0 = S$ and $H_0 = X$. Since X is perfectly normal, there is a continuous function $t : X \to [0,1]$ such that $t^{-1}(0) = X \setminus S$. Define $g : X \to \mathbb{R}$ as

$$g(x) = \begin{cases} \dfrac{1}{t(x)}, & \text{if } x \in S, \\ 0, & \text{if } x \in X \setminus S. \end{cases}$$

Further let $h : X \to \mathbb{R}$ be defined by

$$h(x) = \begin{cases} 0, & \text{if } x \in H, \\ \dfrac{1}{n}, & \text{if } x \in (C_{n-1} \setminus C_n) \setminus \operatorname{Int}(C_{n-1} \setminus C_n), \\ \dfrac{1}{n+m}, & \text{if } x \in X_m \cap \operatorname{Int}(C_{n-1} \setminus C_n), \\ n, & \text{if } x \in (H_{n-1} \setminus H_n) \setminus S. \end{cases}$$

Take

$$f = g + h.$$

We shall show that $C(f) = C$ and $H(f) = H$.

1. Let $x \in C$. Then C_n is a neighbourhood of x and $h(C_n) \subset [0, \frac{1}{n})$. Therefore h is continuous at x. Since g is continuous at x, we have

$$C \subset C(f). \tag{1}$$

2. Let $x \in H \setminus C$. Then by (iii) $x \notin S$ and $f(x) = 0$. Let $n \in \mathbb{N}$. Then there is a neighbourhood U_n of x such that $t(U_n) \subset [0, \frac{1}{n})$. Let $y \in H_n \cap U_n$.
If $y \in S$, then $g(y) \geqq n$, $h(y) \geqq 0$ and hence $f(y) \in [n, \infty)$.
If $y \in H \setminus S$, then $g(y) = h(y) = f(y) = 0$.
If $y \in (H_n \cap U_n) \setminus (H \cup S)$, then $g(y) = 0$, $h(y) \geqq n$ and hence $f(y) \in [n, \infty)$.
Therefore $f(H_n \cap U_n) \subset \{0\} \cup [n, \infty)$. This yields to $C(f, x) \subset \bigcap_{n=1}^{\infty} \operatorname{Cl} f(U_n \cap H_n) \subset \{0\}$, i.e. $C(f, x) = \{0\} = \{f(x)\}$. Therefore we have

$$H \setminus C \subset H(f). \tag{2}$$

3. Let $x \in H \setminus C$. Let U be an open neighbourhood of x. Then $x \notin S$ and $f(x) = 0$. By (iv) there is $y \in U \cap (X \setminus (H \setminus C))$.
If $y \in S$ then $g(y) \geqq 1$, $h(y) \geqq 0$ and hence $f(y) \geqq 1$.
If $y \notin S$ then there is $n \in \mathbb{N}$ such that $y \in (H_{n-1} \setminus H_n) \setminus S$. Then $h(y) = n$ and hence $f(y) \geqq 1$. Therefore $x \notin C(f)$ and

$$H \setminus C \subset X \setminus C(f). \tag{3}$$

4. Let $x \in (X \setminus H) \setminus S$. Then $f(x) \neq 0$. Suppose that $x \notin \operatorname{Cl}(H \setminus C)$. Then there is an open neighbourhood V of x such that $V \cap (H \setminus C) = \emptyset$. Then V is an open set, $V \cup S \neq S$ and $V \cap H \subset C$, thus $(V \cup S) \cap H = C$, a contradiction with the definition of S.

Therefore $x \in \operatorname{Cl}(H \setminus C)$. Then for each neighbourhood U of x we have $U \cap (H \setminus C) \neq \emptyset$. However for $y \in H \setminus C$ we have $f(y) = 0$ and hence $0 \in f(U)$ for each neighbourhood U of x and thus $0 \in C(f, x)$. However $f(x) \neq 0$ and hence $x \notin H(f)$. Therefore we have

$$(X \setminus H) \setminus S \subset X \setminus H(f). \tag{4}$$

5. Let $x \in (X \setminus H) \cap S$. Then there is $n \in \mathbb{N}$ such that $x \in C_{n-1} \setminus C_n$. We shall show that $x \notin H(h)$. We have two possibilities:

a) Let $x \in \operatorname{Int}(C_{n-1} \setminus C_n)$. Then there is $m \in \mathbb{N}$ with $x \in X_m \cap \operatorname{Int}(C_{n-1} \setminus C_n)$ and hence $h(x) = \frac{1}{n+m}$. Since the set $\{\frac{1}{j} : j \in \mathbb{N}\}$ is discrete there is $\varepsilon > 0$ such that $|\frac{1}{n+m} - \frac{1}{j}| > \varepsilon$ for each $j \neq n+m$. Denote by $K = [0, \frac{1}{n+m} - \varepsilon] \cup [\frac{1}{n+m} + \varepsilon, 1]$. Let U be an arbitrary neighbourhood of x. Since $\operatorname{Int} X_m = \emptyset$ there is $k \in \mathbb{N}$ with $k \neq n$ and $U \cap \operatorname{Int}(C_{n-1} \setminus C_n) \cap X_k \neq \emptyset$. We have $h(U \cap \operatorname{Int}(C_{n-1} \setminus C_n) \cap X_k) = \{\frac{1}{n+m}\} \subset K$. Therefore $(\operatorname{Cl} h(U) \cap K)_{U \in \mathcal{U}_x}$ is a family of closed subsets of K with the finite intersection property. Hence $M = \bigcap_{U \in \mathcal{U}_x} \operatorname{Cl} h(U) \cap K \subset C(h,x)$ is a nonempty set. Since $h(x) \notin M$ we have $C(h,x) \neq \{h(x)\}$ and $x \notin H(h)$.

b) Let $x \in (C_{n-1} \setminus C_n) \setminus \operatorname{Int}(C_{n-1} \setminus C_n) = B_n$. Then $h(x) = \frac{1}{n}$. Let $\varepsilon > 0$ be such that $|\frac{1}{n} - \frac{1}{j}| > \varepsilon$ for each $j \neq n$. Denote by $K = [0, \frac{1}{n} - \varepsilon]$. Let U be an arbitrary neighbourhood of x. Since $\operatorname{Int} B_n = \emptyset$ there is $y \in (U \cap C_{n-1}) \setminus B_n$. Then $h(y) < \frac{1}{n}$ and hence $h(y) \in K$. Now $(\operatorname{Cl} h(U) \cap K)_{U \in \mathcal{U}_x}$ is a family of closed subsets of K with the finite intersection property. Similarly as in a) we show that $x \notin H(h)$.

Since $x \in C(g)$ then, according to Lemma 1 we have $x \notin H(f)$. Therefore we see that

$$(X \setminus H) \cap S \subset X \setminus H(f). \tag{5}$$

Combining (1), (2), (3), (4) and (5) we obtain $H = H(f)$ and $C = C(f)$. □

COROLLARY 1. *Let X be a complete metric space without isolated points. If C and H are subsets of X then conditions (A) and (B) are equivalent.*

Remark 3. a) The condition "X is normal" cannot be omitted. In [1] it is shown that there is a closed nowhere dense subset F of the Niemytzki plane X such that $D(f) \neq F$ for each function $f : X \to \mathbb{R}$ with a closed graph. The Niemytzki plane X is a Baire perfect T_1 completely regular almost resolvable space and the sets $H = X$ and $C = X \setminus F$ satisfy (B).

b) The condition "X is perfect" cannot be omitted. Let X be the set of all ordinal numbers which are less than the first uncountable ordinal Ω equipped with the topology $\mathcal{T}$, where $A \in \mathcal{T}$ iff $A = \emptyset$ or $A = X$ or $A = \{x \in X : x > \alpha\}$ for some $\alpha \in X$. Then X is a Baire almost resolvable normal space and $H = X$ and $C = X \setminus \{1\}$ satisfy (B). However, X is compact and hence (by Proposition 5) for every function $f : X \to \mathbb{R}$ we have $H(f) = C(f)$.

c) Evidently the condition "X is almost resolvable "cannot be omitted.

d) We do not known what is in the case of non-Baire spaces.

PROBLEM. Is Theorem 3 true if we omit the condition "X is a Baire space" (and if we replace the condition (iv) with (iv'), where
(iv'): $\operatorname{Int}(H \setminus C)$ is of the first category)?

Remark 4. If $f : X \to \mathbb{R}$ has a closed graph then by [3] the set $D(f)$ $(= H(f) \setminus C(f))$ is of the first category and closed. If moreover X is a

Baire space then $D(f)$ is even nowhere dense. However this is not true for $H(f) \setminus C(f)$, if $H(f) \neq X$. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \\ n, & \text{if } x = q_n, \end{cases}$$

where $\mathbb{Q} = \{q_1, q_2, \dots, q_n, \dots\}$ is a one-to-one sequence. Then $C(f) = \emptyset$ and $H(f) = \mathbb{R} \setminus \mathbb{Q}$, therefore $H(f) \setminus C(f)$ is residual and not closed.

References

[1] I. Baggs, *Nowhere dense sets and real-valued functions with closed graphs*, Internat. J. Math. Math. Sci. 12 (1989), 1–8.

[2] R. Bolstein, *Sets of points of discontinuity*, Proc. Amer. Math. Soc. 38 (1973), 193–197.

[3] J. Doboš, *On the set of points of discontinuity for functions with closed graphs*, Časopis Pěst. Mat. 110 (1985), 60–68.

[4] R. Engelking, *General Topology*, PWN, Warszawa, 1977.

[5] K. R. Gentry, H. B. Hoyle, *c-continuous functions*, Yokohama Math. J. 18 (1970), 71–76.

[6] T. R. Hamlett, L. L. Herrington, *The Closed Graph and P-closed Graph Properties in General Topology*, Vol. 3, Contemporary Mathematics, Amer. Math. Soc., Providence, Rhode Island, 1980.

[7] P. Kostyrko, *A note on the functions with closed graphs*, Časopis Pěst. Mat. 94 (1969), 202–205.

[8] P. Kostyrko, T. Šalát, *On functions, the graphs of which are closed sets*, Časopis Pěst. Mat. 89 (1964), 426–432, (in Russian).

[9] P. E. Long, *Functions with closed graphs*, Amer. Math. Monthly 76 (1969), 930–932.

[10] P. E. Long, M. D. Hendrix, *Properties of c-continuous functions*, Yokohama Math. J. 22 (1974), 119–123.

[11] P. E. Long, L. L. Herrington, *Properties of c-continuous and c*-continuous functions*, Kyungpook Math. J. 15 (1975), 213–221.

[12] T. Neubrunn, *c-continuity and closed graphs*, Časopis Pěst. Mat. 110 (1985), 172–178.

[13] R. J. Pawlak, *On local characterization of closed functions and functions with closed graphs*, Demonstratio Math. 19 (1986), 181–188.

[14] T. Thompson, *Characterizing certain sets with functions having a closed graph*, Boll. Un. Mat. Ital. 12 (1975), 327–329.

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Received March 3, 1995.