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ON A THEOREM BY ROMAN WĘGRZYK

1. Introduction

The concept of generalized metric space was introduced by Jung as follows:

DEFINITION 1.1 ([3]). The pair (X, d) will be called a generalized metric space if X is an arbitrary nonempty set and d is a function $d : X \times X \to [0, \infty]$ which fulfils all the standard conditions for a metric.

In this space, the generalized metric d is allowed to take on $+\infty$ as well. In a generalized metric space, just as in a metric space, we can define open and closed balls, convergences of sequences, completeness of the space, etc.

C.K. Jung gave in [3] the following characterization of the generalized metric space by metric spaces. Let (X_i, d_i), $i \in I$ be a nonempty family of disjoint metric spaces. Then the set

$$X = \bigcup_{i \in I} X_i,$$

with the function d defined by

$$d(x,y) := \begin{cases} d_i(x,y), & \text{if there exists } i \in I \text{ such that } x, y \in X_i \\ +\infty, & \text{otherwise} \end{cases}$$

is a generalized metric space. The converse is also true.

THEOREM 1.2 ([3]). *Let (X, d) be a generalized metric space. Then the relation "ρ" defined as*

$$x\rho y \Leftrightarrow d(x,y) < \infty \text{ for } x, y \in X$$

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is an equivalence relation and if $\{X_i | i \in I\}$ *are the equivalence classes under* ρ, *then* $d(x, y) = +\infty$ *whenever* $x \in X_i$, $y \in X_j$, $i \neq j$. *Also, if we let* $d_i := d|_{X_i \times X_i}$ *then* (X_i, d_i) *is a metric space for each* $i \in I$.

DEFINITION 1.3 ([3]). The partition of a generalized metric space (X, d) into a family of disjoint metric spaces (X_i, d_i), $i \in I$ constructed in Theorem 1.2 will be called the canonical partition of the space (X, d).

A metric space (X, d) is said to be ϵ-chainable (where $\epsilon > 0$ is fixed) if and only if for any given $a, b \in X$ there is an ϵ-chain from a to b; i.e. that is a finite set of points $z_0 = a, z_1, \dots, z_n = b$ such that $d(z_{i-1}, z_i) < \epsilon$ for all $i = 1, 2, \dots, n$.

DEFINITION 1.4. We say that a generalized metric space (X, d) is ϵ-chainable (where $\epsilon > 0$ is a fixed number) iff for any given $x, y \in X$ such that $d(x, y) < \infty$ there is an ϵ-chain from x to y.

DEFINITION 1.5. A generalized metric space is called well-chained if and only if is ϵ-chainable for each $\epsilon > 0$.

The first purpose of this paper is to give a characterization of the ϵ-chainable (respectively well-chainable) generalized metric spaces by ϵ-chainable (respectively well-chainable) metric spaces.

On the other hand, in 1969 Jung proved the following theorem:

THEOREM 1.6 ([3]). *Let* (X, d) *be a complete generalized metric space and let* $f : X \to X$ *be a singlevalued contraction, i.e. there exists a fixed real number* $a \in [0, 1[$ *such that*

$$\forall x, y \in X, \qquad d(x, y) < \infty \Rightarrow d(f(x), f(y)) \leq a d(x, y).$$

If there exists a point $x_0 \in X$ *such that* $d(x_0, f(x_0)) < \infty$ *then* f *has a fixed point* x^* *(i.e.* $x^* \in X$, $x^* = f(x^*)$*).*

The second purpose of this paper is to investigate the problem of existence of such results for a class of multivalued operators.

2. Preliminaries

If (X, d) is a generalized metric space, $Y \subset X$, $x \in X$ and $\epsilon > 0$ then

$\delta(Y) = \sup\{d(a, b) | a, b \in Y\}$

$D(Y, x) = \inf\{d(y, x) | y \in Y\}$

$S(Y, \epsilon) = \{x \in X | D(Y, x) < \epsilon\}$

$P(X) = \{Y \subset X | Y \neq \emptyset\}$

$P_{cl}(X) = \{Y \in P(X) | Y = \bar{Y}\}$

$P_{b,cl}(X) = \{Y \in P_{cl}(X) | \delta(Y) < \infty\}$

$$H(A,B) = \begin{cases} \inf\{\epsilon > 0 | A \subset S(B,\epsilon),\ B \subset S(A,\epsilon)\}, & \text{if the infimum exists} \\ +\infty, & \text{otherwise} \end{cases}$$

The pair $(P_{cl}(X), H)$ is a generalized metric space, and H is called the generalized Hausdorff-Pompeiu distance induced by d.

LEMMA 2.1 ([2]). *If (X,d) is a complete generalized metric space then $(P_{cl}(X), H)$ is a complete generalized metric space.*

DEFINITION 2.2 ([7]). A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a strong comparison function iff:

i) φ is strictly increasing,

ii) $\sum_{n=1}^{\infty} \varphi^n(t) < \infty, \qquad \forall t > 0 \qquad$ (where $\mathbb{R}_+ = [0,\infty[$).

LEMMA 2.3 (see [7], pp. 31). *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a strong comparison function. Then*:

i) $\varphi(t) < t$, *for all* $t > 0$,

ii) $\varphi(0) = 0$,

iii) φ *is continuous from the right in* $t = 0$.

DEFINITION 2.4. Let (X,d) be a generalized metric space and $T : X \to P_{cl}(X)$ be a multivalued operator. Then T is called:

i) a-contraction if there exists an $a \in [0,1[$ such that $\forall x,y \in X$, $d(x,y) < \infty \Rightarrow H(T(x),T(y)) \le ad(x,y)$,

ii) φ-contraction if there exists a strong comparison function $\varphi : \mathbb{R}_+ \to R_+$ such that $\forall x,y \in X,\ d(x,y) < \infty \Rightarrow H(T(x),Ty)) \le \varphi(d(x,y))$.

DEFINITION 2.5. Let (X,d) be a generalized metric space and $T : X \to P(X)$ be a multivalued operator. Then $x^* \in X$ is called a fixed point for T if $x^* \in T(x^*)$. The set of all fixed points will be denoted by Fix T.

The following results are well known in the fixed point theory (see Covitz-Nadler (1970), Rus (1983), Wegrzyk (1982)).

THEOREM 2.6 ([2]). *Let (X,d) be a complete metric space and consider a multivalued a-contraction $T : X \to P_{cl}(X)$. Then* Fix $T \neq \emptyset$.

THEOREM 2.7 ([8]). *Let (X,d) be a complete metric space and consider a multivalued φ-contraction $T : X \to P_{cl}(X)$. Then* Fix $T \neq \emptyset$.

3. Basic results

The first result of this paper is the following:

THEOREM 3.1. *Let (X,d) be a generalized metric space and let (X_i, d_i), $i \in I$ be constructed in Theorem 1.2 the canonical partition of the space (X,d). Then X is ϵ-chainable if and only if X_i is ϵ-chained for each $i \in I$.*

Proof. "$\Rightarrow$" Suppose that (X,d) is a ϵ-chainable generalized metric space. Then, for each $x, y \in X$ such that $d(x,y) < \infty$ there is an ϵ-chain from x to y. Let $i \in I$ and let $x, y \in X_i$. Then $d(x,y) < \infty$. It follows that there is an ϵ-chain from x to y.

"$\Leftarrow$" Suppose that for each $i \in I$, (X_i, d_i) is ϵ-chainable. Let $x, y \in X$ such that $d(x,y) < \infty$. Then there is $i \in I$ such that $x, y \in X_i$. Since X_i is ϵ-chainable there is an ϵ-chain from x to y in $X_i \subset X$.

Using similar arguments as in above we may show

THEOREM 3.2. *Let (X,d) be a generalized metric space and (X_i, d_i), $i \in I$ be the canonical partition of the space (X,d). Then X is well-chained if and only if X_i is well-chained, for each $i \in I$.*

The main result of this paper is the following:

THEOREM 3.3. *Let (X,d) be a complete generalized metric space and let $T : X \to P_{cl}(X)$ be a multivalued φ-contraction. Suppose that there exists a point $x_0 \in X$ such that $D(x_0, T(x_0)) < \infty$. Then* Fix $T \neq \emptyset$.

Proof. Let $X = \cup_{i \in I} X_i$ be the canonical partition of the space (X,d). Let $i \in I$ be such that $x_0 \in X_i$.

We shall prove that for each $x \in X_i : T(x) \cap X_i \neq \emptyset$. Let $x \in X_i$ be an arbitrary point.

Observe that $D(x, T(x)) < \infty \Leftrightarrow$ there is $y \in T(x)$ such that $d(x,y) < \infty \Leftrightarrow y \in T(x) \cap X_i \Leftrightarrow T(x) \cap X_i \neq \emptyset$.

Then, for $x \in X_i : T(x) \cap X_i \neq \emptyset$ if and only if $D(x, T(x)) < \infty$. But

$$\begin{aligned} D(x,T(x)) &\leq D(x,T(x_0)) + H(T(x_0),T(x)) \\ &\leq d(x,x_0) + D(x_0,T(x_0)) + H(T(x_0),T(x)) \\ &\leq d(x,x_0) + D(x_0,T(x_0)) + \varphi(d(x_0,x)) < \infty, \end{aligned}$$

then for each $x \in X_i$: $T(x) \cap X_i \neq \emptyset$.

Consider the multivalued operator

$$T^0 : X_i \to P_{cl}(X_i), \text{ given by } T^0(x) := T(x) \cap X_i$$

and observe that T^0 is a multivalued φ-contraction on a complete metric space (X_i, d_i). Now the conclusion follows from Theorem 2.7. ∎

COROLLARY 3.4. *Let (X,d) be a complete generalized metric space and let $T : X \to P_{cl}(X)$ be a multivalued a-contraction. Suppose that there exists a point $x_0 \in X$ such that $D(x_0, T(x_0)) < \infty$. Then* Fix $T \neq \emptyset$.

Proof. The conclusion follows from Theorem 3.3 by taking $\varphi(t) = at$, for each $t \in \mathbb{R}_+$ (where $a \in [0,1[$). ∎

Remark 3.5. Theorem 3.3 generalizes a result given by Covitz-Nadler (see [2] Corollary 3).

Remark 3.6. Fixed point theorems for a class of locally contractive multivalued operators (see [1] and [4]) are established in a previous paper (see [5]).

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