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ON COLLISION INVARIANTS IN ONE DIMENSION

Consider a system of n particles, with the kinetic energies $\omega_i(\vec{p}_i)$ for $i = 1, \ldots, n$ and the potential energy of the interaction $V(\vec{x}_1, \ldots, \vec{x}_n)$, which is invariant under translations. Then the trajectories of a system lie on level surfaces of the total energy H and momentum, and are solutions of the Hamiltonian system of differential equations of the form

$$\frac{d\vec{x}_i}{dt} = \frac{\partial H}{\partial \vec{p}_i}, \quad \frac{d\vec{p}_i}{dt} = -\frac{\partial H}{\partial \vec{x}_i} \quad \text{for } i = 1, \ldots, n.$$

Let us assume that there is a global trajectory of a system that have the asymptotic free trajectories $\vec{x}_i^{in}(t), \vec{p}_i$ $(i = 1, \ldots, n)$ at $t = -\infty$ and $\vec{x}_i^{out}(t), \vec{q}_i$ $(i = 1, \ldots, n)$ at $t = \infty$, which are solutions of the Hamiltonian system of differential equations with V equal to zero.

We call a set of functions $f_i(\vec{x}_i, \vec{p}_i, t)$ $(i = 1, \ldots, n)$ a collision invariant of a system if

$$f_1(\vec{x}_1^{in}(t), \vec{p}_1, t) + \ldots + f_n(\vec{x}_n^{in}(t), \vec{p}_n, t) = f_1(\vec{x}_1^{out}(t), \vec{q}_1, t) + \ldots + f_n(\vec{x}_n^{out}(t), \vec{q}_n, t)$$

and

$$\frac{df_i(\vec{x}_i^{in}(t), \vec{p}_i, t)}{dt} = 0 = \frac{df_i(\vec{x}_i^{out}(t), \vec{q}_i, t)}{dt} \quad \text{for } i = 1, \ldots, n$$

for all free trajectories that are asymptotic to some global trajectories of a system (cf. [1]). We note that

$$\vec{p}_1 + \ldots + \vec{p}_n = \vec{q}_1 + \ldots + \vec{q}_n \quad \text{and} \quad \omega_1(\vec{p}_1) + \ldots + \omega_n(\vec{p}_n) = \omega_1(\vec{q}_1) + \ldots + \omega_n(\vec{q}_n).$$

The general form of collision invariants of a system of interacting particles in three dimensions was derived in [1]. In particular, it follows that collision invariants that depend only on the momenta must be linear in the kinetic energy and momentum (see also the references quoted in [1]). It was also argued in [1] that there is no analogous result for one dimension (cf. Example 3 of the present paper).

It is the purpose of this paper to show that also in one dimension, collision invariants of a system of more than two interacting particles (which depend only on the momenta) must be linear in the kinetic energy and momentum. This result was already quoted in [2].

1. Our first theorem reads as follows.

THEOREM 1. *Suppose that:*

(a) *$n \in \mathbf{N}$ and $n \geq 3$;*

(b) *$\omega_i : \mathbf{R} \to \mathbf{R}$ are C^1 functions for $i = 1, \ldots, n$ such that at least two of them have nonconstant derivative;*

(c) *M is a subset of $\mathbf{R}^{2n}$ defined by*

$$M = \{(p_1, \ldots, p_n, q_1, \ldots, q_n) \in \mathbf{R}^{2n} : \vec{p}_1 + \ldots + \vec{p}_n = \vec{q}_1 + \ldots + \vec{q}_n, \quad \text{and} \quad \omega_1(\vec{p}_1) + \ldots + \omega_n(\vec{p}_n) = \omega_1(\vec{q}_1) + \ldots + \omega_n(\vec{q}_n)\}.$$

If $f_i : \mathbf{R} \to \mathbf{R}$ are C^1 functions for $i = 1, \ldots, n$ satisfying the functional equation

$$(*) \qquad f_1(p_1) + \ldots + f_n(p_n) = f_1(q_1) + \ldots + f_n(q_n) \quad \text{for } (p_1, \ldots, p_n, q_1, \ldots, q_n) \in M,$$

then exist constants a, b, c_i ($i = 1, \ldots, n$) such that

$$f_i(p) = a\omega_i(p) + bp + c_i \quad \text{for } p \in \mathbf{R} \text{ and } i = 1, \ldots, n.$$

Proof. The assumptions ensure that

$$(1) \qquad dp_1 + \ldots + dp_n = dq_1 + \ldots + dq_n,$$

$$(2) \qquad \omega_1'(p_1)dp_1 + \ldots + \omega_n'(p_n)dp_n = \omega_1'(q_1)dq_1 + \ldots + \omega_n'(q_n)dq_n,$$

$$(3) \qquad f_1'(p_1)dp_1 + \ldots + f_n'(p_n)dp_n = f_1'(q_1)dq_1 + \ldots + f_n'(q_n)dq_n$$

for $(p_1, \ldots, p_n, q_1, \ldots, q_n) \in M$. Solving the system of equations (1) and (2) for dp_i and dq_j, substituting into (3) and equating to zero the coefficient of dq_k, we get

$$(4) \qquad f_i'(p_i) = \frac{f_j'(q_j) - f_k'(q_k)}{\omega_j'(q_j) - \omega_k'(q_k)}\omega_i'(p_i) + \frac{\omega_j'(q_j)f_k'(q_k) - f_j'(q_j)\omega_k'(q_k)}{\omega_j'(q_j) - \omega_k'(q_k)},$$

where p_i, q_j, q_k can range over all real numbers such that

$$\omega_i'(p_i) \neq \omega_j'(q_j), \quad \omega_k'(q_k) \neq \omega_j'(q_j), \quad \text{and} \quad (p_1, \ldots, p_n, q_1, \ldots, q_n) \in M$$

for some $p_l \in \mathbf{R}$ ($l \in \{1, \ldots, n\} \setminus \{i\}$) and $q_m \in \mathbf{R}$ ($m \in \{1, \ldots, n\} \setminus \{j, k\}$).

We note that M contains the diagonal of $\mathbf{R}^{2n} = \mathbf{R}^n \times \mathbf{R}^n$. Thus no problem arises in the last condition when i, j, k are pairwise different, what we from now assume.

Let ω_j' be a nonconstant function and assume the following lemma.

LEMMA 1. *If $\varepsilon : \mathbf{R} \to \mathbf{R}$ is a nonconstant continuous function, then a set of $r \in \varepsilon(\mathbf{R})$ such that the set $\{p \in \mathbf{R} : \varepsilon(p) \neq r\}$ is dense in $\mathbf{R}$, is dense in $\varepsilon(\mathbf{R})$.*

Then one can easily see that for fixed i, k and $q_k \in \mathbf{R}$ there exists a set $J_{ij}(q_k) \subset \omega_j'(\mathbf{R})$, which contains more than one point (even infinitely many points), such that the set

$$\{p \in \mathbf{R} : \omega_i'(p_i) \neq \omega_j'(q_j),\ \omega_k'(q_k) \neq \omega_j'(q_j)\}$$

is dense in $\mathbf{R}$ for all $\omega_j'(q_j) \in J_{ij}(q_k)$. Hence, by (4) and continuity of f_i', we obtain (for $q_k \in \mathbf{R}$ and $\omega_j'(q_j) \in J_{ij}(q_k)$)

$$f_i'(p) = a_{jk}(q_j, q_k)\omega_i'(p) + b_{jk}(q_j, q_k) \quad \text{for all } p \in \mathbf{R}, \tag{5}$$

where

$$\begin{aligned} a_{jk}(q_j, q_k) &= \frac{f_j'(q_j) - f_k'(q_k)}{\omega_j'(q_j) - \omega_k'(q_k)}, \\ b_{jk}(q_j, q_k) &= \frac{\omega_j'(q_j) f_k'(q_k) - f_j'(q_j)\omega_k'(q_k)}{\omega_j'(q_j) - \omega_k'(q_k)}. \end{aligned} \tag{6}$$

By the hypothesis (b) we can (and do) assume that in addition ω_i' is a nonconstant function. Then the functions ω_i' and 1 are linearly independent, so a_{jk} and b_{jk} are constant. Now replacing i by j and j by i we obtain (for $q_k \in \mathbf{R}$ and $\omega_i'(q_i) \in J_{ji}(q_k)$)

$$f_j'(p) = a_{ik}(q_i, q_k)\omega_j'(p) + b_{ik}(q_i, q_k) \quad \text{for all } p \in \mathbf{R}, \tag{7}$$

where

$$\begin{aligned} a_{ik}(q_i, q_k) &= \frac{f_i'(q_i) - f_k'(q_k)}{\omega_i'(q_i) - \omega_k'(q_k)}, \\ b_{ik}(q_i, q_k) &= \frac{\omega_i'(q_i) f_k'(q_k) - f_i'(q_i)\omega_k'(q_k)}{\omega_i'(q_i) - \omega_k'(q_k)}. \end{aligned} \tag{8}$$

Moreover, a_{ik} and b_{ik} are constant too. Therefore, it follows from (6) and (7) that

$$f_k'(q_k) = (a_{ik} - a_{jk})\omega_j'(q_j) + a_{jk}\omega_k'(q_k) + b_{ik} \quad \text{for every } q_k \in \mathbf{R}$$

and $\omega_j'(q_j) \in J_{ij}(q_k)$. But for each $q_k \in \mathbf{R}$ the set $J_{ij}(q_k)$ contains more than one point, so

$$a_{ik} = a_{jk} \quad \text{and} \quad f_k'(p) = a_{jk}\omega_k'(p) + b_{ik} \quad \text{for all } p \in \mathbf{R}. \tag{9}$$

In similar fashion, using (8) and (5), we get additionally

$$f_k'(p) = a_{ik}\omega_k'(p) + b_{jk} \quad \text{for all } p \in \mathbf{R}. \tag{10}$$

Hence $b_{ik} = b_{jk}$. Since $k \in \{1, \ldots, n\} \setminus \{i, j\}$ was arbitrary, the desired result now readily follows from (5), (7), and (10) with $a = a_{ik} = a_{jk}$ and $b = b_{ik} = b_{jk}$. This completes the proof. □

Proof of Lemma 1. The interior of $\varepsilon(\mathbf{R})$ is a nonempty open interval. Therefore, it suffices to verify that each nonempty open interval contains a point r such that the set $\{p \in \mathbf{R} : \varepsilon(p) = r\}$ not contains any nonempty open interval. But to this end it is enough to observe that a set of points r such that the set $\{p \in \mathbf{R} : \varepsilon(p) = r\}$ contains a nonempty open interval is countable. □

Before we pass to the second theorem we are going to give the following remark.

Remark. From the proof it may be seen that for the validity of Theorem 1 the functional equation $(*)$ is not needed on the whole set M. This is important because there exist systems for which not all free trajectories that lie on level surfaces of the energy and momentum are asymptotic to some global trajectories (cf. [1]).

2. In the case of more special assumptions we have the following information about collision invariants.

THEOREM 2. *Suppose that the conditions* (a), (b), *and* (c) *of Theorem 1 are satisfied and in addition for* $i = 1, \ldots, n$ *there is a Lie group* G *left action* $\cdot$ *on* $W_i = \{(p, \omega_i(p)) : p \in \mathbf{R}\}$ *such that*:

(d) *the mapping* $G \times W_i \ni (g, w) \mapsto g \cdot w \in W_i$ *is* C^1;

(e) $e \cdot w = w$ *for all* $w \in W_i$, *where* e *is a unit element in* G;

(f) $(g_1 g_2) \cdot w = g_1 \cdot (g_2 \cdot w)$ *for all* $g_1, g_2 \in G$ *and* $w \in W_i$;

(g) *for* $v, w \in W_i$ *there exists* $g \in G$ *such that* $g \cdot v = w$;

(h) *for a fixed* $v \in W_i$ *there exists a* C^1 *function* $W_i \ni w \mapsto g(w) \in G$ *such that* $g(w) \cdot v = w$ *for all* $w \in W_i$;

(i) *if* $w_i, v_i \in W_i$ $(i = 1, \ldots, n)$, $g \in G$ *and* $w_1 + \ldots + w_n = v_1 + \ldots + v_n$, *then* $g \cdot w_1 + \ldots + g \cdot w_n = g \cdot v_1 + \ldots + g \cdot v_n$.

If $f_i : \mathbf{R} \to \mathbf{R}$ *are* L^1_{loc} *Borel functions for* $i = 1, \ldots, n$ *satisfying the functional equation*

$$(*) \qquad f_1(p_1) + \ldots + f_n(p_n) = f_1(q_1) + \ldots + f_n(q_n) \quad \text{for } (p_1, \ldots, p_n, q_1, \ldots, q_n) \in M,$$

then exist constants a, b, c_i $(i = 1, \ldots, n)$ *such that*

$$f_i(p) = a\omega_i(p) + bp + c_i \quad \text{for almost all } p \in \mathbf{R} \text{ and } i = 1, \ldots, n.$$

Proof. Let for any $g \in G$, $p \in \mathbf{R}$ and $i \in \{1, \ldots, n\}$, $h_i(g)(p)$ means the point in $\mathbf{R}$ such that

$$g \cdot (p, \omega_i(p)) = (h_i(g)(p), \omega_i(h_i(g)(p))).$$

Then it is easy to check that conditions (d) - (i) imply:

(d_1) the mapping $G \times \mathbf{R} \ni (g,p) \mapsto h_i(g)(p) \in \mathbf{R}$ is C^1;

(e_1) $h_i(e)(p) = p$ for all $p \in \mathbf{R}$, where e is a unit element in G;

(f_1) $h_i(g_1g_2)(p) = h_i(g_1)(h_i(g_2)(p))$ for all $g_1, g_2 \in G$ and $p \in \mathbf{R}$;

(g_1) for $q, p \in \mathbf{R}$ there exists $g \in G$ such that $h_i(g)(q) = p$;

(h_1) for a fixed $q \in \mathbf{R}$ there exists a C^1 function $\mathrm{R} \ni p \mapsto g(p,i) \in G$ such that $h_i(g(p,i))(q) = p$ for all $p \in \mathbf{R}$;

(i_1) if $(p_1, \ldots, p_n, q_1, \ldots, q_n) \in M$ and $g \in G$, then

$$(h_1(g)(p_1), \ldots, h_n(g)(p_n), h_1(g)(q_1), \ldots, h_n(g)(q_n)) \in M.$$

Therefore

$$\begin{aligned}(11) \qquad f_1(h_1(g)(p_1)) + \ldots + f_n(h_n(g)(p_n)) = f_1(h_1(g)(q_1)) + \ldots \\ + f_n(h_n(g)(q_n))\end{aligned}$$

for $(p_1, \ldots, p_n, q_1, \ldots, q_n) \in M$ and $g \in G$.

Now the proof depends on the following lemma.

LEMMA 2. *Let $f : \mathbf{R} \to \mathbf{R}$ be a L^1_{loc} Borel function. Let G be a Lie group, and let μ be a right invariant Haar measure on the Borel sets in G. Suppose further that the mapping $G \times \mathbf{R} \ni (g,p) \mapsto h(g)(p) \in \mathbf{R}$ is C^1 and satisfies the conditions* (e_1) - (h_1) *with the index i deleted. Then*

(i) *the mapping $G \times \mathbf{R} \ni (g,p) \mapsto f(h(g)(p)) \in \mathbf{R}$ is a L^1_{loc} (with respect to $\mu \times dp$) Borel function (here dp stands for Lebesgue measure on the Borel sets in* $\mathbf{R}$);

(ii) *for any fixed $g \in G$ the mapping $\mathbf{R} \ni p \mapsto f(h(g)(p)) \in \mathbf{R}$ is a L^1_{loc} Borel function*;

(iii) *for any fixed $\psi \in C^0_0(\mathbf{R})$ the function $G \ni g \mapsto \int_{\mathbf{R}} f(h(g)(p))\psi(p)dp \in \mathbf{R}$ is continuous*;

(iv) *for any fixed $p \in \mathbf{R}$ the mapping $G \ni g \mapsto f(h(g)(p)) \in \mathbf{R}$ is a L^1_{loc} (with repect to μ) Borel function*;

(v) *for any fixed $\phi \in C^1_0(G)$ the function $\mathbf{R} \ni p \mapsto \int_G f(h(g)(p))\phi(g)d\mu(g) \in \mathbf{R}$ is C^1.*

Let us assume the lemma and set

$$F_i^\phi(p) = \int_G f_i(h_i(g)(p))\phi(g)d\mu(g) \quad \text{for } p \in \mathbf{R},\ \phi \in C^1_0(G) \text{ and } i = 1, \ldots, n.$$

Then, by (v) and (11), the functions F_i^ϕ are C^1 for $i = 1, \ldots, n$ and satisfy the functional equation (*). Application of Theorem 1 therefore shows that

$$F_i^\phi(p) = a(\phi)\omega_i(p) + b(\phi)p + c_i(\phi) \quad \text{for } p \in \mathbf{R},\ \phi \in C^1_0(G) \text{ and } i = 1, \ldots, n.$$

or

$$\int_{\mathbf{R}}\int_{G} f_i(h_i(g)(p))\phi(g)\psi(p)d\mu(g)dp = a(\phi)\int_{\mathbf{R}}\omega_i(p)\psi(p)dp + b(\phi)\int_{\mathbf{R}} p\psi(p)dp + c_i(\phi)\int_{\mathbf{R}}\psi(p)dp \tag{12}$$

for $\psi \in C_0^0(\mathbf{R})$, $\phi \in C_0^1(G)$, and $i = 1, \ldots, n$.

Now, let $(U_m)_{m\in\mathbf{N}}$ be an open basis at $e \in G$, and let $(\phi_m)_{m\in\mathbf{N}}$ be a sequence of nonnegative C^1 functions on G such that $\int_G \phi_m(g)d\mu(g) = 1$ and ϕ_m is zero outside U_m for $m \in \mathbf{N}$. Then (ii), (iii), (e_1), and simple estimates show that

$$\lim_{m\to\infty}\int_{\mathbf{R}}\int_{G} f_i(h_i(g)(p))\phi_m(g)\psi(p)d\mu(g)dp = \int_{\mathbf{R}} f_i(p)\psi(p)dp$$

for $\psi \in C_0^0(\mathbf{R})$ and $i = 1, \ldots, n$. Appling this to (12) we obtain

$$\int_{\mathbf{R}} f_i(p)\psi(p)dp = \lim_{m\to\infty}\left[a(\phi_m)\int_{\mathbf{R}}\omega_i(p)\psi(p)dp + b(\phi_m)\int_{\mathbf{R}} p\psi(p)dp + c_i(\phi_m)\int_{\mathbf{R}}\psi(p)dp\right] \tag{13}$$

for $\psi \in C_0^0(\mathbf{R})$ and $i = 1, \ldots, n$.

Now the proof is easy completed. Namely, taking in (13) the function ω_i with nonconstant derivative, $\psi = \chi''$ and $\chi \in C_0^2(\mathbf{R})$ such that $\int_{\mathbf{R}}\omega_i'(p)\chi'(p)dp \neq 0$, we see that the sequence $(a(\phi_m))_{m\in\mathbf{N}}$ converges. Using this and taking in (13) $\psi \in C_0^0(\mathbf{R})$ be such that $\int_{\mathbf{R}} p\psi(p)dp \neq 0$ and $\int_{\mathbf{R}}\psi(p)dp = 0$; we get that the sequence $(b(\phi_m))_{m\in\mathbf{N}}$ converges. Thus the sequence $(c_i(\phi_m))_{m\in\mathbf{N}}$ $(i = 1, \ldots, n)$ converges too. Therefore

$$\int_{\mathbf{R}} f_i(p)\psi(p)dp = a\int_{\mathbf{R}}\omega_i(p)\psi(p)dp + b\int_{\mathbf{R}} p\psi(p)dp + c_i\int_{\mathbf{R}}\psi(p)dp$$

for $\psi \in C_0^0(\mathbf{R})$, where $a = \lim_{m\to\infty} a(\phi_m)$, $b = \lim_{m\to\infty} b(\phi_m)$, and $c_i = \lim_{m\to\infty} c_i(\phi_m)$ $(i = 1, \ldots, n)$.

Since the last property is equivalent to the conclusion of the theorem, the proof is ended. □

Proof of Lemma 2. The composition of Borel functions is a Borel function, so the condition of Borel measurability in (i), (ii), and (iv) follows immediately from our assumptions. Moreover, we can (and do) restrict ourselves to the case when the function f is nonnegative. Then the condition of local integrability with respect to $\mu \times dp$ in (i) is a consequence of (iii).

Now, it is clear that to prove (iii) we can assume that in addition the function ψ is nonnegative. Then, by (d₁) - (f₁) and change of variables formula, we have

$$(14)\quad \int_{\mathbf{R}} f(h(g)(p))\psi(p)dp = \int_{\mathbf{R}} f(p)\psi(h(g^{-1})(p))|J_g(p)|dp < \infty \text{ for } g \in G,$$

where J_g is the Jacobian of the C^1 transformation $\mathbf{R} \ni p \mapsto h(g^{-1})(p) \in \mathbf{R}$.

So, (ii) holds and (iii) easy follows, since for any fixed $p \in \mathbf{R}$ the integrand in the right side of (14) is a continuous function on G and has support contained in a compact subset of $\mathbf{R}$, if g ranges over a compact subset of G.

To see that (iv) holds, it is enough to observe that by (iii), $G \ni g \mapsto f(h(g)(p)) \in \mathbf{R}$ is a L^1_{loc} (with respect to μ) Borel function for almost all $p \in \mathbf{R}$ and, by the right invariance of Haar measure μ, we have

$$\int_G f(h(g)(p))|\phi(g)|d\mu(g) = \int_G f(h(g)(q))|\phi(gg(p)^{-1})|d\mu(g)$$

for any (by (g_1) and (h_1)) fixed $q \in \mathbf{R}$ and all $p \in \mathbf{R}$. Finally, (v) is true since

$$(15)\quad \int_G f(h(g)(p))\phi(g)d\mu(g) = \int_G f(h(g)(q))\phi(gg(p)^{-1})d\mu(g) \text{ for } p \in \mathbf{R},$$

and for any fixed $g \in G$ the integrand in the right side of (15) is a C^1 function on $\mathbf{R}$ and has support contained in a compact subset of G, if p ranges over a compact subset of $\mathbf{R}$. □

3. Now, let us consider some examples

EXAMPLE 1. Put $\omega_i(p) = \frac{p^2}{2m_i}$ for $p \in \mathbf{R}$, $m_i > 0$, $(i = 1, \ldots, n)$ and suppose that $G = \mathbf{R}$ is a Lie group of real numbers under the operation of addition. Let the left action of a group G on $W_i = \{(p, \frac{p^2}{2m_i}) : p \in \mathbf{R}\}$ for $i = 1, \ldots, n$ be given by

$$g \cdot \left(p, \frac{p^2}{2m_i}\right) = \left(p + m_i g, \frac{p^2}{2m_i} + gp + \frac{m_i g^2}{2}\right)$$

for $g, p \in \mathbf{R}$. Then it is easy to verify that the conditions (b) - (i) are satisfied. Consequently, in case $n \geq 3$, Theorem 2 may be employed to obtain

$$f_i(p) = a\frac{p^2}{2m_i} + bp + c_i \quad \text{for almost all } p \in \mathbf{R} \text{ and } i = 1, \ldots, n,$$

for a L^1_{loc} solution of the functional equation (*).

EXAMPLE 2. Take $\omega_i(p) = \sqrt{p^2 + m_i^2}$ for $p \in \mathbf{R}$, $m_i > 0$, $(i = 1, \ldots, n)$ and suppose that $G = \mathbf{R}$ is a Lie group of real numbers under the operation of addition. Let the left action of a group G on $W_i = \{(p, \sqrt{p^2 + m_i^2}) : p \in \mathbf{R}\}$ for $i = 1, \ldots, n$ be given by

$$g \cdot \left(p, \sqrt{p^2 + m_i^2}\right) = \left(p \cosh g + \sqrt{p^2 + m_i^2} \sinh g, p \sinh g + \sqrt{p^2 + m_i^2} \cosh g\right)$$

for $g, p \in \mathbf{R}$. Then the conditions (b) - (i) hold. Therefore, in case $n \geq 3$ we may apply Theorem 2 and get

$$f_i(p) = a\sqrt{p^2 + m_i^2} + bp + c_i \quad \text{for almost all } p \in \mathbf{R} \text{ and } i = 1, \ldots, n,$$

for a L^1_{loc} solution of the functional equation $(*)$.

Remark. It is easy to verify that Theorem 2, in case of Examples 1 and 2 works for f_i, which are only measurable L^1_{loc} functions for $i = 1, \ldots, n$.

The following examples show that hypotheses (a) and (b) are not superfluous in Theorems 1 and 2.

EXAMPLE 3. Let $n = 2$. If $\omega_1(p) = \omega_2(p) = p^2$ for $p \in \mathbf{R}$, then $(p_1, p_2, q_1, q_2) \in M$ implies $(p_1, p_2) = (q_1, q_2)$ or $(p_1, p_2) = (q_2, q_1)$. Therefore, $f_1 = f_2 = f$ is a solution of the functional equation $(*)$ for every function $f : \mathbf{R} \to \mathbf{R}$.

EXAMPLE 4. Let $n = 3$. If $\omega_1(p) = \omega_2(p) = p$ and $\omega_3(p) = \omega(p) + p$ for $p \in \mathbf{R}$, where $\omega : \mathbf{R} \to \mathbf{R}$ is any one-to-one function, then $(p_1, p_2, p_3, q_1, q_2, q_3) \in M$ implies $p_3 = q_3$. Therefore, $f_i(p) = ap + c_i$ for $p \in \mathbf{R}$, $i = 1, 2$ and f_3 is a solution of the functional equation $(*)$ for every function $f_3 : \mathbf{R} \to \mathbf{R}$.

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