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ON A STARLIKENESS PROBLEM FOR CERTAIN CLASS OF MULTIVALENT ANALYTIC FUNCTIONS

1. Introduction

Let $P(\alpha)$, $0 \le \alpha < 1$, denote the class of functions h, with $h(0) = 1$ regular in $K = K(0,1)$, where $K(a,r) = \{z : |z-a| < r\}$, and satisfying the condition $\operatorname{Re} h(z) > \alpha$ for $z \in K$, and let $P(0) = P$.

Let S_p , where p is a positive integer, denote the class of functions f of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a^n z^n, \quad z \in K, \tag{1.1}$$

regular and p-valent in K. In particular, $S_1 = S$ is the class of univalent functions.

We shall also use the following well known notations

$$S_p^*(\alpha) = \left\{ f \in S_p : \operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha, \ z \in K \right\}$$

for the p-valent starlike functions of order α, $0 \le \alpha < p$, and

$$S^c = \left\{ f \in S : \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > 0, \ z \in K \right\}$$

for the class of convex functions.

Definition 1. The function f of the form (1.1), regular in K, belongs to the class $CS_p^*(\alpha)$ of α-close-to-star functions if there exists a function $g \in S_p^*(\alpha)$ such that $\operatorname{Re} \frac{f(z)}{g(z)} > 0$ for $z \in$ K. Especially, denote $CS^*(\alpha) = CS_1^*(\alpha), CS^* = CS^*(0)$.

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The class $CS^*(\alpha)$ was investigated by Al-Amiri [1], Kulkarni and Thakare [4], Sakaguchi [7], Krzyż and Rade [3], MacGregor [5] and others.

Let $\mathcal{A}$ denote a subclass of the class of functions regular in K.

DEFINITION 2. Let $B(\mathcal{A})$ denote the set of all pairs $(|a|, r)$, where $a \in K$, $|a| < r \leq 1 - |a|$, such that any function $f \in \mathcal{A}$ maps the disk $K(a, r)$ onto a domain starlike with respect to the origin.

Putting $a = 0$, we obtain the radius of starlikeness for the class $\mathcal{A}$.

The set $B(\mathcal{A})$ was determined for $\mathcal{A} = S^*$ by Rahmanow [6], for $\mathcal{A} = S^*(\alpha)$, S^c by Stankiewicz and Świtoniak [8], for $\mathcal{A} = S$ by Świtoniak [9] and for $\mathcal{A} = CS^*$ by Dziok [2].

In this paper we determined the set $B(CS_p^*(\alpha))$ for the α-close-to-star functions.

The following lemma is useful for our main result.

LEMMA [9]. *Let f be a regular function in K, $a \in K$, $|a| < r \leq 1 - |a|$. It maps the disk $K(a, r)$ onto a domain starlike with respect to the origin if and only if*

$$\operatorname{Re} \frac{e^{i\theta} f'(a + \mathrm{re}^{i\theta})}{f(a + \mathrm{re}^{i\theta})} \geq 0 \text{ for } 0 \leq \theta \leq 2\Pi. \tag{1.2}$$

2. Main results

THEOREM. Let $f \in CS_p^*(\alpha)$, where p is a positive integer and α is a real number, $0 \leq \alpha <$p. Let

$$\mathcal{B}' = \left\{ (|a|, r) : \begin{cases} |a| < r & \text{for } 0 \leq r \leq r_1 \\ |a| \leq \sqrt{r^2 - (r^2 - (\sqrt{p} - 2\sqrt{r(1+p-\alpha)})^2/(p - 2\alpha)} & \text{for } r_1 < r < r_2 \\ |a| \leq q - r & \text{for } r_2 \leq r < q \end{cases} \right\}$$

where

$$r_1 = \frac{p}{4(1+p-\alpha)}, \tag{2.1}$$

$$r_2 = p(1+p-\alpha)(1+p-\alpha+\sqrt{(1-\alpha)^2+2p})^{-2}, \tag{2.2}$$

$$q = p(1+p-\alpha+\sqrt{(1-\alpha)^2+2p})^{-1}, \tag{2.3}$$

$$\mathcal{B}'' = \{(|a|, r) : |a| < r \leq q - |a|\} \tag{2.4}$$

and let us put

$$\mathcal{B} = \begin{cases} \mathcal{B}' & \text{for } 0 \leq \alpha < p/2 \\ \mathcal{B}'' & \text{for } p/2 \leq \alpha < p \end{cases}. \tag{2.5}$$

If $(|a|, r) \in \mathcal{B}$, then the function f maps the disk $K(a,r)$ onto a domain starlike with respect to the origin.

The result is sharp for $p/2 \le \alpha < p$, and for $0 \le \alpha < p/2$ the set $\mathcal{B}$ can not be larger than $\mathcal{B}$". It means that

$$\mathcal{B}' \subset B(CS_p^*(\alpha)) \subset \mathcal{B}'' \text{ for } 0 \le \alpha < p/2, \tag{2.6}$$

$$B(CS_p^*(\alpha)) = \mathcal{B}'' \text{ for } p/2 \le \alpha < p. \tag{2.7}$$

Proof. Let $z = a + \mathrm{r}e^{i\theta}$, $0 \le \theta \le 2\Pi$. Since the function $e^{-is} f(e^{is} z)$, $s \in R$, $z \in K$, belongs to the class $CS_p^*(\alpha)$ together with the function f, we may assume without the loss of generality that a is real and nonnegative.

Let $f \in CS_p^*(\alpha)$. Thus there exists a function $g \in S_p^*(\alpha)$ which satisfies the $\frac{f(z)}{g(z)} = h(z)$, where $h \in P$, or equivalently

$$f(z) = g(z)h(z). \tag{2.8}$$

Because $g \in S_p^*(\alpha)$, therefore $\frac{zg'(z)}{pg(z)} \in P(\alpha/p)$. Since the domain of variability of the functional h in the class $P(\alpha/p)$ is known, we have

$$\left| \frac{zg'(z)}{pg(z)} - \frac{\alpha}{p} - \left(1 - \frac{\alpha}{p}\right) \frac{1+|z|^2}{1-|z|^2} \right| \le \frac{2(1-\alpha/p)|z|}{1-|z|^2}.$$

Thus, after some calculations, we obtain

$$(1-|z|^2) \operatorname{Re} \frac{e^{i\theta} g'(z)}{pg(z)} \ge \operatorname{Re} \frac{e^{i\theta}(1+(1-2\alpha/p)|z|^2)}{z} - 2(1-\alpha/p). \tag{2.9}$$

Logarithmic differentiation of the equality (2.8) gives

$$\frac{f'(z)}{f(z)} = \frac{g'(z)}{g(z)} + \frac{h'(z)}{h(z)}.$$

Using the well-known estimate for $|h'(z)/h(z)|$ in the class P, we have

$$\operatorname{Re} \frac{e^{i\theta} f'(z)}{f(z)} \ge \operatorname{Re} \frac{e^{i\theta} g'(z)}{g(z)} - \frac{2}{1-|z^2|}.$$

Using (2.9) and setting $z = a + \mathrm{r}e^{i\theta}$, this yields

$$\begin{aligned}(1-|a+\mathrm{r}e^{i\theta}|) \operatorname{Re} \frac{e^{i\theta} f'(a+\mathrm{r}e^{i\theta})}{f(a+\mathrm{r}e^{i\theta})} \\ \ge \operatorname{Re} \frac{e^{i\theta}(p+(p-2\alpha)|a+\mathrm{r}e^{i\theta}|^2)}{a+re^{i\theta}} - 2(p+1-\alpha).\end{aligned} \tag{2.10}$$

We now have to require that the right-hand side of (2.10) must be nonnegative, that is

$$\operatorname{Re}\left(\frac{p}{r+ae^{-i\theta}}+\frac{(p-2\alpha)|r+ae^{-i\theta}|^2}{r+ae^{-i\theta}}\right)\geq 2(1+p-\alpha). \tag{2.11}$$

If we put

$$r+ae^{-i\theta}=x+yi \tag{2.12}$$

into (2.11), we get

$$\frac{px}{x^2+y^2}+(p-2\alpha)x\geq 2(1+p-\alpha). \tag{2.13}$$

Thus, using the equality

$$(x-r)^2+y^2=a^2, \tag{2.14}$$

we obtain

(2.15)
$$2r(p-2\alpha)x^2+[p+(p-2\alpha)(a^2-r^2)-4r(1+p-\alpha)]x+2(1+p-\alpha)(r^2-a^2)\geq 0.$$

Now we require that the inequality (2.15) holds for every $x\in[r-a,r+a]$. Let us denote the quadratic trinomial in the inequality (2.15) by $w(x)$. The determinant Δ of this trinomial is given by

$$\begin{aligned}\Delta&=(p+(p-2\alpha)(a^2-r^2)-4r(1+p-\alpha))^2\\&\quad-16r(p-2\alpha)(1+p-\alpha)(r^2-a^2)=AB,\end{aligned}$$

where

$$\left\{\begin{array}{l}A=(p-2\alpha)(a^2-r^2)+p+4r((1+p-\alpha)+4\sqrt{rp(1+p-\alpha)}\\ B=(p-2\alpha)(a^2-r^2)+p+4r(1+p-\alpha)-4\sqrt{rp(1+p-\alpha)}\end{array}\right.. \tag{2.16}$$

Let $D=\{(a,r)\in R^2:0\leq a<r\leq 1-a\}$. First we discuss the case $0\leq\alpha<p/2$. Thus the inequality (2.15) is satisfied for every $x\in[r-a,r+a]$, if one of the following conditions is satisfied:

1° $\Delta\leq 0$,
2° $\Delta>0$ and $w(r-a)\geq 0$ and $x_0\leq r-a$,
3° $\Delta>0$ and $w(r+a)\geq 0$ and $x_0\geq r+a$,

where

$$x_0=-\frac{(p-2\alpha)(a^2-r^2)-4r(1+p-\alpha)+p}{4(p-2\alpha)r}. \tag{2.17}$$

Ad 1°. Let $\mathcal{B}_1 = \{(a,r) \in D : \Delta \leq 0\}$. Since $A > 0$, the condition $\Delta \leq 0$ is equivalent, by (2.16), to the inequality

$$B = (p-2\alpha)(a^2 - r^2) + p + 4r((1+p-\alpha) - 4\sqrt{r(1+p-\alpha)} \leq 0. \tag{2.18}$$

Let γ denote the boundary of the set $\tilde{\mathcal{B}}_1 = \{(a,r) \in R^2; B \leq 0\}$. But γ is the curve which is tangent to the straight lines $r = a$ and $r = q - a$ at the points $S_1(r_1, r_1)$ and $S_2(r_2, q - r_2)$, respectively, where r_1, r_2, q are defined by (2.1), (2.2) and (2.3), respectively. Moreover γ cuts the straight line $a = 0$ at the points

$$r_3 = \frac{\left[(1+p-\alpha+[p(p-2\alpha)]^{1/2})^{1/2} - (1+p-\alpha)^{1/2}\right]^2}{p-2\alpha},$$

$$r_4 = \frac{[(1+p-\alpha)^{1/2} - (1+p-\alpha-[p(p-2\alpha)]^{1/2})^{1/2}]^2}{p-2\alpha},$$

$$r_0 = \frac{[(1+p-\alpha)^{1/2} + (1+p-\alpha-[p(p-2\alpha)]^{1/2})^{1/2}]^2}{p-2\alpha}.$$

We have $0 < r_3 < r_4 < q$ and $r_0 >$q. Thus

$$\mathcal{B}_1 = \{(a,r) : r_3 \leq r \leq r_4,$$
$$0 \leq a \leq \sqrt{r^2 - (\sqrt{p} - 2\sqrt{r(1+p-\alpha)})^2/(p-2\alpha)}\}.$$

Ad 2°. Let $\mathcal{B}_2 = \{(a,r) \in D : \Delta > 0 \wedge w(r-a) \geq 0 \wedge x_0 \leq r - a\}$. We have $w(r-a) = (r-a)[(p-2\alpha)(r-a)^2 - 2(1+p-\alpha)(r-a) + p] = (p-2\alpha)(r-a)(r-a-q')(r-a-q)$ where q is defined by (2.3) and $q' = p(1+p-\alpha-\sqrt{(1-\alpha)^2+2p})^{-1}$. Since $q' > 1$ and $0 < q < 1$ hold for $0 \leq \alpha < p/2$, we see that $(r-a)(r-a-q') < 0$ and the inequality $w(r-a) \geq 0$ is true, if

$$r \leq a + q. \tag{2.19}$$

The inequality $x_0 \leq r - a$ may be written in the form

$$(p-2\alpha)a^2 + 3(p-2\alpha)r^2 - 4(1+p-\alpha)r - 4(p-2\alpha)ar + p \geq 0. \tag{2.20}$$

The hyperbola h_1, which is the boundary of the set of all pairs $(a,r) \in R^2$ satisfying (2.20), cuts the line $a = r$ at the point S_1 and the line a=0 at the points

$$\begin{aligned} r_5 &= \frac{p}{2(1+p-\alpha) + \sqrt{4(1+p-\alpha)^2 - 3p^2(1-2\alpha)}}, \\ \tilde{r}_5 &= \frac{p}{2(1+p-\alpha) - \sqrt{4(1+p-\alpha)^2 - 3p^2(1-2\alpha)}}. \end{aligned} \tag{2.21}$$

We have $r_3 < r_5 < r_4$, $\tilde{r}_5 > q$. Thus, finally, we describe the set

$$\mathcal{B}_2 = \left\{ (\mathrm{a},\mathrm{r}) : \begin{cases} 0 \le a < r & \text{for } 0 \le r \le r_3 \\ \sqrt{r^2 - (\sqrt{p} - 2\sqrt{r(1+p-\alpha)})^2/(p-2\alpha)} \le a < r & \text{for } r_3 < r < r_1 \end{cases} \right\}$$

3°. Let $\mathcal{B}_3 = \{(a,r) \in R : \Delta > 0 \wedge w(r+a) \ge 0 \wedge x_0 \ge r+a\}$. Since

$$\begin{aligned} w(r+a) &= (r+a)[(p-2\alpha)(r+a)^2 - 2(1+p-\alpha)(r+a) + p] \\ &= (p-2\alpha)(r+a)(r+a-q')(r+a-q) \end{aligned}$$

and $(r+a)(r+a-q') < 0$ hold for $a < r \le 1-a$, we conclude that the inequality $w(r-a) \ge 0$ is true if

$$r \le q - a. \tag{2.22}$$

The inequality $x_0 \le r + a$ may be written in the form

$$(p-2\alpha)a^2 + 3(p-2\alpha)r^2 - 4(1+p-\alpha)r + 4(p-2\alpha)ar + p \le 0. \tag{2.23}$$

The hyperbola h_2, which is the boundary of the set of all pairs $(a,r) \in D$ satisfying (2.23), cuts the line $a + r = q$ at the point S_2 and the line a=0 for $r = r_5$. Thus we determine the set

$$\mathcal{B}_3 = \left\{ (\mathrm{a},\mathrm{r}) : \begin{cases} \sqrt{r^2 - (\sqrt{p} - 2\sqrt{r(1+p-\alpha)})^2/(p-2\alpha)} < a \le q - r & \text{for } r_2 < r < r_4 \\ 0 \le a \le q - r & \text{for } r_4 \le r < q \end{cases} \right\}$$

The union of the sets $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ gives the set $\mathcal{B}'$. Thus

$$\mathcal{B}' \subset B(CS^*(\alpha)) \text{ for } 0 \le \alpha < p/2. \tag{2.24}$$

Now let $p/2 < \alpha <$p. Then the inequality (2.15) is satisfied for every $x \in [r-a, r+a]$ if the following conditions are satisfied

$$w(r-a) \ge 0 \text{ and } w(r+a) \ge 0. \tag{2.25}$$

Since

$$\begin{aligned} w(r+a) &= (p-2\alpha)(r+a)(r+a-q')(r+a-q), \\ w(r-a) &= (p-2\alpha)(r-a)(r-a-q')(r-a-q) \end{aligned}$$

and $q' < 0, 0 < q < 1$ hold for $p/2 < \alpha < p$, then the condition (2.25) is satisfied if

$$r - a - q \leq 0. \tag{2.26}$$

Let $\alpha = p/2$. Taking $\alpha = p/2$ in (2.15) we obtain

$$(p - 4r(1 + p/2))x + 2(1 + p/2)(r^2 - a^2) \geq 0. \tag{2.27}$$

The inequality (2.27) is satisfied for every $x \in [r-a, r+a]$ if $r-a \leq p/(p+2)$ or equivalently (2.26). Thus we obtain

$$\mathcal{B}'' \subset B(CS^*(\alpha)) \text{ for } p/2 \leq \alpha < p. \tag{2.28}$$

Let $0 \leq \alpha < p$. The function $f(z) = \frac{z^p(1-z)}{(1+z)^{2(p-\alpha)+1}}$ belongs to the class $CS_p^*(\alpha)$ and for $z = a + r, \theta = 0, a + r > q$ we have

$$\operatorname{Re} \frac{e^{i\theta} f'(z)}{f(z)} = \frac{p - 2(1 + p - \alpha)(a + r) + (p - 2\alpha)(a + r)^2}{(a + r)(1 - (a + r)^2)} \leq 0,$$

whence by Lemma and Definition 2, we get

$$B(CS^*(\alpha)) \subset \mathcal{B}'' \text{ for } 0 \leq \alpha < p. \tag{2.29}$$

By (2.24) and (2.29) we obtain (2.6). From (2.28) and (2.29) it follows (2.7), which completes the proof.

Remark. Finally let us observe, that taking a=0, we obtain the radius of starlikeness for the class $CS_p^*(\alpha)$, while, taking $\alpha = 0$ and $p = 1$, we have the result due to Dziok [2].

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