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ON THE OSCILLATION OF SOLUTIONS OF PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS OF NEUTRAL TYPE

1. Introduction

In the last few years there has been a growing interest in studying the oscillatory behavior of solutions of parabolic equations with functional arguments. We refer the reader to the papers by Bykov and Kultaev [1], Kreith and Ladas [3], Yoshida [6], [7], Mishev and Bainov [4] and Cui [2]. However, the forced oscillations have been only studied by Yoshida [7].

The purpose of this paper is to extend the work of Yoshida [7] to some nonlinear neutral parabolic equations with functional arguments of the form

$$(1)\quad \frac{\partial}{\partial t}\Big[u(x,t)+\sum_I \lambda_i(t)u(x,\tau_i(t))\Big]-\Big[a(t)\Delta u(x,t)+\sum_J a_j(t)\Delta u(x,\rho_j(t))\Big] +c(x,t,u(x,t),u(x,\sigma(t)))=f(x,t),\quad (x,t)\in\Omega\times R_+\equiv G,$$

where I, J are initial segments of natural numbers, Δ is the Laplacian in Euclidean n-space $R^n, R_+=[0,\infty)$, Ω is a bounded domain in R^n with a smooth boundary $\partial\Omega$.

Now we list a set of assumptions:

(A_1) $a(t), a_j(t), j\in J$, are nonnegative continuous functions on R_+ and $f(x,t)$ is a continuous function on $\bar{\Omega}\times R_+$.

(A_2) $c(x,t,\xi,\eta)\geq 0$ for $(x,t)\in\Omega\times R_+$, $\xi\geq 0$, $\eta\geq 0$, and $c(x,t,\xi,\eta)\leq 0$ for $(x,t)\in\Omega\times R_+$, $\xi\leq 0$, $\eta\leq 0$.

(A_3) $\lambda_i(t)$, $\tau_i(t)$, $\rho_i(t)$ and $\sigma(t)$ are continuous functions on R_+ with $\lim_{t\to\infty}\tau_i(t)=\infty$, $\lim_{t\to\infty}\rho_j(t)=\infty$, $\lim_{t\to\infty}\sigma(t)=\infty$, for $i\in I$, $j\in J$.

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(A_4) $c(x,t,\xi,\eta) \geq P(t)\varphi(\eta)$ for all $(x,t,\xi,\eta) \in \Omega \times R_+^3$, where P and φ are nonnegative functions on R_+ and φ is convex.

(A_5) $c(x,t,\xi,\eta) \geq b(t)\xi$ for all $(x,t,\xi,\eta) \in \Omega \times R_+^3$, where b is a nonnegative continuous functions on R_+.

Our aim is to establish the conditions under which every (classical) solution $u(x,t)$ of (1) satisfying certain boundary condition is oscillatory on $\Omega \times R_+$, in the sense that $u(x,t)$ has a zero on $\Omega \times [t,\infty)$ for any $t > 0$.

Sufficient conditions are given for all solutions of some boundary value problems to be oscillatory in a cylindrical domain. The principal tool employed is an averaging technique which enables us to establish oscillation properties in terms of related functional differential inequalities.

We consider three kinds of boundary conditions:

$$(B_1) \qquad u = \varphi \quad \text{on } \partial\Omega \times R_+,$$

$$(B_2) \qquad \frac{\partial u}{\partial \gamma} = \psi \quad \text{on } \partial\Omega \times R_+,$$

$$(B_3) \qquad \frac{\partial u}{\partial \gamma} + \mu u = 0 \quad \text{on } \partial\Omega \times R_+,$$

where φ, ψ, μ are continuous on $\partial\Omega \times R_+$, γ denotes the unit exterior normal vector to $\partial\Omega$ and $\mu \geq 0$ on $\partial\Omega \times R_+$.

In the domain Ω consider the Dirichlet problem

$$(2) \qquad \Delta u(x) + \alpha u(t) = 0, \quad x \in \Omega,$$

$$(3) \qquad u(t) = 0, \quad x \in \partial\Omega,$$

where $\alpha =$ const. It is well known [5] that the smallest eigenvalue α_1 of the problem (2), (3) is positive and the corresponding eigenfunction $\Phi(x)$ can be chosen so that $\Phi(x) > 0$ for $x \in \Omega$.

2. Oscillation criteria for problem (1)–(B_1)

The following lemma is needed for our main results.

LEMMA 2.1. *Suppose that* (A_1)-(A_4) *hold and that* $u(x,t)$ *is a positive solution of the problem* (1)-(B_1) *on* $\Omega \times [t_0,\infty)$, $t_0 > 0$. *Then the function*

$$(4) \qquad V(t) = \frac{\int_\Omega u(x,t)\Phi(x)dx}{\int_\Omega \Phi(x)dx}, \quad t > 0$$

satisfies the following neutral differential inequality

$$(5) \qquad \frac{d}{dt}\Big[V(t) + \sum_I \lambda_i(t)V(\tau_i)\Big] + \alpha_1\Big[a(t)V(t) + \sum_J a_j(t)V(\rho_j(t))\Big] + P(t)\varphi(V(\sigma(t))) \leq F(t)$$

where

$$(6)\qquad F(t) = \Big[\int_{\Omega} \Phi(x)dx\Big]^{-1}\Big[-a(t)\int_{\partial\Omega} \phi\frac{\partial\Phi}{\partial\gamma}ds - \sum_j a_j(t)\int_{\partial\Omega} \phi(x,\rho_j(t))\frac{\partial\Phi}{\partial\gamma}ds + \int_{\partial\Omega} f(x,t)\Phi(x)dx\Big]$$

and ds is the surface integral element on $\partial\Omega$.

Proof. Suppose that $u(x,t)$ is a positive solution of the problem (1)–(B_1) on $\Omega\times[t_0,\infty)$, $t_0>0$. Since the condition (A_3) holds, there is a number $t_1>t_0$ such that $\tau_i(t)\ge t_0$, $\rho_j(t)\ge t_0$ and $\sigma(t)\ge t_0$ for $t\ge t_1$, $i\in I$, $j\in J$. By condition (A_4) we have

$$c(x,t,u(x,t),u(x,\sigma(t)))\ge P(t)\varphi(u(x,\sigma(t)))\quad \text{on } \Omega\times[t_1,\infty).$$

Therefore

$$(7)\qquad \frac{\partial}{\partial t}\Big[u(x,t)+\sum_I \lambda_i(t)u(x,\tau_i(t))\Big]+P(t)\varphi(u(x,\sigma(t))) \le a(t)\Delta u(x,t)+\sum_J a_j(t)\Delta u(x\rho_j(t))+f(x,t)\quad \text{on } \Omega\times[t_1,\infty).$$

Multiplying (7) by $\Phi(x)$ and integrating over Ω, we obtain

$$(8)\qquad \frac{d}{dt}\Big[\int_{\Omega} u(x,t)\Phi(x)dx+\sum_I \lambda_i(t)\int_{\Omega} u(x,\tau_i(t))\Phi(x)dx\Big] +P(t)\int_{\Omega}\varphi(u(x,\sigma(t)))\Phi(x)dx\le a(t)\int_{\Omega}\Delta u\Phi(x)dx +\sum_J a_j(t)\int_{\Omega}\Delta u(x,\rho_j(t))\Phi(x)dx+\int_{\Omega} f(x,t)\Phi(x)dx,\quad t\ge t_1.$$

It follows from Green's formula that

$$(9)\qquad \int_{\Omega}\Delta u\Phi(x)dx=\int_{\partial\Omega}\Big[\frac{\partial u}{\partial\gamma}\Phi-u\frac{\partial\Phi}{\partial\gamma}\Big]ds+\int_{\Omega} u\Delta\Phi dx = -\int_{\partial\Omega}\phi\frac{\partial\Phi}{\partial\gamma}ds-\alpha_1\int_{\Omega} u\Phi dx,\quad t\ge t_1$$

and

$$(10)\qquad \int_{\Omega}\Delta u(x,\rho_j(t))\Phi(x)dx = -\int_{\partial\Omega}\phi(x,\rho_j(t))\frac{\partial\Phi}{\partial\gamma}ds -\alpha_1\int_{\Omega} u(x,\rho_j(t))\Phi(x)dx,\quad t\ge t_1.$$

It follows from Jensen's inequality that

$$(11)\qquad \int_\Omega \varphi(u(x,\sigma(t)))\Phi(x)dx \\ \geq \Big(\int_\Omega \Phi(x)dx\Big)\varphi\Big(\frac{\int_\Omega u(x,\sigma(t))\Phi(x)dx}{\int_\Omega \Phi(x)dx}\Big), \quad t\geq t_1.$$

Combining (6)–(11) and (4) we find that (5) holds. This completes the proof of the lemma.

THEOREM 2.1. *Suppose that* (A_1)–(A_4) *hold. Then every solution* u *of the problem* (1)–(B_1) *is oscillatory on* $\Omega\times R_+$, *if the following neutral differential inequalities*

$$(12)\qquad \frac{d}{dt}\Big[V(t)+\sum_I \lambda_i(t)V(\tau_i(t))\Big]+\alpha_1\Big[a(t)V(t)+\sum_J a_j(t)V(\rho_j(t))\Big] \\ +P(t)\varphi(V(\sigma(t)))\leq F(t)$$

and

$$(13)\qquad \frac{d}{dt}\Big[V(t)+\sum_I \lambda_i(t)V(\tau_i(t))\Big]+\alpha_1\Big[a(t)V(t)+\sum_J a_j(t)V(\rho_j(t))\Big] \\ +P(t)\varphi(V(\sigma(t)))\leq F(t)$$

have eventually no positive solution.

Proof. Suppose the contrary and let $u(x,t)$ be a nonoscillatory solution of (1)–(B_1) which we may assume to be positive on $\Omega\times[t_1,\infty)$ for some $t_1\geq t_0$. It follows from Lemma 2.1 that the function defined by (4) is eventually an positive solution of (12), which is a contradiction. If $u<0$ on $\Omega\times[t_1,\infty)$, then $\bar u\equiv -u$ is a positive solution of the problem (14)–(15), where

$$(14)\qquad \frac{\partial}{\partial t}\Big[u(x,t)+\sum_I \lambda_i(t)u(x,\tau_i(t))\Big] \\ -\Big[a(t)\Delta u(x,t)+\sum_J a_j(t)\Delta u(x,\rho_j(t))\Big] \\ +c(x,t,u(x,t),u(x,\sigma(t)))=-f(x,t),\quad (x,t)\in G$$

and

$$(15)\qquad u=-\phi \quad \text{on } \partial\Omega\times R_+,$$

and satisfies the neutral differential inequality (13) with

$$\bar V(t)=\Big[\int_\Omega \Phi(x)dx\Big]^{-1}\int_\Omega \bar u(x,t)\Phi(x)dx.$$

Proceeding as in the first case we arrive at a contradiction. The proof of the Theorem 2.1 is complete.

THEOREM 2.2. *Suppose that assumptions* (A_1)–(A_4) *hold and that* $\lambda_i(t) \geq 0$, $i \in I$. *Then every solution* u *of the problem* (1)–(B_1) *is oscillatory on* $\partial\Omega \times R_+$, *if the conditions*

$$\liminf_{t\to\infty} \int_{t_0}^{t} F(s)ds = -\infty \tag{16}$$

and

$$\limsup_{t\to\infty} \int_{t_0}^{t} F(s)ds = +\infty \tag{17}$$

hold for all sufficiently large t_0, *where* F *is defined by* (6).

Proof. Suppose that there is a nonoscillatory solution u of the problem (1)–(B_1). Without loss of generality we assume that $u > 0$ on $\Omega \times [t_0, \infty)$ for some $t_0 > 0$. It follows from Lemma 2.1 that the function defined by (4) satisfies (12) and therefore, we have

$$\frac{d}{dt}\Big[V(t) + \sum_I \lambda_i(t)V(\tau_i(t))\Big] \leq F(t), \quad \text{for } t \geq t_0. \tag{18}$$

Integrating (18) over $[t_0, t]$, we get

$$V(t) + \sum_I \lambda_i(t)V(\tau_i(t)) \leq V(t_0) + \sum_I \lambda_i(t_0)V(\tau_i(t_0)) + \int_{t_0}^{t} F(s)ds. \tag{19}$$

In view of (16) the right-hand side of (19) is not bounded below and hence $V(t) + \sum_I \lambda_i(t)V(\tau_i(t))$ eventually cannot be positive. This is a contradiction. So the inequality (12) has eventually no positive solution. On the other hand, in view of the equality

$$\liminf_{t\to\infty} \int_{t_0}^{t} [-F(s)]ds = -\limsup_{t\to\infty} \int_{t_0}^{t} F(s)ds = -\infty,$$

it follows that the inequality (13) cannot eventually have a positive solution. Thus, the assertion of Theorem 2.2 is true.

THEOREM 2.3. *Suppose that the conditions* (A_1)–(A_3) *and* (A_5) *hold and in addition*

$$0 \leq t - \tau_i(t) \leq T, \quad \lambda_i(t) \leq 0, \quad i \in I, \quad \textit{and} \quad \sum_I \lambda_i(t) \geq -1.$$

Then every solution u of the problem (1)–(B_1) is oscillatory on $\Omega \times R_+$, if the conditions

$$\liminf_{t\to\infty} \int_{t_1}^{t} F(s)\exp[\alpha_1 A(s) + B(s)]ds = -\infty \tag{20}$$

and

$$\limsup_{t\to\infty} \int_{t_1}^{t} F(s)\exp[\alpha_1 A(s) + B(s)]ds = +\infty \tag{21}$$

hold for all sufficiently large t_1, where

$$A(s) = \int_0^s a(r)dr \quad \text{and} \quad B(s) = \int_0^s b(r)dr.$$

Proof. Suppose the contrary and let u be a nonoscillatory solution of the problem (1)–(B_1). Assume that $u > 0$ on $\Omega \times [t_0, \infty)$ for some $t_0 > 0$. Keeping in view (A_5) and arguing as in the proof of Lemma 2.1, one can easily check that the function defined by (4) satisfies the inequality

$$\frac{d}{dt}\Big[V(t) + \sum_I \lambda_i(t)V(\tau_i(t))\Big] + [\alpha_1 a(t) + b(t)]V(t) \le F(t), \quad t \ge t_1, \tag{22}$$

where $t_1 \ge t_0$ and t is chosen such that $V(t) > 0$ and $V(\tau_i(t)) > 0$ for $i \in I$. Let

$$Z(t) = V(t) + \sum_I \lambda_i(t)V(\tau_i(t)), \quad t \ge t_1. \tag{23}$$

We note that $Z(t) \le V(t)$ for $t \ge t_1$. It follows from (22) that

$$\frac{d}{dt}Z(t) + [\alpha_1 a(t) + b(t)]Z(t) \le F(t), \quad t \ge t_1,$$

which is equivalent to

$$\frac{d}{dt}[Z(t)\exp(\alpha_1 A(t) + B(t))] \le F(t)\exp(\alpha_1 A(t) + B(t)). \tag{24}$$

Integrating (24) from t_1 to t, we have

$$Z(t)\exp(\alpha_1 A(t) + B(t)) - Z(t_1)\exp(\alpha_1 A(t_1) + B(t_1)) \le \int_{t_1}^{t} F(s)\exp(\alpha_1 A(s) + B(s))ds.$$

In view of (20), $Z(t)\exp(\alpha_1 A(t) + B(t))$ is not bounded below and hence it follows that $V(t)$ is unbounded. Thus there exists a sequence $\{t_k\}$, $t_k \ge$

$t_1 + T$, such that $V(t_k) = \max_{t_1 \le t \le t_k} V(t)$ and that $\lim_{k\to\infty} V(t_k) = \infty$. We can choose a $t^* \in \{t_k\}$ with $Z(t^*) < 0$. On the other hand

$$Z(t^*) = V(t^*) + \sum_I \lambda_i(t^*)V(\tau_i(t^*) \ge V(t^*)\Big(1 + \sum_I \lambda_i(t^*)\Big) \ge 0$$

which is a contradiction. If $u < 0$ on $\Omega \times [t_0, \infty)$, then $\bar{u} \equiv -u$ is a positive solution of the problem (14)–(15). It is easily verified that the function

$$\bar{V}(t) = \frac{\int_\Omega \bar{u}(x,t)\Phi(x)dx}{\int_\Omega \Phi(x)dx}$$

satisfies the inequality

$$\frac{d}{dt}\Big[V(t) + \sum_I \lambda_i(t)V(\tau_i(t))\Big] + [\alpha_1 a(t) + b(t)]V(t) \le -F(t), \quad t \ge t_I.$$

Proceeding as in the case $u > 0$, we again arrive at a contradiction. This completes the proof of Theorem 2.3.

A special case of the problem (1)–(B_1) is the following

$$(25)\quad \frac{\partial}{\partial t}\Big[u(x,t) + \sum_I \lambda_i(t)u(x,\tau_i(t))\Big] - \Big[\Delta u(x,t) + \sum_J a_j(t)\Delta u(x,\rho_j(t))\Big] + c(x,t,u(x,t),u(x,\sigma(t))) = f(x,t), \quad (x,t) \in \Omega \times R_+ \equiv G$$

and

$$(26)\quad u = 0 \quad \text{on } \partial\Omega \times R_+.$$

COROLLARY. *Suppose that* (A_1)–(A_3) *and* (A_5) *with* $b(t) \equiv 0$ *hold. In addition, let*

$$0 \le t - \tau_i(t) \le T, \quad \lambda_i(T) \le 0, \quad i \in I, \quad \textit{and} \quad \sum_I \lambda_i(t) \ge -1.$$

Then every solution u *of the problem* (25)–(26) *is oscillatory on* $\Omega \times R_+$, *if the conditions*

$$\liminf_{t\to\infty} \int_{t_1}^{t} \exp(\alpha_1 s)\Big[\int_\Omega f(x,s)\Phi(x)dx\Big]ds = -\infty$$

and

$$\limsup_{t\to\infty} \int_{t_1}^{t} \exp(\alpha_1 s)\Big[\int_\Omega f(x,s)\Phi(x)dx\Big]ds = +\infty$$

hold for all sufficiently large t_1.

Proof. Since $A(t) = t$, $B(t) \equiv 0$ and $\phi \equiv 0$, the conclusion follows from Theorem 2.3.

Remark. 1. If $\lambda_i(t) \equiv 0$, $a_j(t) \equiv 0$, $i \in I$, $j \in J$ and $b(t) \equiv 0$, Theorem 2.3 and Corollary reduce to Theorem 1 and Corollary in [7], respectively.

3. Oscillation criteria for problem (1)–(B_2) and (1)–(B_3)

THEOREM 3.1. *Suppose that* (A_1)–(A_4) *hold. Then every solution* u *of the problem* (1)–(B_2) *is oscillatory on* G, *if each of the following neutral differential inequalities*

$$\frac{d}{dt}\Big[V(t) + \sum_I \lambda_i(t)V(\tau_i(t))\Big] + P(t)\varphi(V(\sigma(t))) \leq G(t) \tag{27}$$

and

$$\frac{d}{dt}\Big[V(t) + \sum_I \lambda_i(t)V(\tau_i(t))\Big] + P(t)\varphi(V(\sigma(t))) \leq -G(t) \tag{28}$$

have eventually no positive solution, where

$$G(t) = \frac{1}{|\Omega|}\Big[a(t)\int_{\partial\Omega} \psi ds + \sum_J a_j(t)\int_{\partial\Omega} \psi(x, \rho_j(t))ds + \int_\Omega f(x,t)dx\Big], \tag{29}$$

$$|\Omega| = \int_\Omega dx.$$

Proof. Suppose that there is a nonoscillatory solution u of the problem (1)–(B_2) defined on $\Omega \times [t_0, \infty)$, which we may do to be positive for $t_0 > 0$. As in the proof of Lemma 2.1 we obtain (7). Integrating (7) over Ω, we get

$$\frac{d}{dt}\Big[\int_\Omega u(x,t)dx + \sum_I \lambda_i(t)\int_\Omega u(x,\tau_i(t))dx\Big] + P(t)|\Omega|\varphi\Bigg(\frac{\int_\Omega u(x,\sigma(t))dx}{\int_\Omega dx}\Bigg)$$

$$\leq a(t)\int_\Omega \Delta u dx + \sum_J a_j(t)\int_\Omega \Delta u(x,\rho_j(t))dx + \int_\Omega f(x,t)dx$$

$$= a(t)\int_{\partial\Omega} \psi ds + \sum_J a_j(t)\int_{\partial\Omega} \psi(x,\rho_j(t))ds + \int_\Omega f(x,t)dx$$

for all $(x,t) \in \Omega \times [t_0,\infty)$. We let

$$V(t) = \frac{1}{|\Omega|}\int_\Omega u(x,t)dx, \quad t \geq t_0.$$

There exists a number $t_1 \geq t_0$ such that $\tau_i(t) \geq t_0$, $i \in I$, $\sigma(t) \geq t_0$, for $t \geq t_1$, hence $V(t) > 0$, $V(\sigma(t)) > 0$ and $V(\tau_i(t)) > 0$, $i \in I$. Consequently, we have

$$\frac{d}{dt}\Big[V(t) + \sum_I \lambda_i(t)V(\tau_i(t))\Big] + P(t)\varphi(V(\sigma(t))) \le G(t).$$

This implies that $V(t)$ is an eventually positive solution of (27), which is a contradiction. If $u < 0$ on $\Omega \times [t_0, \infty)$, then arguing as before, we arrive at a contradiction. Thus the proof of Theorem 3.1 is complete.

THEOREM 3.2. *Suppose that* (A_1)–(A_4) *hold and that* $\lambda_i(t) \ge 0$, $i \in I$. *Then every solution* u *of the problem* (1)–(B_2) *is oscillatory on* G, *if the conditions*

$$\liminf_{t\to\infty} \int_{t_0}^{t} G(s)ds = -\infty$$

and

$$\limsup_{t\to\infty} \int_{t_0}^{t} G(s)ds = +\infty$$

hold for all sufficiently large t_0, *where* $G(t)$ *is defined by* (29).

The proof is similar to that of Theorem 2.2 and hence is omitted.

THEOREM 3.3. *Suppose that* (A_1)–(A_3) *and* (A_5) *hold and in addition*

$$0 \le t - \tau_i(t) \le T, \quad \lambda_i(t) \le 0, \quad i \in I, \quad \text{and} \quad \sum_I \lambda_i(t) \ge -1.$$

Then every solution u *of the problem* (1)–(B_2) *is oscillatory on* G, *if the conditions*

$$\liminf_{t\to\infty} \int_{t_1}^{t} G(s)\exp[B(s)]ds = -\infty$$

and

$$\limsup_{t\to\infty} \int_{t_1}^{t} G(s)\exp[B(s)]ds = +\infty$$

hold for all sufficiently large t_1, *where* $B(s) = \int_0^s b(r)dr$ *and* $G(s)$ *is defined by* (29).

Proof. Suppose that there is nonoscillatory solution u of the problem (1)–(B_2) defined on $\Omega \times [t_0, \infty)$ which we may do to be positive for $t_0 > 0$. Using (A_5) instead of (A_4) in the proof of Theorem 3.1, we have

$$\frac{d}{dt}\Big[V(t) + \sum_I \lambda_i(t)V(\tau_i(t))\Big] + b(t)V(t) \le G(t), \tag{30}$$

where

$$V(t) = \frac{1}{|\Omega|} \int_{\Omega} u(x,t)dx, \quad t \geq t_I \geq t_0.$$

Put

$$Z(t) = V(t) + \sum_I \lambda_i(t) V(\tau_i(t)), \quad t \geq t_I.$$

Then $Z(t) \leq V(t)$, $t \geq t_1$. From (30) it follows that

$$\frac{d}{dt} Z(t) + b(t)Z(t) \leq G(t), \quad t \geq t_1,$$

which is equivalent to

$$\frac{d}{dt}[Z(t)\exp[B(t)]] \leq G(t)\exp[B(t)]. \tag{31}$$

Integrating (31) from t_1 to t, we obtain

$$Z(t)\exp[B(t)] \leq Z(t_1)\exp[B(t_1)] + \int_{t_1}^{t} G(s)\exp[B(s)]ds.$$

Arguing as in the proof of Theorem 2.3, we arrive at a contradiction. This completes the proof of Theorem 3.3.

Remark. 2. If $\lambda_i(t) \equiv 0$, $a_j(t) \equiv 0$, $a(t) \equiv 0$ and $b(t) \equiv 0$ for $i \in I$, $j \in J$, then Theorem 3.3 reduces to Theorem 2 in [7].

THEOREM 3.4. *Suppose that* (A_1)–(A_3) *and* (A_5) *hold and in addition*

$$0 \leq t - \tau_i(t) \leq T, \quad \lambda_i(t) \leq 0, \quad i \in I, \quad \text{and} \quad \sum_I \lambda_i(t) \geq -1.$$

Then every solution u *of the problem* (1)–(B_3) *is oscillatory on* G, *if the conditions*

$$\lim_{t\to\infty} \inf \int_{t_1}^{t} \Big[\int_{\Omega} f(x,s)dx \Big] \exp[B(s)]ds = -\infty$$

and

$$\lim_{t\to\infty} \sup \int_{t_1}^{t} \Big[\int_{\Omega} f(x,s)dx \Big] \exp[B(s)]ds = +\infty$$

hold for all sufficiently large t_1.

Proof. Let u be a nonoscillatory solution of the problem (1)–(B_3) having no zeros on $\Omega \times [t_0, \infty)$ for some $t_0 > 0$. Assume that $u > 0$. In view of (A_5), we have

$$\frac{\partial}{\partial t}\Big[u(x,t) + \sum_I \lambda_i(t) u(x,\tau_i(t))\Big] + b(t)u(x,t) \tag{32}$$

$$\leq a(t)\Delta u(x,t) + \sum_J a_j(t)\Delta u(x,\rho_j(t)) + f(x,t)$$

on $\Omega \times [t_0, \infty)$ for $t_1 \geq t_0$. Integrating (32) over Ω and taking into account the condition (B_3), we obtain

$$\frac{d}{dt}\Big[\int_\Omega u(x,t)dx + \sum_I \lambda_i(t)\int_\Omega u(x,\tau_i(t))dx\Big] + b(t)\int_\Omega u(x,t)dx$$

$$\leq a(t)\int_\Omega \Delta u(x,t)dx + \sum_J a_j(t)\int_\Omega \Delta u(x,\rho_j(t))dx + \int_\Omega f(x,t)dx$$

$$= -a(t)\int_{\partial\Omega} \mu u ds - \sum_J a_j(t)\int_{\partial\Omega} \mu(x,\rho_j(t))u(x,\rho_j(t))ds + \int_\Omega f(x,t)dx$$

$$\leq \int_\Omega f(x,t)dx, \quad t \geq t_1.$$

We note that $V(t) = \int_\Omega u(x,t)dx$, $t \geq t_0$, is positive for all $t \geq t_1$. Hence

$$\frac{d}{dt}\Big[V(t) + \sum_I \lambda_i(t)V(\tau_i(t))\Big] + b(t)V(t) \leq \int_\Omega f(x,t)dx.$$

The same argument as used in the proof of Theorem 3.3 leads to a contradiction.

Remark. 3. If $\lambda_i(t) \equiv 0$, $a_j(t) \equiv 0$, $i \in I$, $j \in J$, $a(t) \equiv 0$ and $b(t) \equiv 0$, then Theorem 3.4 reduces to Theorem 3 in [7].

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