DEMONSTRATIO MATHEMATICA
Vol. XXIX No 3 1996

Paola Bandieri

GEOMETRIC GROUP PRESENTATIONS

Introduction

A group presentation Φ is said to be *geometric* (resp. *strongly geometric*) iff there are a 3-manifold (resp. a closed 3-manifold) M and a Heegaard diagram H of M such that the presentation of the fundamental group $\pi_1(M)$, associated with H is exactly Φ. Many authors — by means of different techniques — have dealt with the question if a group presentation is geometric (or strongly geometric). The obtained results concern the case in which the 3-manifold M is orientable. More precisely, Neuwirth ([N]) describes an algorithm to decide if a balanced group presentation (i.e. a presentation with the same number of generators and relators) is strongly geometric. Osborne and Stevens ([OS_1], [OS_2]) solve the same problem by means of particular graphs (the presentation-graphs or P-graphs).

The same result is restated by Montesinos ([Mo]), via branched covering techniques.

Moreover, Grasselli ([Gr]) gives a combinatorial algorithm to construct all orientable, compact and connected 3-manifolds, having the standard complex K_Φ canonically associated to the presentation Φ as a standard spine ([N]); such a standard complex has a unique vertex and its 1-cells (resp. 2-cells) are in one-to-one correspondence with the generators (resp. the relators) of Φ. Subsequently, Grasselli and Piccarreta ([GP]) introduce another combinatorial algorithm to build "normal" crystallizations of all closed, connected, orientable 3-manifolds for which Φ up to additions or deletions of terms either of the form $x_i x_i^{-1}$ or of the form $x_i^{-1} x_i$, for some generators x_i's, in some relations of Φ is a presentation of the fundamental group.

1980 *Mathematics Subject Classification* (1985): Primary 57M15, 05C15; Secondary 57Q05.

Work performed under the auspices of the G.N.S.A.G.A of the C.N.R. (National Research Council of Italy) and within the project "Topologia e Geometria" of the M.U.R.S.T. (Italy).

In this work we construct all compact, connected 3-manifolds having a fixed balanced group presentation Φ (up to additions or deletions of terms either of the form $x_i x_i^{-1}$ or of the form $x_i^{-1}x_i$, for some generators x_i's, in some relations of Φ) as a presentation of the fundamental group associated with a Heegaard diagram. The key-tool of such a construction is the *bijoin - construction* ([BG], [Gr], [B]) over a graph canonically associated with Φ and representing a spine of a 3-manifold.

This process allows to obtain a result concerning both the cases, orientable and non orientable, by means of combinatorial tools. Observe that, since the spines of our construction are not special (in the sense of [Ma]), the 3-manifold cannot be univocally obtained.

1. Preliminaries

In this paper, all spaces and maps will be supposed to be piecewise-linear (PL) in the sense of [Gl] or [RS]; moreover, all 3-manifolds will be closed and connected.

With abuse of language, we shall use the term *graph* instead *multigraph*, whereas the term *pseudograph* indicates that loops are allowed.

Given a pseudograph $\Gamma = (V(\Gamma), E(\Gamma))$, a *generalized coloration* on Γ is any map $\gamma : E(\Gamma) \to \Delta_n = \{0, 1, \ldots, n\}$, where Δ_n is said to be the *colour-set.*

If $B \subset \Delta_n$, and B has cardinality equal to h, then Γ_B denote the subgraph $(V(\Gamma), \gamma^{-1}(B))$ and each connected component of Γ_B will be called B-residue or h-residue coloured B of Γ. If $c \in \Delta_n$, set $\widehat{c} = \Delta_n - \{c\}$.

An $(n+1)$*-dimensional crystallized structure* is a pair (Γ, γ), where Γ is a graph, γ is a generalized coloration with colour-set Δ_n, and, for each $c \in \Delta_n, \Gamma_{\{c\}}$ has cliques as connected components. If each clique of $\Gamma_{\{c\}}$ has length two, for every $c \in \Delta_n$, then (Γ, γ) is said to be an $(n+1)$*-coloured graph.* In this case, γ is a *proper* coloration of Γ, i.e. $\gamma(e) \neq \gamma(f)$, for each pair of adjacent edges of Γ.

An $(n+1)$-coloured graph (Γ, γ) is contracted iff $\Gamma_{\hat{c}}$ is connected, for every $c \in \Delta_n$. For more details on this argument see [FGG], [BM], [L] and [LM].

A coloured complex of dimension n is a pair $(K, \xi), K$ being an n-dimensional pseudocomplex [HW, p. 49] and ξ is a vertex-colouring, that is a map from the set of the vertices of K to Δ_n, whose restriction to each simplex is injective. If K has exactly $(n+1)$ vertices, then it is called a *contracted complex* (see [P_1], [P_2]).

It is well known that to each $(n+1)$-coloured graph (Γ, γ) can be associated a coloured complex $K(\Gamma)$ of dimension n as follows: — for each vertex v of $V(\Gamma)$, take a n-simplex $\sigma(v)$ and label its vertices by the colours of Δ_n;

— for each pair v, w of c-adjacent vertices ($c \in \Delta_n$) identify the $(n-1)$-faces of $\sigma(v)$ and $\sigma(w)$ whose vertices are labelled by $\widehat{c}$.

Conversely, to each coloured complex K of dimension n, can be associated an $(n+1)$-coloured graph $(\Gamma(K), \gamma(K))$, simply by reversing the above construction.

Moreover, it is well known that, for each $(n+1)$-coloured graph (Γ, γ), $\Gamma(K(\Gamma)) = \Gamma$, whereas $K(\Gamma(K)) = K$ iff the disjoint star of each simplex of K is strongly connected, in this case K will be called a *representable complex* (see [FGG] and its bibliography). Finally, K is contracted iff Γ is.

The graph Γ associated to the complex K is said to represent K.

A contracted $(n+1)$-coloured graph representing an n-manifold M is a *crystallization* of M. Each connected compact manifold admits crystallizations (see [P_1] and [P_2]).

We now recall the bijoin algorithm introduced in [B].

Let (H, ζ) be a representable $(n-1)$-dimensional coloured complex, a *pluri-bijoin* over H is an n-dimensional coloured complex (K, ξ), such that $|K|$ is a quasi-manifold and H can be identified with the subcomplex of K generated by the vertices of all colours but one, say c. If K has h vertices coloured c, then K is said to be an $h-$ *bijoin* over H and, if $h = 1$, we simply call it a *bijoin.*

Given any oriented pseudograph $\bar{\Gamma}$ — regular of degree $2n$ — endowed with a generalized coloration $\bar{\gamma} : E(\bar{\Gamma}) \to \Delta_{n-1}$, a *weight* on $\bar{E} = E(\bar{\Gamma})$, relative to the colour $c \in \Delta_{n-1}$ is a map $\omega : \bar{E} \to \{0, 1, 2\}$, with the following properties:

I — for each $i \in \widehat{c}$ and for each $e \in \bar{\gamma}^{-1}(i)$, $\omega(e) = 1$;

II — for each pair of adjacent edges $\bar{e}$ and $\bar{f}$ of $\bar{\gamma}^{-1}(c)$:

— if $\bar{e}(1) = \bar{f}(0)$, then $(\omega(\bar{e}), \omega(\bar{f})) \in \{(1,1), (1,0), (2,0), (0,2), (2,1)\}$;

— if $\bar{e}(1) = \bar{f}(1)$, then $(\omega(\bar{e}), \omega(\bar{f})) \in \{(0,1), (0,2)\}$;

— if $\bar{e}(0) = \bar{f}(0)$, then $(\omega(\bar{e}), \omega(\bar{f})) \in \{(1,2), (2,0)\}$.

The triple $(\bar{\Gamma}, \bar{\gamma}, \omega)$ is called an n-dimensional *pondered structure* if:

a) for each $i \in \Delta_{n-1} - \{c\}$, the connected components of $\gamma^{-1}(i)$ are elementary oriented cycles;

b) the connected components of $\gamma^{-1}(c)$ are elementary cycles, whose edge-orientation is coherent with respect to the weight ω.

Observe that if $\omega(e) = 1$, for each $e \in \bar{E}$, then the triple $(\bar{\Gamma}, \bar{\gamma}, \omega)$ is an oriented structure in the sense of [BG] and [Gr].

To each pondered structure $(\bar{\Gamma}, \bar{\gamma}, \omega)$, it is canonically associated a crystallized structure $(\widetilde{\Gamma}, \widetilde{\gamma})$, obtained by deleting all loops of $\bar{\Gamma}$ and by considering, for each connected component of $\gamma^{-1}(i)$, the clique over the same vertex-set, with all edges coloured i. Set $K(\bar{\Gamma}) = K(\widetilde{\Gamma})$. Obviously, this process cannot be univocally inverted: in fact there are many pondered structures associated with the same crystallized structure. If $(\bar{\Gamma}, \bar{\gamma}, \omega)$ is a pondered structure, the "bijoin" construction over $\bar{\Gamma}$ produces an $(n+1)$-coloured graph (B, β) constructed by the following rules:

i — $V(B) = V(\bar{\Gamma}) \times \{0,1\}$;

ii — for each $v \in V(\bar{\Gamma})$, join $(v,0)$ and $(v,1)$ by an edge e of $E(B)$ and set $\beta(e) = n$;

iii — if $\bar{e} \in \bar{E}$ and v (resp. w) is its first (resp. second) vertex, then join (v,h) and (w,k) by an edge $e \in E(B)$ so that $h \leq k$ and $h + k = \omega(e)$. Set $\beta(e) = \bar{\gamma}(\bar{e})$.

The so obtained graph (B, β) represents an h-bijoin over the (coloured) complex $K(\bar{\Gamma})$.

If $n = 3$, we can state some properties characterizing the h-bijoins over a 3-dimensional pondered structure representing 3-manifolds. A cycle μ, coloured alternately i and j, of a 3-dimensional pondered structure, is said to be a *generalized weak cycle* if: (a) when $\{i,j\} = \Delta_2 - \{c\}$, then for each pair of adjacent edges e, f it is $\omega(e) = \omega(f) = 1$ and either $e(0) = f(0)$ or $e(1) = f(1)$; (b) when two adjacent edges e and f of μ, respectively coloured c and $j \neq c$, have the same first (resp. second) vertex, then $(\omega(e), \omega(f)) \in \{(1,1),(0,1)\}$ (resp. $(\omega(e), \omega(f)) \in \{(1,1),(2,1)\}$); (c) when $e(0) = f(1)$, then $(\omega(e), \omega(f)) = (2,1)$ and when $e(1) = f(0)$, then $(\omega(e), \omega(f)) = (0,1)$.

In this paper, given a 3-dimensional pondered structure $(\bar{\Gamma}, \bar{\gamma})$ (resp. either a crystallized structure $(\widetilde{\Gamma}, \widetilde{\gamma})$ or a h-bijoin (B, β) over $(\bar{\Gamma}, \bar{\gamma})$), we denote by $\bar{g}_i$, $\bar{g}_{\hat{i}}$ (resp. either $\widetilde{g}_i$, $\widetilde{g}_{\hat{i}}$ or $g_i, g_{\hat{i}}$) the number of the connected components of $\bar{\Gamma}_{\{i\}}$ and $\bar{\Gamma}_{\hat{i}}$ (resp. either of $\widetilde{\Gamma}_i$ and $\widetilde{\Gamma}_{\hat{i}}$ or of B_i and $B_{\hat{i}}$) respectively, by q_1 the sum of the $\bar{g}_i$'s, by q_2 the sum of the $\bar{g}_{\hat{i}}$, by v the number of the vertices of $\bar{\Gamma}$ and by $\bar{g}_{ij}$ the number of the generalized weak cycles coloured i and j alternately. With the above notations, the following result holds: *if (B, β) is the h-bijoin constructed over $(\bar{\Gamma}, \bar{\gamma})$, then (B, β) represents a closed 3-manifold iff $\bar{g}_{01} + \bar{g}_{02} + \bar{g}_{12} + q_1 = q_2 + h + 2v$.*

Observe that if B is contracted for each $i \in \Delta_3$, the subcomplex of $K(B)$ constituted by all the $\hat{i}$-coloured 2-simplexes is a spine of $K(B)$.

Moreover, recall that as pointed out in [P$_2$] if (Γ, γ) is a crystallization of a 3-manifold M and $\{i,j\} \cup \{h,k\} = \Delta_3$, the connected components, in fact cycles of two colours, of $\Gamma_{\{i,j\}}$ and $\Gamma_{\{h,k\}}$, but one, are the classes of canonical curves of a Heegaard diagram H of M.

It is well known that, give an Heegaard diagram H of a (closed) 3-manifold M, there is a presentation of $\pi_1(M)$ canonically associated to H, whose generators and relators arise from the (two) classes of canonical curves of H ([S], [He]); hence the presentation must have the same number of generators and relators.

Following [G_1], we describe how to obtain a presentation of the fundamental group of a (closed) 3-manifold M by starting from a crystallization of M. With the above notations, let $\{x_1, \ldots, x_g\}$ be the set of all the connected components of $\Gamma_{\{i,j\}}$, but one, and let $\{y_1, \ldots, y_g\}$ be the set of all the connected components of $\Gamma_{\{h,k\}}$, but one. Choose a fixed vertex and a running direction for each $y_\alpha, \alpha = 1, \ldots, g$, construct a word r_α over the alphabet $\{x_1, \ldots, x_g\}$ as follows:

start from the fixed vertex, walk along y_α in the fixed direction and write step by step all the meeting generators, with exponent $+1$ or -1, according to h or k being the colour of the edge by which you run into the generator.

If $\Phi = \langle x_1, \ldots, x_s / R_1, \ldots, R_r \rangle$ is any group presentation, it is possible to define a new presentation $A(\Phi)$ of the same group, as follows:

$$A(\Phi) = \langle x_1, \ldots, x_s, x_{s+1} / R'_1, \ldots, R'_r, x_{s+1} \rangle$$

where the new generator x_{s+1} is the trivial one, and if R_i, $i = 1, \ldots, r$, is a relator of Φ, say $R_i = x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \ldots x_{i_p}^{\varepsilon_p}$, with $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_p \in \{-1, 1\}$, then in $A(\Phi)$ the corresponding relator has the form $R'_i = x_{i_1}^{\varepsilon_1} x_{s+1}^{\delta_1} x_{i_2}^{\varepsilon_2} x_{s+1}^{\delta_2} \ldots x_{s+1}^{\delta_{p-1}} x_{i_p}^{\varepsilon_p} x_{s+1}^{\delta_p}$, with $\delta_h = 0$, if $\varepsilon_h \neq \varepsilon_{h+1}$ and $\delta_h = -\varepsilon_h$ otherwise.

For the presentation $A(\Phi)$, set $\lambda(x_i)$ the number of the occurrence of the generator x_i and λ_i the length of the relator R'_i. Obviously, $\sum_{i=1}^{s} \lambda(x_i) = \sum_{i=1}^{s} \lambda_i$; set such a number equal to λ.

The presentation $A(\Phi)$ associated to a (balanced) group presentation Φ, with g generators and g relators, is said to be the *alternating presentation* associated with Φ if: 1) for each relator R_i the first esponent ε_1 is $+1$ and the last esponent is -1; 2) each relator R'_i contains neither $x_{s+1}^{+1} x_{s+1}^{-1}$ nor $x_{s+1}^{-1} x_{s+1}^{+1}$; 3) if $R_i = R'_i$, for each $i = 1, \ldots, g$, and $R_i = x_{\alpha(1)}^{+1} x_{\alpha(2)}^{-1} \ldots x_{\alpha(\lambda_1)}^{-1}$, then set $R''_i = x_{s+1}^{+1} x_{\alpha(2)}^{-1} \ldots x_{\alpha(\lambda_1)}^{-1} x_{\alpha(1)}^{+1} x_{s+1}^{-1}$.

2. Seminormal crystallizations

We recall that if (Γ, γ) is an $(n+1)$-coloured graph, ϑ is a subgraph of Γ such that $V(\vartheta) = \{x, y\}$ and x, y are joined by p edges coloured $c_1, \ldots, c_p (1 \leq p \leq n)$, then ϑ is said to be a *dipole of type p* iff x and y lie on distinct connected components of $\Gamma_{\Delta_n - \{c_1, \ldots, c_p\}}$.

Let (Γ, γ) be an $(n+1)$-coloured graph and let $f_1, \ldots, f_p$ be edges (coloured with distinct colours $c_1, \ldots, c_p$) of $E(\Gamma)$ such that: — the graph $(\widetilde{\Gamma}(c_1, \ldots, c_p), \widetilde{\gamma}(c_1, \ldots, c_p))$ obtained by deleting $f_1, \ldots, f_p$ from $\Gamma_{\{c_1, \ldots, c_p\}}$ has many connected components; — there are two components of $\widetilde{\Gamma}(c_1, \ldots, c_p)$, say C^0 and C^1, such that, for each $i = 1, \ldots, p, f_i$ has an end on C^0 and the other on C^1. Let finally v_i^0 (resp. v_i^1), $i = 1, \dots, p$ the end of f_i lying on C^0 (resp. on C^1). Then we can obtain from (Γ, γ) a new graph (Γ', γ') by *adding a dipole of type* $n + 1 - p$ between the edges $f_1, \dots, f_p$, as follows:

(a) $V(\Gamma') = V(\Gamma) \cup \{x, y\}$;

(b) delete the edges $f_1, \dots, f_p$;

(c) join x and y by $n + 1 - p$ distinct edges coloured by distinct colours of $\Delta_n - \{c_1, \dots, c_p\}$;

(d) join x (resp. y) with v_i^0 (resp. v_i^1) by an edge coloured c_i, $i = 1, \dots, p$.

The inverse process is called *"to cancel a dipole"* of type $n + 1 - p$.

It is well known (see [FG]) that if Γ represents an n-manifold M, then Γ' represents again the same n-manifold.

If $\mathfrak{C}^{(\alpha)}$ is a $\{0, 1\}$-coloured 2-residue (in fact a cycle) of a crystallization (Γ, γ), then $\{v_i^{(\alpha)}, i = 1, \dots, t(\alpha)\}$, denote the set of the vertices of $\mathfrak{C}^{(\alpha)}$.

DEFINITION 1. A crystallization (Γ, γ) of a 3-manifold M is said to be $(0, 1, p)$-seminormal, $p \in \{2, 3\}$ iff there exists an ordering $\mathfrak{C}^{(0)}, \mathfrak{C}^{(1)}, \dots, \mathfrak{C}^{(s)}$ of the connected components of $\Gamma_{\{0,1\}}$ with the properties:

a — $v_2^{(i)}, \dots, v_{t(i)-1}^{(i)}$, $i = 1, \dots, s$, are p-adjacent with vertices of $\mathfrak{C}^{(0)}$ and the vertices $\bar{v}_1^{(1)}$ and $\bar{v}_{t(s)}^{(s)}$, which are p-adjacent with the vertices $v_1^{(1)}$ and $v_{t(s)}^{(s)}$ respectively, are 0- (or 1-)adjacent;

b — $v_{t(i)}^{(i)}$ and $v_1^{(i+1)}$, $i = 1, \dots, s - 1$, are p-adjacent;

c — for each vertex v of $\mathfrak{C}^{(0)}$, there is a label i ($i \in \{1, 2, \dots, s\}$), such that the vertex v', p-adjacent with v, lies on $\mathfrak{C}^{(i)}$;

We shall call *base component* (resp. *internal components*) of Γ the connected component $\mathfrak{C}^{(0)}$ (resp. the connected components $\mathfrak{C}^{(1)}, \dots, \mathfrak{C}^{(s)}$).

PROPOSITION 1. *For each* 3*-manifold* M, *there is a crystallization* (Γ, γ) *which is* $(0, 1, p)$*-seminormal.*

Proof. In the following we mean $\{p, q\} = \{2, 3\}$ and $\{a, b\} = \{0, 1\}$. Let (Γ', γ') be any crystallization of M with g $(0, 1)$-coloured cycles. If, on some $\{0, 1\}$-coloured cycles of Γ' there are two vertices x and y joined by a p-coloured edge f, then:

i) if $\vartheta = \{x, y\}$ is a proper 2-dipole, involving colours a and p, then cancel it from Γ';

ii) otherwise, called e an edge of Γ' which is a-coloured and which lies on the same $\{a, p\}$-coloured component as f, then add a 2-dipole involving colours b and q between e and f. Repeat the process in order to eliminate all the pairs of p-adjacent vertices lying on the same $\{0, 1\}$-component.

Choose now two $\{0,1\}$-coloured cycles of Γ' with the property that there are at least a vertex of the first and a vertex of the second which are p-adjacent. Label these two components by $C'^{(0)}$ and by $C'^{(1)}$. Let now $C'^{(2)}$ be a $\{0,1\}$-coloured component other than $C'^{(0)}$ and $C'^{(1)}$ having a vertex $v'^{(2)}_1$ p-adjacent with a vertex $v'^{(1)}_{t(1)}$ of $C'^{(1)}$ by means of an edge ε. Moreover, $C'^{(2)}$ has p-coloured edges $\varepsilon_1, \varepsilon_2, \ldots$, which are p-consecutive ([BDG])and ε_1 lies on the same $\{a,p\}$-coloured cycle as an edge of $C'^{(0)}$.

Note that the existence of such a component is assured, since $C'^{(1)}$ has surely edges lying on $\{a,p\}$-coloured cycles involving edges of $C'^{(0)}$ and having length more than 4. Moreover, label by ε_i the p-coloured edges joining vertices of $C'^{(2)}$ and $C'^{(1)}$ other than ε and add a 2-dipole involving colours a and q (resp. b and q) between such p-coloured edges ε_i's and the suitable edge of $C'^{(0)}$.

Label now by $C'^{(3)}$ a $\{0,1\}$-coloured component other than $C'^{(0)}, C'^{(1)}$ and $C'^{(2)}$ having a vertex p-adjacent with a vertex of $C'^{(2)}$ and with the same properties required to choose $C'^{(2)}$.

As in the case of $C'^{(2)}$, add suitable 2-dipoles to obtain that $C'^{(3)}$ and $C'^{(2)}$ have exactly two p-adjacent vertices.

One can repeat the process to obtain a (finite) sequence $C'^{(1)}, \ldots, C'^{(g)}$ of $\{0,1\}$-coloured cycles such that for each $i = 1, \ldots, g-1$ there is a unique pair of p-adjacent vertices lying respectively on $C'^{(i)}$ and on $C'^{(i+1)}$.

If the so obtained graph is none $(0,1,p)$-seminormal crystallization of M, then there is a p-coloured edge ψ joining a vertex of $C'^{(i)}$ and a vertex of $C'^{(j)}$, with $j \neq i-1, i+1, i = 1, \ldots, g$ and lying on a $\{a,p\}$-coloured cycle involving an a-coloured edge ϕ of $C'^{(0)}$. Add a 2-dipole involving colours b and q between ψ and ϕ. Repeat the process for each pair of edges with the properties of ψ and ϕ to obtain the required $(0,1,p)$-seminormal crystallization of M. ∎

Remark. Since each 3-manifold M admits a $(0,1,p)$-normal crystallization ([BDG]), then the preceding proposition can be obtained by starting from such a crystallization Γ. Let $C^1, C^2, \ldots, C^r$ be the internal components of Γ and denote by $\varepsilon_i, i = 1, \ldots, r$, one of the p-coloured edges having an end on C^i and lying on the (unique) $(1,p)$-coloured 2-cell C of length $4r$ of Γ, then by adding a 2-dipole involving colours 1 and q between ε_i and the 1-coloured edge of C^{i+1} lying on $C(i = 1, \ldots, r-1)$, one obtain a seminormal crystallization of M.

3. The algorithm

Let now $\Phi = \langle X/R \rangle$ be a group presentation with g generators and g relators such that there are no bipartition $X = X_1 \cup X_2$ and no bipartition $R' = R'_1 \cup R'_2$ with $A(\Phi) = \langle X_1/R'_1 \rangle * \langle X_2/R'_2 \rangle$ (where "*" denotes the free product of groups). Then it is possible to associate to $A(\Phi)$ a representable 2-complex as follows: for each relation $R'_i, i = 1, \dots, g$, consider a 2-cell B_i whose boundary is subdivided according with the word corresponding to R'_i. Obviously, the relation R'_i induces a natural ordering of the vertices of the boundary of B_i and it is natural to label the edges of ∂B_i by the name of the corresponding generator; moreover, label by 0 (resp. 1) each vertex of the resulting subdivision lying between two edges labelled, say, x_i and x_j with the property that in the corresponding word the generator x_i has exponent 1 (resp.-1) and the generator x_j has exponent -1 (resp. 1). From another point (labelled 2) internal to B_i, construct the join on the boundary of B_i. Let now $K(\Phi)$ be the disjoint union of the B_i's and in $K(\Phi)$ label each 2-simplex incident with an edge labelled x_α, with the same label. Since the group G of which Φ is a balanced presentation is not a free product, then there is a sequence $\tau_1, \tau_2, \dots, \tau_g$ of 2-simplexes such that: a. — for each $j = 1, \dots, g$, there is a label $i(j), i(j) = 1, \dots, g$, such that τ_j lies in $B_{i(j)}$; b. — for each $j = 1, \dots, g-1, \tau_j$ and τ_{j+1} have the same label. Let now $K'(\Phi)$ be the ball complex obtained from $K(\Phi)$ by identifying all the $\{0,1\}$-coloured edge with the same label and (pairwise) the $\{0,2\}$-coloured (resp. $\{1,2\}$-coloured) edges of τ_i and $\tau_{i+1}, i = 1, \dots, g-1$.

$K'(\Phi)$ is representable and, more precisely, it is represented by the crystallized structure $(\widetilde{\Gamma}_\Phi, \widetilde{\gamma}_\Phi)$, so defined:

a — for each relator R'_i $(i = 1, 2, \dots, s)$, let μ_i be a cycle with edges alternately coloured 0 and 1 and whose vertices are labelled by the generators occurring in R'_i, so that, if $R'_i = \dots x_\alpha^{\varepsilon_\alpha} x_\beta^{\varepsilon_\beta} \dots$, then in μ_i the vertices labelled x_α and x_β are joined by a 0-coloured (resp. 1-coloured) edge if $\varepsilon_\alpha = -\varepsilon_\beta = 1$ (resp. $\varepsilon_\alpha = -\varepsilon_\beta = -1$).

b — Since the group is not a free product, then there is a sequence $\mu_1, \mu_2, \dots, \mu_g$ such that, for each $i = 1, \dots, g-1$, there exist two vertices x_i and x_{i+1} of μ_i and μ_{i+1} respectively with the same label. Then identify x_i and x_{i+1} to a unique vertex z_i and, if $x_i(0)$ (resp. $x_i(1)$) and $x_{i+1}(0)$ (resp. $x_{i+1}(1)$) are the vertices 0-adjacent (resp. 1-adjacent) to x_i and x_{i+1}, join $x_i(0)$ and $x_{i+1}(0)$ (resp. $x_i(1)$ and $x_{i+1}(1)$) by means of a 0-coloured (resp. a 1-coloured) edge.

c — consider a 2-coloured clique for each class of vertices labelled by the same generator.

Note that, by the same construction, the so obtained crystallized structure has $\lambda - g + 1$ vertices.

Observe that, if Φ is a canonical group presentation, then the above construction works again, since in this case $A(\Phi)$ is simply obtained via condition 3) of the definition; hence $K'(\Phi)$ and $K'(A(\Phi))$ are combinatorially equivalent.

Note that, if $\widetilde{g}_B$ (resp. $\bar{g}_B$) denotes the number of connected components of $(\widetilde{\Gamma}_\Phi)_B$ (resp. of any pondered structure $(\bar{\Gamma}_\Phi, \bar{\gamma}_\Phi)$ associated with $(\widetilde{\Gamma}_\Phi, \widetilde{\gamma}_\Phi)$) and by writing i instead $\{i\}$, then, for each $i \in \Delta_2$, it results:

$$-\widetilde{g}_0 = \widetilde{g}_1 = \bar{g}_0 = \bar{g}_1 = (\lambda - 2g + 2)/2;$$
$$-\widetilde{g}_2 = \bar{g}_2 = g + 1 \text{ (the generator's number)};$$
$$-\widetilde{g}_{\hat{2}} = \bar{g}_{\hat{2}} = 1;$$

hence $\chi(\widetilde{\Gamma}_\Phi) = 1$ iff $\widetilde{g}_{\hat{0}} + \widetilde{g}_{\hat{1}} = 2$ iff $\widetilde{g}_{\hat{0}} = \widetilde{g}_{\hat{1}} = 1$.

From now on, we consider the pondered structures associated to $\widetilde{\Gamma}_\Phi$ with the following property (property (SN)): each 0- and 1-coloured edge becomes weight 1 and it is assigned an orientation to each of such cycles so that the vertex $z_i (i = 1, \ldots, g)$ lies on two different generalized weak cycles; obviously under these hypotesis it is:

$$-\bar{g}_{01} = g + 1.$$

Let (B, β) be the $(h-)$ bijoin constructed on any pondered structure $\bar{\Gamma}_\Phi$ with the property (SN) associated to $\widetilde{\Gamma}^*_\Phi$, then we can state the following:

PROPOSITION 2. *With the above notations*, (B, β) *is a crystallization of a (closed) 3-manifold M iff*: $\bar{g}_{i2} = \lambda/2 - g + 1$, *for* $i = 0, 1$ *and* $h = 1$; *moreover, in this case*, (B, β) *is a seminormal crystallization of M.*

Proof. If (B, β) is a crystallization of a (closed) 3-manifold, then $\widetilde{\Gamma}^*_\Phi$ is a spine of M, hence $\widetilde{g}_{\hat{0}} = \widetilde{g}_{\hat{1}} = 1$ and the result is an easy calculation on the Euler characteristic.

Conversely, if $\bar{g}_{12} = \lambda/2 - g + 1$, for $i = 0, 1$ and $h = 1$, then for $i = 0, 1$ and j such that $\{i, j\} = \{0, 1\}$, we obtain:

$$\chi(B_{\hat{i}}) = g_{j2} + g_{23} + g_{j3} - (\lambda - g + 1) = \bar{g}_{j2} + \widetilde{g}_j + \widetilde{g}_2 - (\lambda - g + 1)$$
$$= \bar{g}_{j2} + (g + 1) + (\lambda/2 - g + 1) - (\lambda - g + 1) = 2 = 2\widetilde{g}_{\hat{i}};$$

moreover, $\chi(B_{\hat{2}}) = g_{01} + g_{03} + g_{13} - (\lambda - g + 1) = \bar{g}_{01} + \widetilde{g}_0 + \widetilde{g}_1 - (\lambda - g + 1) = (g + 1) + 2(\lambda/2 - g + 1) - (\lambda - g + 1) = 2$; finally,

$$\chi(B_{\hat{3}}) = g_{01} + g_{02} + g_{12} - (\lambda - g + 1) = \bar{g}_{01} + \bar{g}_{02} + \bar{g}_{12} - (\lambda - g + 1)$$
$$= (g + 1) + 2(\lambda/2 - g + 1) - (\lambda - g + 1) = 2 = 2h.$$

Since $h = 1, (B, \beta)$ is contracted and hence a crystallization of M.

Moreover, for the same "bijoin" algorithm, applied to $\bar{\Gamma}_\Phi$, in B there are $g + 1$ $(0, 1)$-coloured cycles $C^{(0)}, C^{(1)}, \ldots, C^{(g)}$ corresponding to the cycles $\mathfrak{C}^0, \mathfrak{C}^1, \ldots, \mathfrak{C}^g$ of $\bar{\Gamma}_\Phi$, such that all the vertices, less the two (resp. less the one) of the components $C^{(2)}, \ldots, C^{(g-1)}$ (resp. $C^{(1)}$ and $C^{(g)}$) arising from the vertices $z_1, \ldots, z_{g-1}$ of $\bar{\Gamma}_\Phi$, are 3-adjacent to vertices of the component $C^{(0)}$ and the vertices $(z_i, 0)$ and $(z_{i+1}, 1), i = 1, \ldots, g-1$, are 3-adjacent. Hence (B, β) is a (0.1;3)-seminormal crystallization of M. ■

Remark. If H is an Heegaard diagram of a closed connected 3-manifold M and Φ is the presentation of $\pi_1(M)$ associated with Γ, then Φ is again the presentation associated with the crystallization (Γ, γ) of M obtained from H via the construction described in [G_2] by respect (say) to the pair $\{i, j\}$ of colours of Γ. The above algorithm applied to Φ produces a seminormal crystallization (Γ', γ') of M which is obtained from (Γ, γ) by adding 2-dipoles involving colours h, k, with $h \in \{i, j\}$ and $k \in \Delta_3 - \{i, j\}$; hence the Heegaard diagram associated with Γ' is obtained from H by means of isotopic transformations.

References

[B] P. Bandieri, *Constructing 3-manifolds from spines*, (to appear).

[BDG] P. Bandieri, A. Donati, L. Grasselli, *Normal crystallizations of 3-manifolds*, Geom. Dedicata 14 (1983), 405–418.

[BG] P. Bandieri, C. Gagliardi, *Generating all orientable n-manifolds from (n-1)-complexes*, Rend. Circ. Mat. Palermo 31 (1982), 233–246.

[BM] J. Bracho, L. Montejano, *The combinatorics of colored triangulations of manifolds*, Geom. Dedicata 22 (1987), 303–328.

[FGG] M. Ferri, C. Gagliardi, L. Grasselli, *A graph-theoretical representation of PL-manifolds — A survey on crystallizations*, Aequationes Math. 31 (1986), 121–141.

[Gl] L.C. Glaser, *Geometrical Combinatorial Topology*. Van Nostrand Reinhold Math. Studies, New York 1970.

[G_1] C. Gagliardi, *How to deduce the fundamental group of a closed n-manifold from a contracted triangulation*, J. Comb. Inf. Sist. Sci. 4 (1979), 237–252.

[G_2] C. Gagliardi, *Extending the concept of genus to dimension N*, Proc. Amer. Math. Soc. 81 (1981), 473–481.

[Gr] L. Grasselli, *3-manifold spines and bijoins*, Rev. Mat. Univ. Complutense Madrid 3 (1990), 165–179.

[GP] L. Grasselli, S. Piccarreta, *Crystallizations of generalized Neuwirth manifolds*, (to appear).

[He] J. Hempel, *3-Manifolds*. Princeton Univ. Press 1976.

[HW] P.J. Hilton, S. Wylie, *An Introduction to Algebraic Topology — Homology Theory*. Cambridge University Press, Cambridge 1960.

[L] S. Lins, *Graph-encoded maps*, J. Comb. Theory Ser. B 32 (1982), 171–181.

[LM] S. Lins, A. Mandel, *Graph — encoded 3-manifolds*, Discrete Math. 57 (1985), 261–284.

[Ma] S. Matveev, *Special spines of piecewise linear manifolds*, Math. URSS Sbornik, 21 (1973), 279–291.

[Mo] J.M. Montesinos, *Representing 3-manifolds by a universal branching set*, Math. Proc. Cambridge Phil. Soc. 94 (1983), 109–123.

[N] L. Neuwirth, *An algorithm for the construction of 3-manifolds from 2-complexes*, Proc. Cambridge Phil. Soc. 64 (1968), 603–613.

[OS_1] R.P. Osborne, R.S. Stevens, *Group presentations corresponding to spines of 3-manifolds I*, Amer. J. Math. 96 (1974), 454–471.

[OS_2] R.P. Osborne, R.S. Stevens, *Group presentations corresponding to spines of 3-manifolds II*, Trans. Amer. Math. Soc. 234 (1977), 213–243.

[P_1] M. Pezzana, *Sulla struttura topologica delle varietè compatte*, Atti Sem. Mat. Fis. Univ. Modena 23 (1974), 269–277.

[P_2] M. Pezzana, *Diagrammi di Heegaard e triangolazione contratta*, Boll. Un. Mat. Ital. 12 Suppl. Fasc. 3 (1975), 98–105.

[RS] C.P. Rourke, B.J. Sanderson, *Introduction to Piecewise — Linear Topology*. Springer Verlag, New York — Heidelberg, 1972.

[S] J. Singer, *Three-dimensional manifolds and their Heegaard diagrams*, Trans. Amer. Math. Soc. 35 (1933), 88–111.

DIPARTIMENTO DI MATEMATICA
Via Campi, 213/b
I-41100 MODENA, ITALY

Received January 23, 1995.