DEMONSTRATIO MATHEMATICA
Vol. XXIX No 3 1996

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ON SOLUTIONS OF SYSTEMS OF FUNCTIONAL EQUATIONS DETERMINING SOME SUBSEMIGROUPS OF THE GROUP $\mathbf{L}_6^1$

1. Let $\mathbb{R}$ be the set of real numbers, and $\mathbb{R}_0 = \mathbb{R}\setminus\{0\}$. In the set $\mathbb{R}_0\times\mathbb{R}^5$ we introduce the following operation

$$
\begin{aligned}
(1.1)\quad & \langle y_1, y_2, y_3, y_4, y_5, y_6\rangle \cdot \langle x_1, x_2, x_3, x_4, x_5, x_6\rangle = \\
&= \langle y_1x_1, y_1x_2 + y_2x_1^2, y_1x_3 + 3y_2x_1x_2 + y_3x_1^3, y_1x_4 + y_4x_1^4 + \\
&+ 4y_2x_1x_3 + 6y_3x_1^2x_2 + 3y_2x_2^2, y_1x_5 + y_5x_1^5 + 10x_1^3x_2y_4 + \\
&+ 15x_1x_2^2y_3 + 10x_1^2x_3y_3 + 10x_2x_3y_2 + 5x_1x_4y_2, y_1x_6 + y_6x_1^6 + \\
&+ 6y_2x_1x_5 + 15y_2x_2x_4 + 10y_2x_3^2 + 15y_3x_1^2x_4 + 60y_3x_1x_2x_3 + \\
&+ 20y_4x_1^3x_3 + 45y_4x_1^2x_2^2 + 15y_5x_1^4x_2\rangle.
\end{aligned}
$$

The set $\mathbb{R}_0\times\mathbb{R}^5$ with the operation (1.1) is a group, which is denoted by $\mathbf{L}_6^1$ (cf. [2]).

It is known that the right hand side of (1.1) expresses the derivatives of order n, $n = 1,\ldots,6$, for some composed function $f(t) = F(\phi(t))$ by means of the derivatives $y_1,\ldots,y_6$ respectively of the order $1,\ldots,6$ of the exterior function $F(u)$ $\left(y_i = \frac{d^iF(u)}{du_i}\right)$ and the derivatives $x_1,\ldots,x_6$ of the order $1,\ldots,6$ of the inner function $\phi(t)$ $\left(x_i = \frac{d_i\phi(t)}{dt^i}\right)$.

In the first part of the paper we shall deal with the determination of subsemigroups of the group $\mathbf{L}_6^1$, in which three parameters are functions of the others or some others.

In the second and third part of the paper we determine subsemigroups of the group $\mathbf{L}_6^1$ in which four parameters are functions of the others.

In the paper [12] there were detemined subsemigroups of the group $\mathbf{L}_6^1$ in which the last parameter is a function of the others or some of the others. Subgroups or subsemigroups of the groups $\mathbf{L}_s^1$ $s = 2,3,4,5$, were determined by means of functional equations in the paper from [2] to [15] besides [6].

Let f, g, h be functions mapping $\mathbb{R}_0 \times \mathbb{R}^2$ into reals. Let us denote

$$(1.2)\quad \mathbf{P}_{456} = \{\langle x_1, x_2, x_3, f(x_1,x_2,x_3), g(x_1,x_2,x_3), h(x_1,x_2,x_3)\rangle : x_1 \in \mathbb{R}_0, x_2, x_3 \in \mathbb{R}\}.$$

The set $\mathbf{P}_{456}$ is closed with respect to the operation (1.1) if and only if for any x_1, x_2, x_3 and y_1, y_2, y_3

$$\begin{aligned}
&\langle y_1, y_2, y_3, f(y_1,y_2,y_3), g(y_1,y_2,y_3), h(y_1,y_2,y_3)\rangle \cdot \\
&\langle x_1, x_2, x_3, f(x_1,x_2,x_3), g(x_1,x_2,x_3), h(x_1,x_2,x_3)\rangle = \\
&= \langle y_1x_1, y_1x_2 + y_2x_1^2, y_1x_3 + 3y_2x_1x_2 + y_3x_1^3, y_1 f(x_1,x_2,x_3) + \\
&\quad + f(y_1,y_2,y_3)x_1^4 + 4y_2x_1x_3 + 6y_3x_1^2x_2 + 3y_2x_2^2, y_1 g(x_1,x_2,x_3) + \\
&\quad + g(y_1,y_2,y_3)x_1^5 + 10x_1^3x_2 g(y_1,y_2,y_3) + 15x_1x_2^2y_3 + 10x_1^2x_3y_3 + \\
&\quad + 10x_2x_3y_2 + 5x_1 f(x_1,x_2,x_3)y_2, y_1 h(x_1,x_2,x_3) + h(y_1,y_2,y_3)x_1^6 + \\
&\quad + 6y_2x_1 g(x_1,x_2,x_3) + 15y_2x_2 f(x_1,x_2,x_3) + 10y_2x_3^2 + \\
&\quad + 15y_3x_1^2 f(x_1,x_2,x_3) + 60y_3x_1x_2x_3 + 20x_1^3x_3 f(y_1,y_2,y_3) + \\
&\quad + 45x_1^2x_2^2 f(y_1,y_2,y_3) + 15x_1^4x_2 g(y_1,y_2,y_3)\rangle \in \mathbf{P}_{456}.
\end{aligned}$$

Hence and by the definition of $\mathbf{P}_{456}$ get that the functions f, g, h satisfy the following system of functional equations

$$\begin{aligned}
(1.3)\quad & f(y_1x_1, y_1x_2 + y_2x_1^2, y_1x_3 + 3y_2x_1x_2 + y_3x_1^3) = \\
& \quad = y_1 f(x_1,x_2,x_3) + f(y_1,y_2,y_3)x_1^4 + 4y_2x_1x_3 + 6y_3x_1^2x_2 + 3y_2x_2^2, \\
(1.4)\quad & g(y_1x_1, y_1x_2 + y_2x_1^2, y_1x_3 + 3y_2x_1x_2 + y_3x_1^3) = \\
& \quad = y_1 g(x_1,x_2,x_3) + x_1^5 g(y_1,y_2,y_3) + 10x_1^3x_2 g(y_1,y_2,y_3) + \\
& \quad + 15x_1x_2^2y_3 + 10x_1^2x_3y_3 + 10x_2x_3y_2 + 5x_1y_2 g(x_1,x_2,x_3), \\
(1.5)\quad & h(y_1x_1, y_1x_2 + y_2x_1^2, y_1x_3 + 3y_2x_1x_2 + y_3x_1^3) = \\
& \quad = y_1 h(x_1,x_2,x_3) + x_1^6 h(y_1,y_2,y_3) + 6y_2x_1 g(x_1,x_2,x_3) + \\
& \quad + 15y_2x_2 f(x_1,x_2,x_3) + 10y_2x_3^2 + 15y_3x_1^2 f(x_1,x_2,x_3) + \\
& \quad + 60y_3x_1x_2x_3 + 20 f(y_1,y_2,y_3)x_1^3x_3 + 45x_1^2x_2^2 f(y_1,y_2,y_3) + \\
& \quad + 15 g(y_1,y_2,y_3)x_1^4x_2.
\end{aligned}$$

In [3] (Theorem 1) it was proved that the equation (1.3) has no solutions. Therefore the system (1.3)–(1.5) has no solutions, and the group $\mathbf{L}_6^1$ has no subsemigroups of the form $\mathbf{P}_{456}$.

Denote

$$(1.6)\quad \mathbf{P}_{456}^{\circ} = \{\langle x_1, 0, x_3, F(x_1,x_3), G(x_1,x_3), H(x_1,x_3)\rangle : x_1 \in \mathbb{R}_0, x_3 \in \mathbb{R}\}.$$

The set $\mathbf{P}^{\circ}_{456}$ is closed with respect to (1.1) if and only if the functions F, G, H satisfy the following system of functional equations. (One can obtain it from (1.3)–(1.5) putting $x_2 = y_2 = 0$ i $F(x_1, x_3) := f(x_1, 0, x_3)$, $G(x_1, x_3) := g(x_1, 0, x_3)$, $H(x_1, x_3) := h(x_1, 0, x_3)$).

$$(1.7)\quad F(x_1y_1, y_1x_3 + y_3x_1^3) = y_1F(x_1, x_3) + x_1^4F(y_1, y_3),$$

$$(1.8)\quad G(x_1y_1, y_1x_3 + y_3x_1^3) = y_1G(x_1, x_3) + x_1^5G(y_1, y_3) + 10x_1^2x_3y_3,$$

$$(1.9)\quad H(x_1y_1, y_1x_3 + y_3x_1^3) = y_1H(x_1, x_3) + x_1^6H(y_1, y_3) + 15y_3x_1^2F(x_1, x_3) + 20F(y_1, y_3)x_1^3x_3.$$

In view of Theorem 2 from [3] the general solution of the equation (1.7) is a family of functions of the form

$$(1.10)\qquad F(x_1, x_3) = p(x_1^4 - x_1),$$

where p is an abitrary real number.

By Lemma 1 in [9] it follows that the general solution of the equation (1.8) is a family of functions of the form

$$(1.11)\qquad G(x_1, x_3) = k(x_1^5 - x_1) + 5\frac{x_3^2}{x_1}.$$

where k is an abitrary real number.

If in (1.9) we substitute (1.10), we obtain

$$(1.12)\quad H(x_1y_1, y_1x_3 + y_3x_1^3) = y_1H(x_1, x_3) + x_1^6H(y_1, y_3) + 15y_3x_1^2p(x_1^4 - x_1) + 20p(y_1^4 - y_1)x_1^3x_3.$$

If in (1.12) we put $x_1 = 1$ and $y_3 = 0$, then we get

$$(1.13)\qquad H(y_1, y_1x_3) = y_1H(1, x_3) + H(y_1, 0) + 20p(y_1^4 - y_1)x_3.$$

Next setting $y_1 = 1$ and $x_3 = 0$ in (1.12) we have

$$(1.14)\qquad H(x_1, y_3x_1^3) = H(x_1, 0) + x_1^6H(1, y_3) + 15y_3x_1^2p(x_1^4 - x_1),$$

and setting $x_3 = y_3 = 0$ in (1.12) we get

$$(1.15)\qquad H(x_1y_1, 0) = y_1H(x_1, 0) + x_1^6H(y_1, 0).$$

By Theorem 1 in [2] it follows that the general solution of (1.15) is a family

$$(1.16)\qquad H(x_1, 0) = l(x_1^6 - x_1),$$

where l is an arbitrary real number.

If in (1.12) we put $x_1 = y_1 = 1$, then we get

$$(1.17)\qquad H(1, x_3 + y_3) = H(1, x_3) + H(1, y_3).$$

From the last equality it follows that the function

$$\phi(x) := H(1.x) \tag{1.18}$$

is additive.

If in (1.13) we assume $u_3 = y_1 x_3 (x_3 = \frac{u_3}{y_1})$ then due to (1.16) and (1.18) we get

$$H(y_1, u_3) = y_1 \phi\left(\frac{u_3}{y_1}\right) + l(y_1^6 - y_1) + 20p(y_1^3 - 1)u_3. \tag{1.19}$$

If in (1.14) we set $u_3 = y_3 x_1^3 (y_3 = \frac{u_3}{x_1^3})$ and $x_1 = y_1$, then in view of (1.16) and (1.18) we have

$$H(y_1, u_3) = l(y_1^6 - y_1) + y_1^6 \phi\left(\frac{u_3}{y_1^3}\right) + 15u_3 p(y_1^3 - 1). \tag{1.20}$$

Note that the left hand sides of (1.19) and (1.20) are equal, and so are the right hand sides. Hence

$$y_1 \phi\left(\frac{u_3}{y_1}\right) + 20p(y_1^3 - 1)u_3 = y_1^6 \phi\left(\frac{u_3}{y_1^3}\right) + 15p(y_1^3 - 1)u_3. \tag{1.21}$$

Let us consider the above equality only for y_1 from the set of rational numbers. From this and from the fact that ϕ is additive it follows that (1.21) takes the form

$$\phi(u_3) + 20p(y_1^3 - 1)u_3 = y_1^3 \phi(u_3) + 15p(y_1^3 - 1)u_3. \tag{1.22}$$

Therefore

$$\phi(u_3)(y_1^3 - 1) = 5p(y_1^3 - 1)u_3.$$

By the above equality it follows that

$$\phi(u_3) = 5pu_3. \tag{1.23}$$

Hence (1.19) takes the form

$$H(x_1, x_3) = 5px_3 + l(x_1^6 - x_1) + 20p(x_1^3 - 1)x_3 \tag{1.24}$$

It is easily checked that every function of the form (1.24) satisfies the equation (1.12).

We then proved

LEMMA 1. *The general solution of the equation* (1.12) *is a family of functions of the form* (1.24), *where l is a real number.*

THEOREM 1. *The general solution of the system* (1.7)–(1.9) *is a family of triples of the form*

$$F(x_1, x_3) = p(x_1^4 - x_1),$$
$$G(x_1, x_3) = k(x_1^5 - x_1) + 5\frac{x_3^2}{x_1},$$

$$H(x_1, x_3) = l(x_1^6 - x_1) + 5px_3 + 20p(x_1^3 - 1)x_3,$$

where p, k, l belong to reals.

THEOREM 1′. *Unique subsemigroups of the group $\mathbf{L}_6^1$ of the form $\mathbf{P}_{456}^o$ are the sets*

$$\left\{\left\langle x_1, 0, x_3, p(x_1^4 - x_1), k(x_1^5 - x_1) + 5\frac{x_3}{x_1}, l(x_1^6 - x_1) + 5px_3 + 20p(x_1^3 - 1)x_3\right\rangle : x_1 \in \mathbb{R}_0, x_3 \in \mathbb{R}\right\},$$

where p, k, l are reals.

2. By f, g, h, H we denote functions such that $f, g, h, H : \mathbb{R}_0 \times \mathbb{R} \to \mathbb{R}$. Denote

$$\mathbf{P}_{2456} = \{\langle x_1, f(x_1, x_3), x_3, g(x_1, x_3), h(x_1, x_3), H(x_1, x_3)\rangle : x_1 \in \mathbb{R}_0, x_3 \in \mathbb{R}\}. \tag{2.1}$$

Analogously as in the case of $\mathbf{P}_{456}$ one proves that the set $\mathbf{P}_{2456}$ is closed with respect to the operation (1.1) if and only if the functions f, g, h, H verify the following system of functional equations

$$f(y_1x_1, y_1x_3 + 3x_1f(x_1, x_3)f(y_1, y_3) + y_3x_1^3) = = y_1f(x_1, x_3) + f(y_1, y_3)x_1^2, \tag{2.2}$$

$$g(y_1x_1, y_1x_3 + 3x_1f(x_1, x_3)f(y_1, y_3) + y_3x_1^3) = = y_1g(x_1, x_3) + x_1^4g(y_1, y_3) + 4f(y_1, y_3)x_1x_3 + + 6y_3x_1^2f(x_1, x_3) + 3f(y_1, y_3)f^2(x_1, x_3), \tag{2.3}$$

$$h(y_1x_1, y_1x_3 + 3x_1f(x_1, x_3)f(y_1, y_3) + y_3x_1^3) = = y_1h(x_1, x_3) + x_1^5h(y_1, y_3) + 10x_1^3f(x_1, x_3)g(y_1, y_3) + + 15x_1f^2(x_1, x_3)y_3 + 10x_1^2x_3y_3 + 10x_3f(x_1, x_3)f(y_1, y_3) + + 5x_1g(x_1, x_3)f(y_1, y_3), \tag{2.4}$$

$$H(y_1x_1, y_1x_3 + 3x_1f(x_1, x_3)f(y_1, y_3) + y_3x_1^3) = = y_1H(x_1, x_3) + x_1^6H(y_1, y_3) + 6x_1f(y_1, y_3)h(x_1, x_3) + + 15f(x_1, x_3)f(y_1, y_3)g(x_1, x_3) + 10x_3^2f(y_1, y_3) + + 15x_1^2y_3g(x_1, x_3) + 60y_3x_1x_3f(x_1, x_3) + 20x_1^3x_3g(y_1, y_3) + + 45x_1^2f^2(x_1, x_3)g(y_1, y_3) + 15x_1^4f(x_1, x_3)h(y_1, y_3). \tag{2.5}$$

In virtue of Theorem 1 in [9] the general solution of the system (2.2)–(2.4) are the family of triples of functions

$$f(x_1, x_3) = 0, \tag{2.6}$$

$$g(x_1, x_3) = p(x_1^4 - x_1), \tag{2.7}$$

$$h(x_1,x_3)=k(x_1^5-x_1)+5\frac{x_3^2}{x_1}, \tag{2.8}$$

where k and p are arbitrary real constans.

If in (2.5) we substitute (2.6), (2.7) and (2.8) then we get

$$\begin{aligned}(2.9)\ H(y_1x_1, y_1x_3+y_3x_1^3) = y_1H(x_1,x_3)+x_1^6H(y_1,y_3)+\\ +15x_1^2y_3p(x_1^4-x_1)+20x_1^3x_3p(y_1^4-y_1)\end{aligned}$$

Which is actually the equation (1.12).

It follows by Lemma 1 that the general solution of (2.9) constitutes a family of functions of the form

$$H(x_1,x_3)=l(x_1^6-x_1)+20p(x_1^3-1)x_3+5px_3 \tag{2.10}$$

where l is arbitrary real.

THEOREM 2. *The general solution of the system* (2.2)–(2.5) *is a family of quadruples of functions* (2.6)–(2.8) *and* (2.10), *where* p, k *and* l *are arbitrary real numbers.*

THEOREM 2′. *Unique subsemigroups of the group* $\mathbf{L}_6^1$ *of the form* $\mathbf{P}_{2456}$ *are sets of the form*

$$\left\{\left\langle x_1,0,x_3,p(x_1^4-x_1),k(x_1^5-x_1)+5\frac{x_3}{x_1},l(x_1^6-x_1)+20p(x_1^3-1)x_3+5px_3\right\rangle :\right.$$
$$\left. x_1\in\mathbb{R}_0, x_3\in\mathbb{R}\right\},$$

where p, k *and* l *are abitrary real numbers.*

3. Let us denote

$$\begin{aligned}(3.1)\qquad \mathbf{P}_{1456}=\{\langle f_1(x_2,x_3),x_2,x_3,f_4(x_2,x_3),f_5(x_2,x_3),f_6(x_2,x_3)\rangle :\\ x_2,x_3\in\mathbb{R}\},\end{aligned}$$

where $f_1:\mathbb{R}\times\mathbb{R}\to\mathbb{R}_o$, $f_k:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ for $k=4,5,6$.

Analogously as in case $\mathbf{P}_{456}$ one shows that $\mathbf{P}_{1456}$ is closed with respect (1.1) if and only if the functions f_1, f_4, f_5 and f_6 satisfy the system of functional equations:

$$\begin{aligned}(3.2)\ & f_1(x_2f_1(y_2,y_3)+y_2f_1^2(x_2,x_3),x_3f_1(y_2,y_3)+\\ &+3x_2y_2f_1(x_2,x_3)+y_3f_1^3(x_2,x_3))=f_1(x_2,x_3)f_1(y_2,y_3),\\ (3.3)\ & f_4(x_2f_1(y_2,y_3)+y_2f_1^2(x_2,x_3),x_3f_1(y_2,y_3)+\\ &+3x_2y_2f_1(x_2,x_3)+y_3f_1^3(x_2,x_3))=f_1(y_2,y_3)f_4(x_2,x_3)+\\ &+f_4(y_2,y_3)f_1^4(x_2,x_3)+4y_2f_1(x_2,x_3)x_3+6y_3x_2f_1^2(x_2,x_3)+3y_2x_2^2,\end{aligned}$$

$$(3.4)\quad f_5(x_2f_1(y_2,y_3)+y_2f_1^2(x_2,x_3),x_3f_1(y_2,y_3)+ \\ +3x_2y_2f_1(x_2,x_3)+y_3f_1^3(x_2,x_3)) = f_5(x_2,x_3)f_1(y_2,y_3)+ \\ +f_5(y_2,y_3)f_1^5(x_2,x_3)+10f_1^3(x_2,x_3)x_2f_4(y_2,y_3)+ \\ +15f_1(x_2,x_3)x_2^2y_3+10x_3y_3f_1^2(x_2,x_3)+10x_2x_3y_2+ \\ +5f_1(x_2,x_3)y_2f_4(x_2,x_3),$$

$$(3.5)\quad f_6(x_2f_1(y_2,y_3)+y_2f_1^2(x_2,x_3),x_3f_1(y_2,y_3)+ \\ +3x_2y_2f_1(x_2,x_3)+y_3f_1^3(x_2,x_3)) = f_1(y_2,y_3)f_6(x_2,x_3)+ \\ +f_6(y_2,y_3)f_1^6(x_2,x_3)+6y_2f_1(x_2,x_3)f_5(x_2,x_3)+ \\ +15y_2x_2f_4(x_2,x_3)+10y_2x_3^2+15y_3f_1^2(x_2,x_3)f_4(x_2,x_3)+ \\ +60y_3x_2x_3f_1(x_2,x_3)+20x_3f_1^3(x_2,x_3)f_4(y_2,y_3)+ \\ +45x_2^2f_1^2(x_2,x_3)f_4(y_2,y_3)+15x_2f_1^4(x_2,x_3)f_5(y_2,y_3).$$

Let us consider the system (3.2)–(3.5) in the class of functions

$$(3.6)\qquad \text{functions} f_1(\bullet,0), f_1(0,\bullet) \text{arecontinuous.}$$

Under the above assumptions in view of Theoremm 4 from [7] the system (3.2)–(3.3) has no solutions. Therefore one has

THEOREM 3. *In the class of functions* (3.6) *the system* (3.2)–(3.5) *does not have any solutions.*

THEOREM 3′. *There are no subsemigroups of the group* $\mathbf{L}_1^6$ *of the form* $\mathbf{P}_{1456}$ *in the set of functions satisfying the*, *assumption* (3.6).

Let us denote

$$(3.7)\qquad \mathbf{P}^{\circ}_{1456} = \{\langle F_1(x_3),0,x_3,F_4(x_3),F_5(x_3),F_6(x_3)\rangle : x_3\in\mathbb{R}\},$$

where $F_i:\mathbb{R}\to\mathbb{R}$ for $i=4,5,6$ and $F_1:\mathbb{R}\to\mathbb{R}_0$.

The set $\mathbf{P}^{\circ}_{1456}$ is closed with respect to the operation (1.1) if and only if the functions F_1, F_4, F_5 and F_6 verify the following system of functional equations (one can obtain it from (3.2)–(3.5) putting $x_2=y_2=0$ and $F_i=f_i$ for $i=1,4,5,6$).

$$(3.8)\quad F_1(x_3F_1(y_3)+y_3F_1^3(x_3)) = F_1(x_3)F_1(y_3),$$

$$(3.9)\quad F_4(x_3F_1(y_3)+y_3F_1^3(x_3)) = F_1(y_3)F_4(x_3)+F_4(y_3)F_1^4(x_3),$$

$$(3.10)\quad F_5(x_3F_1(y_3)+y_3F_1^3(x_3)) = F_5(x_3)F_1(y_3)+F_5(y_3)F_1^5(x_3)+ \\ +10x_3y_3F_1^2(x_3),$$

$$(3.11)\quad F_6(x_3F_1(y_3)+y_3F_1^3(x_3)) = F_1(y_3)F_6(x_3)+F_6(y_3)F_1^6(x_3)+ \\ +15y_3F_1^2(x_3)F_4(x_3)+20x_3F_1^3(x_3)F_4(y_3).$$

Consider the system (3.8)–(3.11) in the class of functions verifying the assumptions

(3.12) function F_1 is continuous,

(3.13) function F_4 is continuous at at least one point.

By (3.12) and Theorem 1 in [5] it follows that a unique solution of (3.8) is a constant function

$$F_1(x_3) = 1. \tag{3.14}$$

In view of (3.14) the equation (3.9) takes the form

$$F_4(x_3 + y_3) = F_4(x_3) + F_4(y_3). \tag{3.15}$$

The general solution of (3.15) in the class of functions satisfying (3.13) is a family of functions of the form

$$F_4(x_3) = cy_3, \tag{3.16}$$

where c is an arbitrary real number.

In view of (3.14), the equation (3.10) assumes the form

$$F_5(x_3 + y_3) = F_5(x_3) + F_5(y_3) + 10x_3y_3. \tag{3.17}$$

Let us consider

$$F(x + y) = F(x) + F(y) + bxy, \tag{3.18}$$

where $F : \mathbb{R} \to \mathbb{R}$ is an unknown function, and $b \in \mathbb{R}$.

It is easily checked that if F satisfies (3.18) then the function

$$\phi(x) = F(x) - \frac{b}{2}x^2$$

is additive. Each function of the form

$$F(x) = \phi(x) + \frac{b}{2}x^2, \tag{3.19}$$

where ϕ is an arbitrary additive function verifies (3.18).

We have shown

LEMMA 2. *The general solution of the equation (3.18) is a family of functions of the form (3.19) where ϕ is an additive function.*

In the general solution of the equation (3.17) is a family of functions of the form

$$F_5(x_3) = \phi(x_3) + 5x_3^2, \tag{3.20}$$

where ϕ is an additive function.

Substituting (3.14), (3.16) in (3.11) we get

$$F_6(x_3 + y_3) = F_6(x_3) + F_6(y_3) + 35cx_3y_3. \tag{3.21}$$

It follows by Lemma 2 that the general solution of (3.21) is a family of functions of the form

$$F_6(x) = \psi(x) + 17,5x^2, \tag{3.22}$$

where ψ is an additive function.

We have proved

THEOREM 4. *The general solution of the system* (2.8)–(2.11) *in the function class* (3.12) *and* (3.13) *is a family of quadraples of functions of the form* (3.14), (3.16), (3.20), (3.22), *where c is an arbitrary real constant, and ϕ, ψ are arbitrary additive functions.*

THEOREM 4′. *Unique sets of the form* (3.7) *closed with respect to* (1.1) *in the class of functions satisfying* (3.12) *and* (3.13) *are the sets*

$$\{\langle 1, 0, x_3, cx_3, \phi(x_3) + 5x_3^2, \psi(x_3) + 17.5x_3^2\rangle : x_3 \in \mathbb{R}\},$$

where c is a constant, and ϕ, ψ are additive functions.

4. Denote

$$\mathbf{P}_{1356} = \{\langle f_1(x_2,x_4), x_2, f_3(x_2,x_4), x_4, f_5(x_2,x_4), f_6(x_2,x_4)\rangle : x_2, x_4 \in \mathbb{R}\}. \tag{4.1}$$

where $f_l : \mathbb{R}^2 \to \mathbb{R}$ for $l = 3,5,6$, $f_1 : \mathbb{R}^2 \to \mathbb{R}_0$.

Similarly as in case of the set P_{456} one proves that the set $\mathbf{P}_{1356}$ are closed with respect to (1.1) if and only if the functions f_1, f_3, f_5 and f_6 verify the system of functional equations:

$$\begin{aligned} &f_1(x_2f_1(y_2,y_4) + y_2f_1^2(x_2,x_4), x_4f_1(y_2,y_4) + y_4f_1^4(x_2,x_4)+\\ &+ 4y_2f_1(x_2,x_4)f_3(x_2,x_4) + 6f_3(y_2,y_4)f_1^2(x_2,x_4)x_2 + 3y_2x_2^2) =\\ &= f_1(x_2,x_4)f_1(y_2,y_4), \end{aligned} \tag{4.2}$$

$$\begin{aligned} &f_3(x_2f_1(y_2,y_4) + y_2f_1^2(x_2,x_4), x_4f_1(y_2,y_4) + y_4f_1^4(x_2,x_4)+\\ &+ 4y_2f_1(x_2,x_4)f_3(x_2,x_4) + 6f_3(y_2,y_4)f_1^2(x_2,x_4)x_2 + 3y_2x_2^2) =\\ &= f_1(y_2,y_4)f_3(x_2,x_4) + 3y_2x_2f_1(x_2,x_4) + f_3(y_2,y_4)f_1^3(x_2,x_4), \end{aligned} \tag{4.3}$$

$$\begin{aligned} &f_5(x_2f_1(y_2,y_4) + y_2f_1^2(x_2,x_4), x_4f_1(y_2,y_4) + y_4f_1^4(x_2,x_4)+\\ &+ 4y_2f_1(x_2,x_4)f_3(x_2,x_4) + 6f_3(y_2,y_4)f_1^2(x_2,x_4)x_2 + 3y_2x_2^2) =\\ &+ f_1(y_2,y_4)f_5(x_2,x_4) + f_1^5(x_2,x_4)f_5(y_2,y_4) + 10f_1^3(x_2,x_4)x_2y_4+\\ &+ 15f_1(x_2,x_4)x_2^2f_3(y_2,y_4) + 10f_1^2(x_2,x_4)f_3(x_2,x_4)f_3(y_2,y_4)+\\ &+ 10x_2y_2f_3(x_2,x_4) + 5y_2x_4f_1(x_2,x_4), \end{aligned} \tag{4.4}$$

$$\begin{aligned} &f_6(x_2f_1(y_2,y_4) + y_2f_1^2(x_2,x_4), x_4f_1(y_2,y_4) + y_4f_1^4(x_2,x_4)+\\ &+ 4y_2f_1(x_2,x_4)f_3(x_2,x_4) + 6f_3(y_2,y_4)f_1^2(x_2,x_4)x_2 + 3y_2x_2^2) =\\ &= f_1(y_2,y_4)f_6(x_2,x_4) + f_6(y_2,y_4)f_1^6(x_2,x_4)+\\ &+ 6y_2f_1(x_2,x_4)f_5(x_2,x_4) + 15y_2x_2x_4 + 10y_2f_3^2(x_2,x_4)+\\ &+ 15f_1^2(x_2,x_4)f_3(y_2,y_4)x_4 + 60x_2f_3(y_2,y_4)f_1(x_2,x_4)f_3(x_2,x_4)+\\ &+ 20f_1^3(x_2,x_4)f_3(x_2,x_4)y_4 + 45x_2^2f_4(y_2,y_4)f_1^2(x_2,x_4)+\\ &+ 15x_2f_1^4(x_2,x_4)x_2f_5(y_2,y_4). \end{aligned} \tag{4.5}$$

Let us consider (4.2)–(4.5) under the assumptions

(4.6) functions $f_1(0,\cdot), f_1(\cdot,0)$ are continuous,

(4.7) function $f_3(0,\cdot)$ is continuous at a point.

In view of Theorem 3 in [9] the system (4.2)–(4.4) under the assunmptiomns (4.6)–(4.7) has no solutions. Hence the following holds true.

THEOREM 5. *The system* (4.2)–(4.5) *in the class satisfying* (4.6) *and* (4.7) *has no solutions.*

THEOREM 5′. *The group* $\mathbf{L}_6^1$ *has no subsemigroups of the form* (4.1) *in the class of functions satisfying* (4.6) *and* (4.7).

Now we are concerned with the determination of subsemigroups of the group $\mathbf{L}_6^1$ of the form

$$\mathbf{P}_{1356}^0 = \{\langle F_1(x_4), 0, F_3(x_4), x_4, F_5(x_4), F_6(x_4)\rangle : x_4 \in \mathbb{R}\}. \tag{4.8}$$

where $F_i : \mathbb{R} \to \mathbb{R}$ for $i = 3,5,6$, $f_1 : \mathbb{R} \to \mathbb{R}_0$.

The set $\mathbf{P}_{1356}^0$ is closed with respect to (1.1) if and only if the functions F_1, F_3, F_5, F_6 satisfy the following system of functional equations (with results from (4.2)–(4.5) by putting $x_2 = y_2 = 0$, $x_4 = x$, $y_4 = y$, and $F_i(x) = f_i(0,x)$ for $i = 1,3,5,6$).

$$F_1(xF_1(y) + yF_1^4(x)) = F_1(x)F_1(y), \tag{4.9}$$

$$F_3(xF_1(y) + yF_1^4(x)) = F_1(y)F_3(x) + F_1^3(x)F_3(y), \tag{4.10}$$

$$F_5(xF_1(y) + yF_1^4(x)) = F_1(y)F_5(x) + F_1^5(x)F_5(y) + 10F_1^2(x)F_3(x)F_3(y), \tag{4.11}$$

$$F_6(xF_1(y) + yF_1^4(x)) = F_1(y)F_6(x) + F_1^6(x)F_6(y) + 15F_1^2(x)F_3(y)x + 20F_1^3(x)F_3(x)y, \tag{4.12}$$

Let us consider (4.9)–(4.12) in the class of functions satisfying

(4.13) F_1 is continuous,

(4.14) F_3 is continuous at at least one point.

By (4.13) and Theorem 1 in [5] one has that a unique solution of the equation (4.9) is a constant function

$$F_1(x) = 1. \tag{4.15}$$

Then (4.10) takes the form

$$F_3(x + y) = F_3(x) + F_3(y). \tag{4.16}$$

A solution of the equation (4.16) in the class (4.14) is a family of functions of the form

$$F_3(x) = dx, \tag{4.17}$$

where d is any real.

Substituting (4.15) and (4.17) in (4.11) we have

$$F_5(x+y) = F_5(x) + F_5(y) + 10d^2xy. \tag{4.18}$$

In view of Lemma 2 the general solution of (4.18) is a family of functions of the form

$$F_5(x) = a_1(x) + 5d^2x^2, \tag{4.19}$$

where a_1 is an additive function.

If we substitute (4.17) in (4.12) we get

$$F_6(x+y) = F_6(x) + F_6(y) + 35dxy. \tag{4.20}$$

By Lemma 2 the general solution of (4.20) is a family of functions of the form

$$F_6(x) = a_2(x) + 17{,}5dx^2, \tag{4.21}$$

where a_2 is an additive function.

Observe that under the assumptions (4.13) and (4.14) we solved the system (4.9)–(4.12) similarly as (3.8)–(3.11) under the assumptions (3.12) and (3.13).

We then have

THEOREM 6. *The general solution of the system* (4.9)–(4.12) *in the class of functions satisfying* (4.13), (4.14) *is a family of functional quadruples* (4.15), (4.17), (4.19) *and* (4.21), *where* $d \in \mathbb{R}$ *and* a_1, a_2 *are additive functions.*

THEOREM 6′. *Unique sets of the form* (4.8) *closed with respect to the operation* (1.1) *in the class of functions satisfying* (4.13) *and* (4.14) *are the sets*

$$\{\langle 1, 0, dx, x, a_1(x) + 5d^2x^2, a_2(x) + 17{,}5dx^2\rangle : x \in \mathbb{R}\},$$

where d is a constant, and a_1, a_2 are additive functions.

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Received January 12, 1995.