DEMONSTRATIO MATHEMATICA
Vol. XXIX No 3 1996

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A NOTE ON TREES COLOURING

In this paper, we show that the number of k-colouring S of a tree with n vertices is equal to $S(n-1,k-1)$ for all $n,k \geq 2$, where $S(\cdot,\cdot)$ denotes the Stirling numbers of the second kind.

Let $T=(V,E)$ be a tree (e.g., see [1]), with V ($|V|=n$) the set of vertices and E the set of edges. A *k-colouring* of T is a partition of V into k classes, such that two vertices belonging to the same class are not adjacent. We denote by $C(T,n,k)$ the number of k-colourings of T, where $n,k \geq 2$. The main result of this note is the following

THEOREM. $C(T,n,k)=S(n-1,k-1)$, *for all* $n,k \geq 2$.

Proof. Clearly, $C(T,n,n)=C(T,n,2)=1$ for $n \geq 2$, and $C(T,n,k)=0$ for $k>n$. We shall prove the theorem by induction on n. Obviously, for $n=2$, the theorem is true. So, suppose that it is true for any tree with at most $n-1$ vertices. Thus, we have

$$(1) \qquad C(T,n,k)=S(n-2,k-2)+(k-1)\cdot S(n-2,k-1).$$

Indeed, T contains at least a vertex $v \in V$ of degree one, so that $T-v$, obtained from T by deleting v, is a tree with $n-1$ vertices. Let A be the set of k-colourings of T for which v is single in a class of the partition. Thus, $|A|=S(n-2,k-2)$, that is, the number of $(k-1)$-colourings of $T-v$. On the other hand, let B be the set of k-colourings of T for which v appears in a class together with other vertices of T. Obviously, the set of $C(T,n,k)$ colourings of T can be written as the union of A and B. Since v is adjacent to a single vertex of T, it follows that v can be added to exactly $k-1$ classes of a k-colouring of a tree with $n-1$ vertices. Thus, B contains $(k-1)\cdot S(n-2,k-1)$ colourings, by induction hypothesis.

1980 *Mathematics Subject Classification*: 05C40.

From [2], we have

$$(2) \qquad \begin{cases} S(p+1,m) = S(p,m-1) + m \cdot S(p,m), & \text{and} \\ S(p,1) = S(p,p) = 1. \end{cases}$$

Hence, from (1) and (2), we obtain $C(T,n,k) = S(n-1,k-1)$, the theorem being proved.

References

[1] C. Berge, *Graphes et Hypergraphes*. Dunod, Paris, 1970.
[2] I. Tomescu, *Introduction to Combinatorics*, Collet's (Publishers) Ltd., London and Wellingborough, 1975.

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Received January 9, 1995.