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ON GROUPS OF ORDER p^n WITH AUTOMORPHISMS OF ORDER p^{n-2}

Let p be a prime and let G be a finite p-group. If φ is a $p-$automorphism of G then, clearly, its order divides $\frac{|G|}{p}$. In this paper we study the following question: what is the structure of G, when the order of φ is possibly large. In [3] V.G. Berkovich classified all finite $p-$groups G with automorphisms of order $\frac{|G|}{p}$. Here we present an alternative approach to this classification and give a complete description of $p-$groups possessing automorphisms of order $\frac{|G|}{p^2}$.

The problem was communicated to the first author by J.G. Berkovich in a personal correspondence. After writing the paper it appeared however that the results of [3] and the results of the paper were obtained in [8] in a slightly weaker form and in a completely different way.

In the first section we note some basic facts and give the alternative proof of the main result of [3] (Theorem 1). The essential part of the proof is that for $p = 2$. In the further considerations the case $p > 2$ needs also a different approach than the case $p = 2$, so we study them separately in sections two and three respectively.

Throughout the paper terminology and notation will be standard and follow [1], [2], [7].

1. Preliminary results

LEMMA 1. *Let H be a normal subgroup of a finite $p-$group G. Let $\varphi \in AutG$ and $H^\varphi = H$. If φ induces the identity on G/H and for every $h \in H$*

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$$h^{\varphi^{p-1}} \dots h^{\varphi} h = 1$$

then φ^p is the identity on G.

Proof. In fact, if $g \in G$ then $g^{\varphi} = gh$ for a suitable $h \in H$. Hence $g^{\varphi^p} = gh^{\varphi^{p-1}} \dots h^{\varphi} h = g$.

LEMMA 2. *Let H be a p–group of order $\leq p^{p-1}$ and of exponent p. If φ is a p–automorphism of H then for every $h \in H$*

$$h^{\varphi^{p-1}} \dots h^{\varphi} h = 1.$$

Proof. Let $K = \langle \varphi, H \rangle$ be a subgroup of the holomorph of H. The nilpotency class of K is less than p, so K is regular and then for every $h \in H$ we have $(\varphi h)^p = \varphi^p h^p c^p$, where $c \in K'$. But $K' \subseteq H$. Therefore $(\varphi h)^p = \varphi^p$. On the other hand $(\varphi h)^p = \varphi^p h^{\varphi^{p-1}} \dots h^{\varphi} h$. This ends the proof.

Let G be a finite p–group and let H be its subgroup. We say that H is p–characteristic if there exists a Sylow p–group P of $AutG$ such that $H^{\varphi} = H$ for all $\varphi \in P$. It is clear that p–characteristic subgroups are normal in G and all characteristic subgroups are p–characteristic.

LEMMA 3. *If G contains a p–characteristic subgroup H of order p^k, $k < p$, and of exponent p then the exponent of a Sylow p–subgroup of $AutG$ is not greater than $\frac{|G|}{p^k}$.*

Proof. Let P be a Sylow p–group of $AutG$ fixing H and let A be a subgroup of P consisting of all automorphisms inducing the identity on G/H. The natural epimorphism of G onto G/H induces a homomorphism of P into $Aut(G/H)$. It is easily seen that A is the kernel of this homomorphism. By Lemmas 1 and 2 the exponent of A equals p. Hence $\exp(P) \leq p \cdot \exp(P/A) < p \cdot |G/H|$.

LEMMA 4. ([6], III.13.10) *Let G be a noncyclic p–group. If every abelian characteristic subgroup of G is cyclic then one of the following cases holds:*

a) If $p > 2$ then G is a central product of an extraspecial p–group B of exponent p and a cyclic group $Z(G)$ with $Z(B) = \Omega_1(Z(G))$.

b) If $p = 2$ then

(1) G is extraspecial or

(2) G is of maximal class (i.e. dihedral, generalized quaternion or semidihedral), or

(3) G is a central product of two groups B and Q with $Z(B) = \Omega_1(Z(Q))$, where B is extraspecial and Q is either cyclic or of maximal class.

Let, as in the proof of Lemma 3, P be a Sylow p–subgroup of $AutG$ and let H be a subgroup of G such that $H^{\varphi} = H$ for all $\varphi \in P$. It is clear that all groups of the sequence $H = H_1 > H_2 > \dots > H_k > \dots$, where

$H_{i+1} = [P, H_i]$ are p–characteristic as well as all groups of its arbitrary refinement. Therefore

LEMMA 5. *If G contains a p–characteristic subgroup of order $\geq p^2$ and of exponent p then it contains also such a p–characteristic subgroup of order p^2.*

LEMMA 6. *If G is a noncyclic p–group, $p > 2$, then G contains a p–characteristic elementary abelian subgroup of order p^2.*

Proof. By Lemma 5, we need only to show that G contains a p–characteristic subgroup of order $\geq p^2$ and of exponent p. If the centre of $\gamma_2(G)$ is not cyclic then such a subgroup obviously exists. If $Z(\gamma_2(G))$ is cyclic then by ([6], III.7.8) so is $\gamma_2(G)$ and by ([6], III.10.2) G is regular. But in regular p–groups the subgroup $\Omega_1(G)$ has exponent p. This ends the proof.

In many places we shall use also the following obvious fact.

LEMMA 7. *Let φ be a p–automorphism of a group G and let M, N be different maximal subgroups which are invariant under the action of φ. If the restrictions of φ to M and N have orders smaller than k then so is the order of φ.*

Now we are ready to give an alternative proof of the main result of [3].

THEOREM 1. *A p–group G of order p^n, $n \geq 1$, has an automorphism of order p^{n-1} if and only if one of the following cases holds:*

a) G is of order $\leq p^2$

b) G is cyclic and $p > 2$;

c) G is a noncyclic group of order 8;

d) G is a dihedral 2–group, $n > 3$;

e) G is a generalized quaternion group, $n > 3$.

Proof. It is easily seen that all groups listed above have automorphisms of desired order. We show that there are no other p–groups satisfying this condition. We shall consider independently the cases p odd and $p = 2$.

Let us assume first that p is odd. For groups of order p^2 the theorem is obviously true. Now assume that $|G| > p^2$ and suppose G is not cyclic. So by Lemma 6 G contains an elementary abelian p–characteristic subgroup of order p^2. By Lemma 3 the exponent of a Sylow p–group of $AutG$ is less then $\frac{|G|}{p}$. A contradiction.

Now let $p = 2$. By induction on $|G|$ we show that if $n \geq 4$ and G has an automorphism of order 2^{n-1} then G is either dihedral or generalized quaternion. Let G be of order 2^4. If G possesses a characteristic elementary abelian subgroup H of order 4 and $\varphi \in AutG$ is a 2–automorphism then the automorphism $\overline{\varphi}$ of G/H induced by φ has order ≤ 2. Since φ^2 acts trivially on H and induces the identity on G/H, by Lemma 1 $\varphi^4 = id_G$. Now assume that G does not contain a characteristic elementary abelian subgroup of order 4 and G is not of maximal class. Then, since extraspecial 2–groups have orders 2^{2r+1}, G is a central product of a nonabelian group of order 8 and a cyclic group of order 4 (Lemma 4). Let $K_i = \{x \in G : o(x) = 2^i\}, i = 1, 2$. It can be easily shown that each set generates G. If $\varphi \in AutG$ has order 8 then in both sets K_i there exist elements x_i such that the sets $\{x_i^{\varphi^k} \mid k = 1, 2, \ldots\}$ have 8 elements. But it is not possible as one of the sets K_i has less than 8 elements.

Let G be of order $2^n, n > 4$, and let H be a 2-characteristic subgroup of G of order 2 contained in $\gamma_2(G)$. If φ is an automorphism of order 2^{n-1} then the automorphism $\overline{\varphi}$ of G/H induced by φ has order 2^{n-2}. By induction G/H is a 2–group of maximal class. Since

$$|G : \gamma_2(G)| = |(G/H) : (\gamma_2(G)/H)| = 4$$

by ([1],Corollary to Th.3.9.) G is of maximal class, too. Since all maximal subgroups of the semidihedral group of order $2^n, n \geq 4$, are characteristic and then by Lemma 7 this group has no automorphism of order 2^{n-1}, G is either dihedral or generalized quaternion.

2. On p–groups of odd order

First we study p–groups G of order $> p^4$, $p > 2$. It is a little surprising that this general case needs less effort than the case $|G| = p^4$.

THEOREM 2. *Let p be an odd prime and let G be a p–group of order $p^n, n > 4$. The following conditions are equivalent:*

a) G has an automorphism of order p^{n-2};

b) G contains a cyclic subgroup of order p^{n-1};

c) G is cyclic of order p^n or G is a direct product of a cyclic group of order p^{n-1} and a group of order p or $G = \langle x, y \mid x^{p^{n-1}} = 1, y^p = 1, y^{-1}xy = x^{p^{n-2}+1}\rangle$.

Proof. The equivalence of the statements b) and c) follows immediately from ([6],I.14.9). We shall show the equivalence of a) and b).

Since for cyclic groups the statements are clearly true we assume that G is not cyclic. Now let G has an automorphism of order p^{n-2}. By Lemma 6 G contains an elementary abelian p–characteristic subgroup H of order p^2. Since $|G/H| > p^2$ and G/H has an automorphism of order $\frac{|G|}{p}$ by Theorem 1 G/H must be cyclic. Let $x \in G$ be such that $G = \langle x, H\rangle$. The automorphism of H induced by the conjugation by x is of course of order p, therefore $x^p \in Z(G)$ and then we can assume that the subgroup H was chosen such that $\Omega_1(\langle x\rangle) \subseteq H$. This means that $\langle x\rangle$ is a maximal subgroup of G.

If G contains a maximal cyclic subgroup and is not cyclic itself then by c) it is generated by two elements x and y with $\langle x\rangle$ maximal in G and

y of order p. As it is easily seen the correspondence $x \longrightarrow x^{p+1}$, y $\longrightarrow y$ determines the automorphism of G of order p^{n-2}.

Now we consider groups of order p^4. In our study we use the list of this groups from [4] (pages 145-146) with the numbering as given, replacing the letters P, Q, R, S, E respecively by x, y, z, t and 1.

THEOREM 3. *If $p > 2$ then a $p-$group G of order p^4 has an automorphism of order p^2 if and only if one of the following cases holds:*

a) G *is abelian of the type* (4) *or* $(3,1)$ *or* $(2,2)$;

b) $G = \langle x, y \mid x^{p^3} = 1, y^p = 1, y^{-1}xy = x^{1+p^2} \rangle$;

c) $G = \langle x, y \mid x^{p^2} = 1, y^{p^2} = 1, y^{-1}xy = x^{1+p} \rangle$;

d) G *is abelian of the type* $(2,1,1)$ *or* $(1,1,1,1)$ *and* $p = 3$.

e) $G = \langle x, y, z \mid x^9 = y^3 = z^3 = 1, z^{-1}xz = x^4, y^{-1}xy = x, z^{-1}yz = y \rangle$;

f) $G = \langle x, y, z \mid x^9 = y^3 = 1, z^{-1}xz = xy, y^{-1}xy = x^4, z^{-1}yz = y, z^3 = x^{-3} \rangle$;

g) $G = \langle x, y, z, t \mid x^3 = y^3 = z^3 = t^3 = 1, t^{-1}zt = zx, t^{-1}yt = y, t^{-1}xt = x, z^{-1}xz = x, y^{-1}xy = x, z^{-1}yz = y \rangle$;

h) $G = \langle x, y, z \mid x^9 = y^3 = z^3 = 1, z^{-1}xz = xy, y^{-1}xy = x, z^{-1}yz = x^{-3}y \rangle$.

Proof. By Lemma 3, if G has an automorphism of order p^2 then G does not contain a $p-$characteristic subgroup H of order p^3 and of exponent p. In particular $|\Omega_1(G)| \leq p^2$ and by ([6],11.6) G must be a metacyclic group and only groups (i), (ii), (iii), (vi), (viii) should be considered. First three groups are abelian and one can easily find for them automorphisms of order p^2. The correspondence

$$x \longrightarrow x$$
$$y \longrightarrow xy$$

determines a $p-$automorphism of the group (viii) and symilarly the correspondence

$$x \longrightarrow x^{p+1}$$
$$y \longrightarrow y$$

determines a $p-$automorphism of the groups (vi). Remark that for these five groups the above is also true when $p = 3$.

For other groups of order 3^4 the situation is slighty more complicated. We prove that the only groups that do not have automorphisms of order 3^2 are the groups (vii), (x), (xi) and (xii).

Let G be of type (vii). Every $p-$automorphism φ of G fixes $Z(G)$ and $\Omega_1(G)$. The restriction of φ to both subgroups have orders 3. But $Z(G)\Omega_1(G) = G$, hence φ is of order 3.

It is easily seen that for a group G of the type (x) every $p-$automorphism φ of G must act trivially on its Frattini subgroup. Actually, in this case $\Phi(G) = G' \times G^3$. Hence for every $h \in \Omega_1(G)$ $h^{\varphi^2}h^{\varphi}h = 1$ and then by Lemma 1 3$-$automorphisms of G have order 3.

Assume now that G is of the type (xi) or (xii) that is $G = \langle x, y, z \mid x^{3^2} = 1, y^3 = 1, y^{-1}xy = x^4, z^{-1}xz = xy, z^{-1}yz = y, z^3 = x^{3\alpha}\rangle$, where $\alpha = 0$ or $\alpha = 1$. Let φ be a 3$-$automorphism of G. Notice that $Z(G) = \langle x^3\rangle$ and $G' = \langle x^3, y\rangle$. Hence $\varphi(y) = yx^{3m}$ and $\varphi(x) = xx^{3i}y^jz^k$. Since $x^3 = \varphi(x^3) = \varphi(x)^3 = (xx^{3i}y^jz^k)^3 = x^{3+3k}z^{3k}$ in both cases we have $k \equiv 0(mod3)$ and then the subgroup $\langle x, y\rangle$ is invariant under the action of φ. On the other hand the subgroup $\langle x^3, y, z\rangle$ is the unique maximal abelian subgroup of G so it is also invariant. This means that φ induces the identity on G/G'. By Lemma 1 φ must have order 3.

Now we show that other groups of order 3^4 have automorphisms of order 3^2. For groups of the types (vi) and (viii) the solution is analogous as in the case $p > 3$. For groups of the types (xiii),(ix),(xiv) and (xv) we define automorphisms of order 3^2 symilarly as in the previous section by indicating images of generators. We have then

group	x	y	z	t
(ix)	xz	yx^3	zy	
(xiii)	xz	y	z	
(xiv)	x	yx	zt	ty
(xv)	x	y	zx	

Table 1.

3. On 2$-$groups

Our study of 2$-$groups of small order (i.e. of order $< 2^6$) rests upon explicit computations symilarly as for $p-$groups of small order for $p > 2$. In the end of the section we collect all the needed information about groups of order 2^5 leaving computations and the case of smaller 2$-$groups to the reader. We begin the study of the general case with some lemmas.

LEMMA 8. *If a group G of order 2^n, $n > 5$, has an automorphism of order 2^{n-2} then $\gamma_2(G)$ is cyclic.*

Proof. Suppose by way of contradiction that $\gamma_2(G)$ is not cyclic. Then by ([6],III.7.8.) $Z(\gamma_2(G))$ is also not cyclic and by Lemma 5 G contains a 2$-$characteristic elementary abelian subgroup A of order 4, which lies in $\gamma_2(G)$. If $\varphi \in AutG$ is of order 2^{n-2} then the automorphism $\overline{\varphi}$ of G/A induced by φ is of order 2^{n-3}. Hence by Theorem 1 G/A is of maximal class and

$$|G:\gamma_2(G)| = |(G/A):(\gamma_2(G)/A)| = |(G/A)/\gamma_2(G/A)| = 4.$$

This means by ([1],Corollary of Th.3.9.) that G is of maximal class and then $\gamma_2(G)$ is cyclic. This is a contradiction.

LEMMA 9. *Let $|G| = 2^n, n > 5$, and let A be a 2−characteristic elementary abelian subgroup of order 4 such that G/A is neither dihedral nor generalized quaternion. Then for every 2−automorphism φ of G $o(\varphi) < 2^{n-2}$.*

Proof. By Theorem 1 every automorphism of G/A is of order smaller than 2^{n-3}. In particular if φ is a 2−automorphism of a group G and $\overline{\varphi}$ is the automorphism of the group $\overline{G} = G/A$ induced by φ then $\overline{\varphi}^{2^{n-4}} = id_{\overline{G}}$. Therefore, as $\varphi^{2^{n-4}}$ restricted to A is the identity on A, for every $g \in G$ there exists $a \in A$ such that $\varphi^{2^{n-4}}(g) = ga$. Hence $\varphi^{2^{n-3}}(g) = g$.

Following Blackburn [1], we say that a group G belongs to the class CF(n-1,n,2) if its class is n-2 and for $2 \leq i \leq n-2$ $|\gamma_i(G):\gamma_{i+1}(G)| = 2$. By D_n, S_n and Q_n we denote respectively the dihedral, semidihedral and quaternion 2−groups of order 2^n. By Z_n we denote the cyclic group of order n.

LEMMA 10. *If a group G of order $2^n, n > 5$, has an automorphism of order 2^{n-2} and G is neither cyclic nor of maximal class then $G \in CF(n-1,n,2)$.*

Proof. First assume that G contains a 2−characteristic elementary abelian subgroup A of order 4. Then by Lemma 9 G/A is either dihedral or generalized quaternion. Since we can choose A such that $A \cap \gamma_2(G) \neq 1$ we get $|G:\gamma_2(G)| = 2^3$ and then $G/\gamma_2(G)$ is elementary abelian or of the type (4,2). In both cases by ([1],1.5.) $|\gamma_2(G):\gamma_3(G)| = 2$ and $|\gamma_i(G):\gamma_{i+1}(G)| = 2$ for $i = 1,2,\ldots,n-2$. Hence $G \in CF(n-1,n,2)$.

Now we show that there are no other groups satisfying the assumptions. Assume by the foregoing that G does not contain a 2−characteristic elementary abelian subgroup of order 4. Since no group of order 2^6 is extraspecial we may assume that if $|G| = 2^6$ then G is a central product of D_3 and one of the following four groups: Z_{16}, D_4, Q_4, S_4, or G is the central product of two copies of D_3 and Z_4 or the central product of D_3,Q_3 and Z_4 (Lemma 4 and [9],2.2).

If G is the central product of D_3 and Z_{16} or a central product of three mentioned groups then $\Omega_2(G)$ and $\Omega_2(Z(G))$ are of exponent 4. Hence a 2−characteristic subgroup A such that $\Omega_2(Z(G)) \subsetneq A \subsetneq \Omega_2(G)$ and $|A:\Omega_2(Z(G))| = 2$ is noncyclic and abelian. If G is one of the reminder groups then the subgroup $H = C_G(\gamma_2(G))$ is characteristic in G and $|G:H| = 2$. The subgroup $Z(H)$ is cyclic of order 8 and of course also characteristic. Now a 2−characteristic subgroup A of G such that $A \supseteq Z(H)$ and $|A : Z(H)| = 2$ is noncyclic and abelian. Hence in all cases we can find a 2−characteristic noncyclic abelian subgroup A such that $\Omega_1(A)$ is elementary abelian of order 4 and clearly 2−characteristic in G. A contradiction. Therefore $|G| > 2^6$.

Assume now that G is either extraspecial or a central product of an extraspecial and a cyclic groups (Lemma 4b). Since the group $\widehat{G} = G/Z(G)$ is abelian and noncyclic, there exists a 2−characteristic noncyclic subgroup $\widehat{H} = H/Z(G)$ of $\widehat{G}$ of order 4. The group $\overline{G} = G/H$ is abelian and $|\overline{G}| = 2^{n-3}, n > 6$. Let $\varphi \in AutG$ be a 2−automorphism from a Sylow p−subgroup of $AutG$ preserving H and let $\overline{\varphi}$ be the 2−automorphism of $\overline{G}$ induced by φ. Hence by Theorem 1 $\overline{\varphi}^{2^{n-5}} = id_{\overline{G}}$. Since $\varphi^{2^{n-3}}$ acts trivially on H for every $g \in G$ there exists $h \in H$ such that $g^{\varphi^{2^{n-5}}} = gh$. But $\exp H \leq 4$, so $g^{\varphi^{2^{n-3}}} = g$.

Now let G be a central product of an extraspecial 2−group A and a 2−group of maximal class B (Lemma 4b). We consider the group $\overline{G} = G/Z(G)$ which is a direct product of the elementary abelian group $\overline{A} = A/Z(G)$ and the dihedral group $\overline{B} = B/Z(G)$. Since $|\overline{A}| > 2$ there exists a 2−characteristic noncyclic subgroup $\overline{H}$ of order 4 in $Z(\overline{G}) = \langle \overline{A}, \gamma_{n-2}(\overline{B})\rangle$ such that $\gamma_{n-2}(\overline{B}) \subseteq \overline{H}$. Then by Lemma 9 $\overline{G}$ does not have a 2−automorphism of order 2^{n-3}. Hence for every 2−automorphism of G $o(\varphi) < 2^{n-2}$.

THEOREM 4. *A group G of order $2^n, n > 5$, has an automorphism of order 2^{n-2} if and only if one of the following cases holds:*

a) G is cyclic;

b) G is of maximal class;

c) $G = A \times B$ where A is dihedral or generalized quaternion and B is of order 2;

d) G is a central product of a dihedral 2−group and a cyclic group of order 4;

e) $G = \langle s, s_1 \mid s^4 = 1, s_1^s = s_1^{-1}, s_1^{2^{n-2}} = 1\rangle$ or $G = \langle s, s_1 \mid s^4 = 1, s_1^s = s_1^{-1+2^{n-3}}, s_1^{2^{n-2}} = 1\rangle$ or $G = \langle s, s_1 \mid s^4 = s_1^{2^{n-3}}, s_1^s = s_1^{-1}, s_1^{2^{n-2}} = 1\rangle$.

Proof. Clearly by the previous results all groups listed in a)-d) have automorphisms of desired orders. It can be also easily checked that the correspondence

$$s \longrightarrow ss_1$$
$$s_1 \longrightarrow s_1$$

determines the p−automorphisms of the groups e).

Now we show that there are no other 2−groups satisfying this condition. By Lemma 8 we have to consider only groups of almost maximal class with cyclic derived subgroups. All these groups are described in Theorems 5.1, 5.2 and 5.3 of [7]. So if we assume that G is not among ones listed in the theorem we have:

a) $G = A \times B$ where A is semidihedral and B is of order 2;

b) G is one of six groups G of order 2^n and class $n-2$ with $\gamma_1(G)/\gamma_2(G)$ cyclic;

c) $G = \langle s, s_1 \mid s^4 = 1, s_1^s = s_1^{-1+2^{n-4}}, s_1^{2^{n-2}} = 1\rangle$.

In all these groups we can find two characteristic maximal subgroups M and N which are neither dihedral nor generalized quaternion which by Lemma 7 and Theorem 1 will mean that non of these groups have an automorphism of order 2^{n-2}. With the notation of [7] we have for these cases respectively:

a) $M = \langle s, t, s_1^2\rangle$, $N = \langle s_1, t\rangle$;

b) $M = \gamma_1(G)$, $N = \langle s, \Phi(G)\rangle$;

c) $M = \langle s, \Phi(G)\rangle$, $N = \langle s_1, \Phi(G)\rangle$.

In the end we classify all 2–groups of order 2^5 with automorphisms of order 8. There are 51 isomorphism types of groups of order 2^5. The numbering scheme of [5] is used to reference these groups. We obtain the result by calculating automorphisms with the use of the following two easy lemmas which we give without proofs.

LEMMA 11. *Let H be an abelian 2–characteristic subgroup of a 2–group G such that H is of the type $(2,2)$, G/H is abelian of the type $(4,2)$ and $H \subseteq Z(G)$. Then G does not have an automorphism of order 8.*

LEMMA 12. *Let H be an elementary abelian 2–characteristic subgroup of a 2–group G and let φ be a 2–automorphism of G. If the restriction of φ to H and the automorphism of G/H induced by φ have order ≤ 2 then $o(\varphi) \leq 4$.*

Table 1 contains all nonabelian groups of order 2^5, besides three 2–groups of maximal class, having automorphisms of order 8. We define automorphisms by indicating images of generators. In the first row there are given generators of the groups. In first column — the number of a group.

The groups 49-51 are respectively the dihedral, semidihedral and generalized quaternion groups.

Table 3 contains the list of all groups without automorphisms of order 8. In the first column we have the number of a group G. In the second and the third there are 2–characteristic subgroups M and N generating G with the following property: if φ is a 2–automorphism of G fixing M and N then the restrictions of φ to both subgroups have order smaller than 8. The fourth column contains the subgroup H such as in Lemma 11 or 12.

group	α_1	α_2	α_3	α_4	α_5	β	β_1	β_2	β_3
8		$\alpha_2\beta_2$	$\alpha_3\alpha_2$					$\beta_2\beta_3$	$\beta_3\alpha_2^2$
9		$\alpha_2\alpha_3$	$\alpha_3\beta_2$					$\beta_2\beta_3$	$\beta_3\alpha_2^2$
11		$\alpha_2\beta_3$	$\alpha_3\alpha_2$				β_1		$\beta_3\alpha_3^2$
23, 25			α_3	$\alpha_4\alpha_3$				β_2	
26	$\alpha_1\alpha_3$		α_3			β			
29, 30, 32			α_3	$\alpha_4\alpha_3$					
34			$\alpha_3\alpha_4$	α_4^{-1}	$\alpha_5\alpha_3$				
35, 40			α_3^{-1}	$\alpha_4\alpha_3$	$\alpha_5\alpha_4$				
42		α_3	α_5	α_2^3	α_4				
43		α_3	α_2	$\alpha_3\alpha_5\alpha_2$	$\alpha_2\alpha_4$				
47, 48		α_2		$\alpha_4\alpha_5^2$	$\alpha_4\alpha_5^{-1}$				

Table 2. Groups of order 2^5 with automorphism of order 8

group	M	N	H	
10	$Z(G)$	$\langle\alpha_2, \alpha_3, \Omega_1(Z(G))\rangle$		Lemma 7
12			$Z(G)$	Lemma 12
13			$\Omega_1(Z(G))$	Lemma 11
14	$\langle\alpha_2, Z(G)\rangle$	$\langle\alpha_2, \alpha_3, \Omega_1(Z(G))\rangle$		Lemma 7
15, 16			$\Omega_1(Z(G))$	Lemma 12
17	$Z(G)$	$\Omega_1(G)$		Lemma 7
18			$\langle\beta_1, \alpha_3^2\rangle$	Lemma 11
19, 20, 21			$\Omega_1(Z(G))$	Lemma 11
22			$\langle\alpha_2, \alpha_3^8\rangle$	Lemma 12
24, 27, 28	$\langle\alpha_3, Z(G)\rangle$	$\langle\alpha_4, \alpha_3^2, Z(G)\rangle$		Lemma 7
31	$\langle\alpha_3, \beta\rangle$	$\langle\alpha_4, \alpha_3^2, \beta\rangle$		Lemma 7
33			$\langle\alpha_1, \alpha_2\rangle$	Lemma 12
36, 37			$\langle\alpha_3^2, \alpha_2\rangle$	Lemma 12
38			$\langle\alpha_4^2, \alpha_2\rangle$	Lemma 12
41			$\langle\alpha_3^2, \alpha_4^2\rangle$	Lemma 12
39, 44, 45	$\langle\alpha_3, \alpha_4\rangle$	$\langle\alpha_3, \alpha_5, \alpha_4^2\rangle$		Lemma 7
46			$\langle\alpha_1, \alpha_2\rangle$	Lemma 12

Table 3. Groups of order 2^5 without automorphisms of order 8.

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