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SOME FIXED POINT THEOREMS IN TOPOLOGICAL VECTOR SPACES

1. A fixed point theorem for nonself mappings

Let A be a subset of a sequentially complete Hausdorff locally convex topological vector space E (over the field $\Re$) with calibration Γ. By the terminology of R.T. Moore [6], a calibration Γ for E means a collection of continuous seminorms p on E which induce the topology of E. Let f, g be nonself mappings from A into E. Let a_p, b_p, c_p, d_p and e_p be nonnegative real numbers such that $a_p + b_p + c_p + d_p + e_p < 1$ and for any x, y in A, and $p \in \Gamma$

$$(1)\quad p(f(x) - g(y)) \leq a_p p(x-y) + b_p p(x - f(x)) + c_p p(y - g(y)) + d_p p(x - g(y)) + e_p p(y - f(x)).$$

Włodarczyk [9] proved that f has a unique fixed point if $f = g$. In this section, we prove that f, g have a unique common fixed point if $b_p = c_p$ and $d_p = e_p$. When $f = g$, because of $p(x-y) = p(y-x)$, one can, without loss of generality, assume $b_p = c_p$ and $d_p = e_p$. So our result generalizes the result of Włodarczyk [9]. Since our Theorem includes Theorem 3.3 of Włodarczyk [9], it also includes the corresponding theorems in: Hardy and Rogers [2], Goebel, Kirk and Shimi [1], Kannan [4], Nova [7] and Wong [10].

DEFINITION. Let $\Gamma_0 \subset \Gamma$, $\Gamma_0 \neq \{0\}$. A subset A of E is said to be of *type* Γ_0 with respect to $x_0 \in A$, if the inequality $p(y) \leq p(x)$, for some $x \in A - x_0$ and for all $p \in \Gamma_0$ implies that $y \in A - x_0$.

THEOREM 1. *Let E be a sequentially complete Hausdorff locally convex topological vector space with calibration Γ, let A be a subset of E and let $f : A \to E$, $g : A \to E$ be two nonself mappings. Assume A is of type Γ_0 ($\Gamma_0 \subset \Gamma$), with respect to $x_0 \in A, f$ and g satisfy* (1), *such that a_p, b_p, c_p, d_p, e_p are non-negative real-valued functions on $E \times E$ for $p \in \Gamma$. If*

(i) $\gamma \equiv \sup_{x,y \in E}\{a_p(x,y) + b_p(x,y) + c_p(x,y) + 2d_p(x,y)\} < 1$; *for* $p \in \Gamma$.

(ii) $b_p \equiv c_p, d_p \equiv e_p$ *for* $p \in \Gamma$,

(iii) $f(x_0) - x_0 \in \frac{1-a_p-b_p-c_p-2d_p}{1-c_p-d_p}(A - x_0)$, *for all* $p \in \Gamma_0$,

(iv) $(g \circ f)(x_0) - x_0 \in \frac{1-a_p-b_p-c_p-2d_p}{1-c_p-d_p}(A - x_0)$ *for all* $p \in \Gamma_0$, *where* a_p, b_p, c_p *and* d_p *are evaluated at* (x, y).

Then $x_n \to u$, *and* u *is the fixed point of* f *or* g *in* A. *If both* f *and* g *have fixed points, then each of* f, g *has a unique fixed point and these two fixed points coincide.*

Proof. Let the sequence $\{x_n\}$ be defined as follows

$$x_{2n+1} = f(x_{2n}), \quad x_{2n+2} = g(x_{2n+1}), \quad n = 0, 1, 2, \dots$$

We show that $x_n \in A$, $n \in \aleph$. Indeed, since A is of type Γ_0, the set $A - x_0$ is balanced and, since $\frac{1-a_p-b_p-c_p-2d_p}{1-c_p-d_p} < 1$, $p \in \Gamma_0$, then

$$f(x_0) - x_0 \in \frac{1 - a_p - b_p - c_p - 2d_p}{1 - c_p - d_p}(A - x_0) \subset (A - x_0),$$

$$g(x_1) - x_0 \in \frac{1 - a_p - b_p - c_p - 2d_p}{1 - c_p - d_p}(A - x_0) \subset (A - x_0),$$

for all $p \in \Gamma_0$. Consequently, $f(x_0) = x_1 \in A$, i.e., $x_n \in A$ for $n = 0, 1$. Suppose it is true for $n = k$. We show that it is true for $n = k + 1$.

CASE I. For x_{2n+1}, where $n = k + 1$,

$$p(x_{2(k+1)+1} - x_0) = p(x_{2k+3} - x_0) \le \sum_{m=0}^{2k+2} p(x_{m+1} - x_m). \tag{2}$$

If m is even then for all $p \in \Gamma$,

$$\begin{aligned} p(x_{m+1} - x_m) &= p(f(x_m) - g(x_{m-1})) \\ &\le a_p p(x_m - x_{m-1}) + b_p p(x_m - f(x_m)) + c_p p(x_{m-1} - g(x_{m-1})) \\ &\quad + d_p p(x_m - g(x_{m-1})) + e_p p(x_{m-1} - f(x_m)) \\ &= a_p p(x_m - x_{m-1}) + b_p p(x_m - x_{m+1}) + c_p p(x_{m-1} - x_m) \\ &\quad + d_p p(x_m - x_m) + e_p p(x_{m-1} - x_{m+1}) \\ &\le (a_p + c_p + e_p) p(x_{m-1} - x_m) + (b_p + e_p) p(x_m - x_{m+1}). \end{aligned}$$

It implies,

$$p(x_{m+1} - x_m) \le \frac{a_p + c_p + e_p}{1 - b_p - e_p} p(x_m - x_{m-1}).$$

Also,

$$\begin{aligned} &p(x_m - x_{m-1}) = p(f(x_{m-2}) - g(x_{m-1})) \\ &\le a_p p(x_{m-2} - x_{m-1}) + b_p p(x_{m-2} - f(x_{m-2})) \\ &\quad + c_p p(x_{m-1} - g(x_{m-1})) + d_p p(x_{m-2} - g(x_{m-1})) + e_p p(x_{m-1} - f(x_{m-2})) \end{aligned}$$

$$= a_p p(x_{m-2} - x_{m-1}) + b_p p(x_{m-2} - x_{m-1}) + c_p p(x_{m-1} - x_m)$$
$$+ d_p p(x_{m-2} - x_m) + e_p p(x_{m-1} - x_{m-1})$$
$$\leq (a_p + b_p + d_p) p(x_{m-2} - x_{m-1}) + (c_p + d_p) p(x_{m-1} - x_m), \quad \text{for all } p \in \Gamma.$$

It further implies,

$$p(x_m - x_{m-1}) \leq \frac{a_p + b_p + d_p}{1 - c_p - d_p} p(x_{m-1} - x_{m-2}).$$

Using (ii), we get,

$$p(x_{m+1} - x_m) \leq \left(\frac{a_p + b_p + d_p}{1 - c_p - d_p}\right)^2 p(x_{m-1} - x_{m-2})$$

for all $p \in \Gamma$.

So by induction, we obtain,

$$p(x_{m+1} - x_m) \leq \left(\frac{a_p + b_p + d_p}{1 - c_p - d_p}\right)^m p(x_1 - x_0). \tag{3}$$

Similarly, if m is odd,

$$p(x_{m+1} - x_m) \leq \left(\frac{a_p + b_p + d_p}{1 - c_p - d_p}\right)^m p(x_1 - x_0).$$

Therefore,

$$p(x_{2(k+1)+1} - x_0) \leq \sum_{m=0}^{2k+2} p(x_{m+1} - x_m)$$
$$\leq \sum_{m=0}^{2k+2} p(x_{m+1} - x_m)\left(\frac{a_p + b_p + d_p}{1 - c_p - d_p}\right)^m p(x_1 - x_0)$$
$$= \frac{1 - \left(\frac{a_p + b_p + d_p}{1 - c_p - d_p}\right)^{2k+3}}{1 - \left(\frac{a_p + b_p + d_p}{1 - c_p - d_p}\right)} p(x_1 - x_0)$$
$$\leq \frac{1 - c_p - d_p}{1 - a_p - b_p - c_p - 2d_p} p(x_1 - x_0) \quad \text{for all } p \in \Gamma.$$

Since A is of type Γ_0 with respect to x_0, hence

$$x_{2(k+1)+1} - x_0 \in A - x_0 \quad \text{and so} \quad x_{2(k+1)+1} \in A.$$

CASE II. For x_{2n+2}, where $n = k + 1$,

$$P(x_{2(k+1)+2} - x_0) = p(x_{2k+4} - x_0) \leq \sum_{m=0}^{2k+3} p(x_{m+1} - x_m).$$

Using (3), we get,

$$p(x_{2(k+1)+2} - x_0) \leq \sum_{m=0}^{2k+3} \left(\frac{a_p + b_p + d_p}{1 - c_p - d_p}\right)^m p(x_1 - x_0),$$
$$\leq \frac{1 - c_p - d_p}{1 - a_p - b_p - c_p - 2d_p} p(x_1 - x_0),$$

since $a_p + b_p + d_p < 1 - c_p - d_p$.

Since A is of type Γ_0 with respect to x_0, therefore $x_{2(k+1)+2} \in A$. By the induction argument $x_n \in A$, $(\forall) n \in \aleph$.

The inequality (3), implies that $\{x_n\}$ is a Cauchy sequence. Hence it converges to some point u in E. Without loss of generality, we can assume that $x_{n+1} \neq x_n$ for each n, either $x_{2n-1} \neq u$ for infinitely many n or $x_{2n} \neq u$ for infinitely many n. By the symmetry we may assume that $x_{2n} \neq u$ for infinitely many n. Thus there is a subsequence $\{k(n)\}$ of $\{n\}$ such that $x_{2k(n)} \neq u$ for each n.

For any $n \geq 1$ and all $p \in \Gamma$ we have

$$p(u - f(u)) \leq p(u - x_{2k(n)}) + p(x_{2k(n)} - f(u)) \tag{4}$$
$$= p(u - x_{2k(n)}) + p(g(x_{2k(n)-1}) - f(u)).$$

Now, $p(f(u) - g(x_{2k(n)-1})) \leq a_p p(u - x_{2k(n)-1}) + b_p p(u - f(u)) + c_p p(x_{2k(n)-1} - g(x_{2k(n)-1})) + d_p p(u - g(x_{2k(n)-1})) + e_p p(x_{2k(n)-1} - f(u)) = a_p p(x_{2k(n)-1} - u) + b_p p(u - f(u)) + c_p p(x_{2k(n)-1} - x_{2k(n)}) + d_p p(u - x_{2k(n)}) + e_p p(x_{2k(n)-1} - f(u)) \leq \gamma \max\{p(x_{2k(n)-1} - u), p(u - f(u)), p(x_{2k(n)-1} - x_{2k(n)}), p(u - x_{2k(n)}), p(x_{2k(n)-1} - f(u))\} \leq \gamma p(f(u) - u)$ as n is sufficently large.

Thus

$$p(f(u) - g(x_{2k(n)-1})) \leq \gamma p(f(u) - u). \tag{5}$$

Since $\gamma < 1$. So $f(u) = u$.

Further we have to show that $u \in A$. But

$$p(u - x_0) = p(\lim_m x_m - x_0) = \lim_m p(x_m - x_0)$$

$$\leq \lim_m \sum_{i=0}^{m-1} p(x_{i+1} - x_i) = \lim_m \sum_{i=0}^{m-1} p(x_{i+1} - x_i)\left(\frac{a_p + b_p + d_p}{1 - c_p - d_p}\right)^i p(x_1 - x_0)$$

for all $p \in \Gamma$ (using 3). So, by passing to the limit,

$$p(u - x_0) \leq \frac{1 - b_p - d_p}{1 - a_p - b_p - c_p - 2d_p} p(x_1 - x_0)$$

for all $p \in \Gamma$. Since A is of type Γ_0 with respect to x_0, so $u \in A$. Hence u is the fixed point of f in A. If u, v are the fixed points of f and g respectively, such that $u \neq v$, then $p(u-v) = p(f(u)-g(v) \leq (a_p+2d_p)p(u-v) < p(u-v)$ for all $p \in \Gamma$, what is a contradiction. So $u = v$.

2. A Meir-Keeler type fixed point theorem

In 1969, Meir and Keeler [5] obtained a remarkable generalization of the Banach's results. Park and Bae [8] extended the Meir-Keeler theorem to two commuting maps by adopting Jungck's method. Consequently, a number of new results in this line followed. Recently, Hicks and Kubicek [3] and Włodarczyk [9] studied fixed point theorems in locally convex topological vector spaces. In This section a Meir-Keller type fixed point theorem for a pair of maps on locally convex topological vector spaces is given.

THEOREM 2. *Let E be a sequentially complete Hausdorff locally convex topological vector spaces with calibration Γ. Consider two mappings f, g from E into E satisfying a condition: for any given $\epsilon > 0$, there exits $\delta > 0$ such that the inequality*

$$\epsilon \leq p(x-y) < \epsilon + \delta \text{ implies } p(f(x) - g(y)) < \epsilon \text{ for all } p \in \Gamma.$$

If at least one of f and g is continuous then f or g has a fixed point. If both f and g have fixed points, then each of them has a unique fixed point and these two points coincide.

Proof. Fix $x_0 \in E$ and define $\{x_n\}$ by $x_{2n+1} = f(x_{2n})$, $x_{2n+2} = g(x_{2n+1})$. Then $\{x_n\}$ is a Cauchy sequence. Indeed, if otherwise, then there exists $\epsilon > 0$, such that $\limsup p(x_m - x_n) > 2\epsilon$, for all $p \in \Gamma$. By hypothesis, there exits $\delta > 0$, such that,

$$(7) \qquad \epsilon \leq p(x-y) < \epsilon + \delta \quad \text{and so } p(f(x) - g(y)) < \epsilon \text{ for all } p \in \Gamma.$$

Replace δ by $\delta' = \min\{\delta, \epsilon\}$. Firstly, we show that $\lim p(x_n - x_{n+1}) \downarrow 0$, $(\forall)p \in \Gamma$. Let $C_n = p(x_n - x_{n+1})$. Since from (6) C_n, is a decreasing sequence, then (6) fails for $C_{m+1}, p \in \Gamma$, where C_m is chosen less than $\epsilon + \delta$. Hence

$$(8) \qquad \lim_n C_n \downarrow 0 \quad \text{for all } p \in \Gamma.$$

By (8), we can find an M so that $C_M < \delta'/3$. Pick $m, n > M$, so that

$$(9) \qquad p(x_m - x_n) > 2\epsilon,\ p \in \Gamma,\ |p(x_m - x_j) - p(x_m - x_{j+1})| \leq C_j < \frac{\delta'}{3}$$

for all $p \in \Gamma$.

Since $C_m < \epsilon$ and $p(x_m - x_n) > \epsilon + \delta'$, for all $p \in \Gamma$, therefore there exists an integer $j \in [m, n]$ with $\epsilon + \frac{2\delta'}{3} < p(x_m - x_j) < \epsilon + \delta'$, for all $p \in \Gamma$. Indeed from (9), $p(x_m - x_{j+1}) - C_j \leq p(x_m - x_j)$. It gives, $\epsilon + \delta' - \frac{\delta'}{3} = \epsilon + \frac{2\delta'}{3} < p(x_m - x_j)$. Also $p(x_m - x_j) < \epsilon + \delta'$ for all $p \in \Gamma$. Hence

$\epsilon + \frac{2\delta'}{3} < p(x_m - x_j) < \epsilon + \delta'$. Using (7), we conclude that for all m and j,

$$p(x_m - x_j) \leq p(x_m - x_{m+1}) + p(x_{m+1} - x_{j+1}) + p(x_{j+1} - x_j)$$
$$\leq C_m + \epsilon + C_j < \frac{2\delta'}{3} + \epsilon, \quad \text{for all } p \in \Gamma.$$

Hence it is a contradiction. So $\{x_n\}$ is a Cauchy sequence. Since E is sequentially complete, $\{x_n\}$ converges to some point $x \in E$. Thus $f(x_{2n}) \to x$ and $g(x_{2n+1}) \to x$. If f is continuous, then

$$f(x) = f(\lim_{n\to\infty} g(x_{2n+1})) = \lim_{n\to\infty} f(x_{2n+2}) = x.$$

So x is a fixed point of f. Let u and v be the fixed points of f and g respectively such that $u \neq v$. Then by using (7), we have that $p(u - v) = p(f(u) - g(v)) < p(u - v)$, for all $p \in \Gamma$, a contradiction. Therefore $u = v$.

COROLLARY 3. *Let E be a sequentially complete Hausdorff locally convex topological vector space with calibration Γ. Let f be a mapping from E into E satisfying: for given $\epsilon > 0$, there exists $\delta > 0$ such that the condition $\epsilon \leq p(x - y) < \epsilon + \delta$ implies $p(f(x) - f(y)) < \epsilon$, for all $p \in \Gamma$. Then f has a unique fixed point.*

COROLLARY 4. *Let E be a sequentially complete Hausdorff locally convex topological vector space with calibration Γ. Let f be a surjective mapping from E into E satisfying a condition: for given $\epsilon > 0$, there exists $\delta > 0$ such that, the inequality*

$$p(x - y) < \epsilon \quad \textit{implies } \epsilon \leq p(f(x) - f(y)) < \epsilon + \delta, \tag{10}$$

for all $p \in \Gamma$.

Then f has a unique fixed point.

Proof. We shall show that f is a one-to-one mapping. Indeed, let $x \neq y$ and $p(x - y) < \epsilon$ but $f(x) = f(y)$. Using (10), we obtain $0 \leq p(x - y) < p(f(x) - f(y)) = 0$, $p \in \Gamma$, what is impossible.

Let g be the inverse of f. Then (10) becomes $p(g(x) - g(y)) < \epsilon$, whenever $\epsilon \leq p(x - y) < \epsilon + \delta$. By Corollary (3), g has the unique fixed point u. Thus $g(u) = u = f(g(u)) = f(u)$. So u is the unique fixed point of f.

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