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ON THE MONOTONIC CONTINUOUS SOLUTIONS OF SOME ITERATED EQUATION

1. Introduction

On the iterated equation

$$f^N(x) = \sum_{n=0}^{N-1} A_n f^n(x) \tag{1.1}$$

(where $f^0(x) = x$, $f^k(x) = f \circ f^{k-1}(x)$, $A_n \in R$) some wonderful results have been given in [1–4]. In [1], the general continuous solution of $f^N(x) = x$ was found for any N, and in [2, 3], the general continuous solution $f(x)$ was given in completely explicitly form. In 1986, P. J. McCarthy [4] studied the more general iterated equation (1.1). The purpose of this paper is to prove that under suitable conditions equation (1.1) possesses infinitely many solutions that are continuous and strictly increasing on a certain interval.

2. Main result

THEOREM. *Suppose that $A_0 > 0$, $A_n > 0$ $(n = 0, 1, \ldots, N-1)$ and $\sum_{n=0}^{N-1} A_n < 1$. Then for arbitrary fixed $x_0 \in (0, +\infty)$ the equation (1.1) possesses infinitely many solutions which are continuous and strictly increasing on the interval $[0, x_0]$.*

Proof. Take an arbitrary $x_0 \in (0, +\infty)$ and let $x_1, x_2, \ldots, x_{N-1} \in (A_0 x_0, x_0)$ satisfy the inequalities

$$x_{N-1} < x_{N-2} < \ldots < x_2 < x_1.$$

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We define the sequence $\{x_m\}$ by the recurrence formula

$$x_{m+N} = \sum_{n=0}^{N-1} A_n x_{m+n}, \quad m = 0, 1, 2, \ldots. \tag{2.1}$$

Put

$$D = \{(y_{N-1}, y_{N-2}, \ldots, y_1, x) : x \in (0, x_0], y_n \in (A_0 x, x),$$
$$n = 1, 2, \ldots, N-1, y_{N-1} < y_{N-2} < \ldots < y_2 < y_1\}.$$

We shall show that

$$(x_{m+N-1}, x_{m+N-2}, \ldots, x_{m+2}, x_{m+1}, x_m) \in D \tag{2.2}$$

for $m = 0, 1, 2, \ldots$.

For $m = 0$ it is evident. Assume that (2.2) is valid as $m \geq 0$, then we have

$$x_0 > x_{m+N-1} = \sum_{n=0}^{N-1} A_n x_{m+n-1} > \sum_{n=0}^{N-1} A_n x_{m+n}$$
$$= x_{m+N} \geq A_0 x_m > A_0 x_{m+1} > 0.$$

By the induction argument this proves (2.2).

Notice that the sequence $\{x_m\}$ is strictly decreasing and then convergent. Let

$$\lim_{m\to\infty} x_m = \alpha.$$

Passing to a limit in the recurrence formula (2.1) we obtain $\alpha = \sum_{n=0}^{N-1} A_n \alpha$. Since $\sum_{n=0}^{N-1} A_n < 1$ we have that $\alpha = 0$.

Let $f_0(x), f_1(x), \ldots, f_{N-2}(x)$ be arbitrary strictly increasing continuous functions defined on the intervals $[x_1, x_0], [x_2, x_1], \ldots, [x_{N-1}, x_{N-2}]$, respectively, and satisfying the following conditions:

$$f_m(x_m) = x_{m+1}, m = 0, 1, \ldots, N-2;$$
$$f_{m-1}(x_m) = x_{m+1}, m = 1, 2, \ldots, N-1;$$
$$(f_{N-2} \circ f_{N-\alpha} \circ \ldots \circ f_0(x), f_{N-3} \circ f_{N-4} \circ \ldots \circ f_0(x), \ldots, f_1 \circ f_0(x), f_0(x), x) \in D,$$
$$x \in (x_1, x_0).$$

Put now

$$\begin{aligned} f_{m+N-1}(x) = {} & A_{N-1} x + A_{N-2} f^{-1}_{m+N-2}(x) + \ldots \\ & + A_2 f^{-1}_{m+2} \circ f^{-1}_{m+3} \circ \ldots \circ f^{-1}_{m+N-2}(x) \\ & + A_1 f^{-1}_{m+1} \circ f^{-1}_{m+2} \circ \ldots \circ f^{-1}_{m+N-2}(x) \\ & + A_0 f^{-1}_{m} \circ f^{-1}_{m+1} \circ \ldots \circ f^{-1}_{m+N-2}(x), \end{aligned} \tag{2.3}$$

$$x \in [x_{m+N}, x_{m+N-1}], m = 0, 1, 2, \dots$$

We shall prove that for every m the function $f_m(x)$ is continuous, strictly increasing on the interval $[x_{m+1}, x_m]$,

$$f_m(x_m) = x_{m+1}, f_m(x_{m+1}) = x_{m+2}, \tag{2.4}$$

$$(f_{m+N-2} \circ f_{m+N-3} \circ \dots \circ f_m(x), f_{m+N-3} \circ f_{m+N-4} \circ \dots \circ f_m(x), \dots, f_{m+1} \circ f_m(x), f_m(x), x) \in D. \tag{2.5}$$

In fact, for $n = 0$, it is so by the hypothesis. Let us suppose that it is true for $k = 0, 1, \dots, m + N - 2$. Then the function $f_{m+N-2}(x)$ is invertible on the interval $[x_{m+N}, x_{m+N-1}]$, and $f^{-1}_{m+N-2}(x) \in [x_{m+N-1}, x_{m+N-2}]$, for $x \in [x_{m+N}, x_{m+N-1}]$. From (2.4) and (2.5) it follows that

$(x, f^{-1}_{m+N-2}(x), f^{-1}_{m+N-3} \circ f^{-1}_{m+N-2}(x), \dots, f^{-1}_{m+1} \circ f^{-1}_{m+2} \circ \dots \circ f^{-1}_{m+N-2}(x),$

$f^{-1}_m \circ f^{-1}_{m+1} \circ \dots \circ f^{-1}_{m+N-2}(x)) \in D$, for $x \in [x_{m+N}, x_{m+N-1}]$. Thus the function $f_{m+N-1}(x)$ is, by (2.3), defined for $x \in [x_{m+N}, x_{m+N-1}]$. It is continuous and strictly increasing on the interval $[x_{m+N}, x_{m+N-1}]$. Further,

$$0 < x_{m+N} \le x \le x_{m+N-1} < x_0. \tag{2.6}$$

Moreover, according to (2.3) and (2.5) we have

$$\begin{aligned} x &= f_m(f^{-1}_m(x)) > f_{m+N-2} \circ f_{m+N-3} \circ \dots \circ f_{m+1}(x) \\ &> f_{m+N-1} \circ f_{m+N-2} \circ \dots \circ f_{m+1}(x) \\ &= A_{N-1} f_{m+N-2} \circ f_{m+N-3} \circ \dots \circ f_{m+1}(x) \\ &\quad + A_{N-2} f^{-1}_{m+N-2} \circ f^{-1}_{m+N-2} \circ f^{-1}_{m+N-3} \circ \dots \circ f_{m+1}(x) + \dots \\ &\quad + A_2 f^{-1}_{m+2} \circ f^{-1}_{m+3} \circ \dots \circ f^{-1}_{m+N-2} \circ f_{m+N-2} \circ f_{m+N-3} \circ \dots \circ f_{m+1}(x) \\ &\quad + A_1 f^{-1}_{m+1} \circ f^{-1}_{m+2} \circ \dots \circ f^{-1}_{m+N-2} \circ f_{m+N-2} \circ f_{m+N-3} \circ \dots \circ f_{m+1}(x) \\ &\quad + A_0 f^{-1}_m \circ f^{-1}_{m+1} \circ \dots \circ f^{-1}_{m+N-2} \circ f_{m+N-2} \circ \dots \circ f_{m+1}(x) \\ &= A_{N-1} f_{m+N-2} \circ f_{m+N-3} \circ \dots \circ f_{m+1}(x) \\ &\quad + A_{N-2} f_{m+N-3} \circ f_{m+N-4} \circ \dots \circ f_{m+1}(x) \\ &\quad + \dots + A_2 f_{m+1}(x) + A_1 x + A_0 f^{-1}_m(x) > A_0 f^{-1}_m(x) > A_0 x. \end{aligned} \tag{2.7}$$

From relations (2.6) and (2.7) it follows that the point

$$(f_{m+N-1} \circ f_{m+N-2} \circ \dots \circ f_{m+1}(x), f_{m+N-2} \circ f_{m+N-3} \circ \dots \circ f_{m+1}(x), \dots, f_{m+2} \circ f_{m+1}(x), f(x), x) \in D \quad \text{for any } x \in [x_{m+2}, x_{m+1}]$$

Therefore, we get by (2.4)

$$\begin{aligned} f_{m+N-1}(x_{m+N-1}) &= A_{N-1} x_{m+N-1} + A_{N-2} f^{-1}_{m+N-2}(x_{m+N-1}) \\ &\quad + \dots + A_2 f^{-1}_{m+2} \circ f^{-1}_{m+3} \circ \dots \circ f^{-1}_{m+N-2}(x_{m+N-1}) \end{aligned}$$

$$
\begin{aligned}
&+ A_1 f_{m+1}^{-1} \circ f_{m+2}^{-1} \circ \ldots \circ f_{m+N-2}^{-1}(x_{m+N-1}) \\
&+ A_0 f_m^{-1} \circ f_{m+1}^{-1} \circ \ldots \circ f_{m+N-2}^{-1}(x_{m+N-1}) \\
&= A_{N-1} x_{m+N-1} + A_{N-2} x_{m+N-2} + \ldots \\
&+ A_2 x_{m+2} + A_1 x_{m+1} + A_0 x_m = X_{m+N},
\end{aligned}
$$

$$
\begin{aligned}
f_{m+N-1}(x_{m+N}) &= A_{N-1} x_{m+N} + A_{N-2} f_{m+N-2}^{-1}(x_{m+N}) \\
&+ \ldots + A_2 f_{m+2}^{-1} \circ f_{m+3}^{-1} \circ \ldots \circ f_{m+N-2}^{-1}(x_{m+N}) \\
&+ A_1 f_{m+1}^{-1} \circ f_{m+2}^{-1} \circ \ldots \circ f_{m+N-2}^{-1}(x_{m+N}) \\
&+ A_0 f_m^{-1} \circ f_{m+1}^{-1} \circ \ldots \circ f_{m+N-2}^{-1}(x_{m+N}) \\
&= A_{N-1} x_{m+N} + A_{N-2} x_{m+N-1} + \ldots \\
&+ A_2 x_{m+3} + A_1 x_{m+2} + A_0 x_{m+1} = x_{m+N-1}.
\end{aligned}
$$

Finally, let

$$
(2.8) \qquad f(x) = \begin{cases} 0, & x = 0 \\ f_{m+N-1}(x), & x \in [x_{m+N}, x_{m+N-1}],\ m = 0, 1, 2, \ldots, \\ f_n(x), & x \in [x_{n+1}, x_n],\ n = 0, 1, 2, \ldots, N-2. \end{cases}
$$

One we can easily prove that the function $f(x)$ is defined, continuous and strictly increasing on the interval $[0, x_0]$. We shall show that it satisfies the equation (1.1).

For an arbitrary $x \in (0, x_0]$, there exists an m such that $x \in (x_{m+1}, x_m]$, because of (2.2) and the fact that $lim_{n\to\infty} x_n = 0$. Thus $f(x) \in (x_{m+1}, x_m]$, $f^2(x) \in (x_{m+3}, x_{m+2}]$, $\ldots$, $f^N(x) \in (x_{m+N}, x_{m+N-1}]$. We have by (2.3) and (2.8) that

$$
f(x) = f_m(x).
$$

Therefore,

$$
\begin{aligned}
f^N(x) &= \overbrace{f \circ f \circ \ldots \circ f}^{N}(x) \\
&= f_{m+N-1} \circ f_{m+N-2} \circ \ldots \circ f_{m+1} \circ f_m(x) \\
&= A_{N-1} f_{m+N-2} \circ \ldots \circ f_{m+1} \circ f_m(x) + A_{N-2} f_{m+N-3} \circ \ldots \circ f_m(x) \\
&\quad + \ldots + A_2 f_{m+1} \circ f_m(x) + A_1 f_m(x) + A_0 x \\
&= A_{N-1} f^{N-1}(x) + A_{N-2} f^{N-2}(x) + \ldots + A_2 f^2(x) + A_1 f(x) + A_0 x.
\end{aligned}
$$

Moreover, we easily see that $f(x)$ satisfies equation (1.1) for $x = 0$.

Since x_n $(n = 1, \ldots, N-1)$ and the functions $f_m(x)$ $(m = 0, 1, \ldots, N-2)$ can be chosen in infinitely many ways we obtain thus infinitely many solutions. The proof of the theorem is finished.

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