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# ON THE NEUTRIX PRODUCT OF DISTRIBUTIONS ON $C^\infty$-MANIFOLDS

Let $\mathcal{D}'(M)$ be the space of distributions on a smooth $m$-manifold $M$, each defined by a collection of 'compatible' ordinary distributions (components) given on the charts of some $C^\infty$-atlas on $M$. Here we extend the definition of the neutrix distribution product, based on van der Corput's notion of neutrix limits, onto the space $\mathcal{D}'(M)$. We prove two theorems concerning the existence of the neutrix distribution product in the space $\mathcal{D}'(M)$ under different hypotheses for the neutrix product of the components.

**1.** Recall first the definition we accept of generalized functions (distributions) on an arbitrary smooth $m$-dimensional real manifold, which we will be referring to as a 'manifold'. For any manifold $M$ and some $C^\infty$-atlas $\{\kappa_i, M_i\}_{i\in I}$ on it, we shall use the notation: $\widetilde{M_i} = \kappa_i(M_i) \subseteq \mathbf{R}^m$, $M_{ij} = M_i \cap M_j$ and $\kappa_{ij} := \kappa_i(\kappa_j^{-1}) : \kappa_j(M_{ij}) \to \kappa_i(M_{ij})$ for the (coordinate) diffeomorphic maps of class $C^\infty$ of open sets in $\mathbf{R}^m$ $(i, j \in I)$. Further, for arbitrary open subset $U$ of $M$ we shall denote: $U_i = U \cap M_i$, $U_{ij} = U_i \cap U_j$ and $\widetilde{U}_i = \kappa_i(U_i) \subseteq \mathbf{R}^m$ $(i, j \in I)$. Later on we shall often need the following.

THEOREM 1. *Let $\kappa : U_1 \to U_2$ be a $C^\infty$-diffeomorphic map of open sets in $\mathbf{R}^m$. Then there is a unique continuous linear map of the distribution spaces $\kappa^* : \mathcal{D}'(U_2) \to \mathcal{D}'(U_1) : F \mapsto \kappa^* F$ (pull-back of $F$ by $\kappa$) coinciding with the composition of functions $F(\kappa(x))$ whenever $F$ is in $C^0(U_2)$ and it holds for any test-function $\phi$ in $\mathcal{D}(U_1)$*

$$(1) \qquad \langle \kappa^* F, \phi\rangle = \langle F, \psi\rangle, \quad \textit{where } \psi = \phi(\kappa^{-1})\,|\det D\kappa^{-1}| \in \mathcal{D}(U_2).$$

*Further, for each function $f$ in $C^\infty(U_1)$*

$$(2) \qquad \kappa^*(F\cdot f) = \kappa^* F \cdot f(\kappa),$$

*and for any open subset $V_2$ of $U_2$ and $V_1 = \kappa^{-1}(V_2)$ open in $U_1$*

$$\kappa^*(F\,|_{V_2}) = (\kappa^* F)\,|_{V_1}, \tag{3}$$

*where the restriction $F\,|_V$ is defined by $\langle F\,|_V, \psi\rangle = \langle F, \bar{\psi}\rangle$ for each $\psi$ in $\mathcal{D}(U)$ and $\bar{\psi} = \{\psi$ on $V$, $0$ on $U\backslash V\}$.*

Proof. For an arbitrary distribution $F$ in $\mathcal{D}'(U_2)$, let $\{F_n(x)\}$ be a sequence of infinitely-differentiable functions in $U_2$ converging weakly to $F$, as $n \to \infty$. Then, on making the substitution $t = \kappa(x)$, we have for any $\phi$ in $\mathcal{D}(V_1)$:

$$\int_{U_1} F_n(\kappa(x))\phi(x)dx = \int_{U_2} F_n(t)\phi(\kappa^{-1}(t))\,|\det D\kappa^{-1}|\,dt = \int_{U_2} F_n(t)\psi(t)dt$$

with a test-function $\psi$ in $\mathcal{D}'(U_2)$ defined as in (1). Now taking the weak limits as $n \to \infty$, we see that the sequence $\{F_n(\kappa(x))\}$ converges to the unique distribution $\kappa^* F$ in $\mathcal{D}'(U_1)$ given by (1). Moreover, the map $\kappa^* : \mathcal{D}'(U_2) \to \mathcal{D}'(U_1) : F \mapsto \kappa^* F$ is linear, continuous and coincides with the ordinary composition of functions in $C^0(U_2)$, by its construction.

Further, equation (2) readily follows on noting that

$$\int_{U_1} (F_n.f)(\kappa(x))\phi(x)dx = \int_{U_1} F_n(\kappa(x)).f(\kappa(x))\phi(x)dx.$$

Both expressions in this equation clearly lead to the same distribution in $\mathcal{D}'(U_1)$, when we make the substitution $t = \kappa(x)$ and pass to the weak limits, as $n \to \infty$.

In order to prove (3), we get the following chain of equations for arbitrary sequence $\{F_n(x)\}$ weakly converging to $F$, on making the due substitutions

$$\begin{aligned}
\int_{V_1} (F_n\,|_{V_2})(\kappa)\phi(x)dx &= \int_{V_2} (F_n\,|_{V_2})(t)\psi(t)dt && [\text{with } \psi \text{ defined as in (1)}]\\
&= \int_{U_2} F_n(t)\bar{\psi}(t)dt && [\bar{\psi} = \psi \text{ on } V_2,\ 0 \text{ on } U_2\backslash V_2]\\
&= \int_{U_1} (F_n(\kappa(x)))\,\bar{\phi}(x)dx && [\bar{\phi} = \phi \text{ on } U_1,\ 0 \text{ on } U_1\backslash V_1]\\
&= \int_{V_1} (F_n(\kappa(x)))\,|_{V_1}\,\phi(x)dx.
\end{aligned}$$

Since the restriction map $R_V : F \mapsto F\,|_V$ is linear and continuous, we obtain on passing to the weak limits as $n \to \infty$, that $\langle \kappa^*(F\,|_{V_2}), \phi\rangle = \langle (\kappa^* F)\,|_{V_1}, \phi\rangle$ for any $\phi$ in $\mathcal{D}(V_1)$. This completes the proof of the theorem.

We note that a stronger version of the main claim of this theorem concerning the pull-back map by a $C^\infty$-differentiable function with surjective derivative, is proved in ([7] §6.1).

DEFINITION 1. For each coordinate chart $(\kappa_i, M_i)$ (or briefly, $\kappa_i$) in an atlas $\{\kappa_i, M_i\}_{i\in I}$ on a given manifold $M$, let $F_i$ be a distribution in $\mathcal{D}'(\widetilde{M}_i)$, such that for any other $\kappa_j$ and any distribution $F_j$ in $\mathcal{D}'(\widetilde{M}_j)$ the pull-back by the map $\kappa_{ij}$ satisfies

$$F_j = \kappa_{ij}^* F_i \qquad \text{on } \kappa_j(M_{ij}) \subseteq \mathbf{R}^m. \tag{4}$$

Then we call the collection (of components) $\{F_i\}_{i\in I}$ a *distribution $F$ on $M$*.

We point out that although this definition of distributions on a manifold $M$ is not as elegant as the global one (i.e. as linear forms on the $C_0^\infty$-densities on the manifold), it is preferable when there are concrete calculations or applications in mind (cf. [7]).

Note also that this is a direct extension of an alternative definition of a $C^r$-function $f$ on a given manifold $M$, viz. as collection of functions $\{f_i \in C^r(\widetilde{M}_i)\}_{i\in I}$ satisfying a consistency condition as in (4). Definition 1 needs to be justified by the following.

LEMMA. *Suppose $\{\mu_i, N_i\}_{i\in I}$ is a second atlas on $M$ which is $C^\infty$-compatible with $\{\kappa_i, M_i\}_{i\in I}$. We say that a collection $\{G_i\}_{i\in I}$ defines the same distribution on $M$ as $\{F_i\}_{i\in I}$ if*

$$G_j = \left(\kappa_i(\mu_j^{-1})\right)^* (F_i) \qquad on\ \mu_j(M_i \cap N_j)$$

*for all $i$ and $j$ in $I$. This then defines an equivalence relation on the set of all collections satisfying* (4), *each given on some $C^\infty$-atlas within the maximal one.*

Proof. It is immediately verified that this relation is reflexive, symmetric and transitive with respect to all collections in consideration.

Now we specify the definition of a distribution $F$ on a given manifold $M$ as the equivalence class of collections $\{F_i\}_{i\in I}$ satisfying equation (4) on any $C^\infty$-compatible atlas in the maximal atlas on $M$. Thus, a distribution on $M$ is uniquely defined by a collection $\{F_i\}_{i\in I}$ given on *some* atlas $\{\kappa_i\}_{i\in I}$ and satisfying (4); we write $F = G$ iff, for *some* atlas $\{\kappa_i\}_{i\in I}$, $F_i = G_i$ for all $i$ in $I$. The vector space of all distributions on $M$ with component-wise $\mathbf{C}$-linear operations will be denoted by $\mathcal{D}'(M)$.

**2.** Next we recall the definition of neutrix product for the distributions, starting with that of a neutrix given in ([1]).

DEFINITION 2. A *neutrix $N$* is an additive group of functions $\nu(\xi)$ defined on domain $N'$ with values in an additive group $N''$, where if, for some $\nu$ in

$N$, $\nu(\xi) = \gamma$ for all $\xi$ in $N'$, then $\gamma = 0$. Let further $N'$ be a set contained in a topological space with a limit point $b$ which does not belong to $N'$. If $f(\xi)$ is a function on $N'$ with values in $N''$ and it is possible to find a constant $\beta$ such that $f(\xi) - \beta$ is negligible in $N$, then $\beta$ is called the *neutrix limit* of $f$ as $\xi$ tends to $b$. This is written as $\mathrm{N} - \lim_{\xi \to b} f(\xi) = \beta$.

The elements of the neutrix chosen are viewed as 'negligible functions' and basically the neutrix-limit approach represents a systematic method to neglect infinite quantities of certain type. We note that if a neutrix limit $\beta$ exists, it is unique; also, if the limit exists in the normal sense, it exists in the neutrix sense as well, the two limits being identical.

In what follows, we fix $N$ to be the set of all finite linear sums of the functions

$$n^\lambda \ln^{r-1}(n), \quad \ln^r(n) : \lambda > 0,\ r, n \in \mathbb{N}$$

and all functions $f(n)$ which converge to 0 in the usual sense, as $n \to \infty$. It is straightforward to check that $N$ is a neutrix with domain $\mathbb{N}$ and range $\mathbf{R}^1$.

Further, we let $\rho$ be a fixed function in $\mathbf{R}^1$, such that:

$$\rho(x) = 0, \quad \text{for } |x| > 1, \quad \rho(x) \geq 0, \quad \rho(-x) = \rho(x), \quad \int_{-1}^{1} \rho(x) = 1.$$

Then we define the function $\delta_n$ by $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$. The sequence $\{\delta_n\}$ of functions in $\mathcal{D}$ (short for $\mathcal{D}(\mathbf{R}^1)$) is convergent in $\mathcal{D}'$ to the Dirac $\delta$-function. For any distribution $G$ in $\mathcal{D}'$, the convolution $G_n(x) = (G * \delta_n)(x) = \langle G(x-t), \delta_n(t)\rangle$, gives rise to a sequence $\{G_n\}$ of $C^\infty$-functions weakly converging to $G$, as $n \to \infty$.

The next definition of neutrix distribution product was given in ([3]).

DEFINITION 3. Let $F$ and $G$ be arbitrary distributions in $\mathcal{D}'$ and let $G_n = G * \delta_n$. We say that the *neutrix product* $F \circ G$ exists and is equal to $H$ on the open interval $(a, b)$ if

$$\mathrm{N} - \lim_{n \to \infty} \langle FG_n, \phi\rangle = \mathrm{N} - \lim_{n \to \infty} \langle F, G_n \phi\rangle = \langle H, \phi\rangle$$

for all test-functions $\phi$ with support contained in $(a, b)$.

It was shown in ([3]) that if $\lim_{n \to \infty} \langle FG_n, \phi\rangle$ exists and is equal to $H$ on the open interval $(a, b)$, then the same holds for the neutrix product $F \circ G$.

Note that neutrix products of certain pairs of distributions depend on the particular choice of the function $\rho$. This is like introducing an arbitrary constants into the product, with different products containing the same constants. Güttinger [6] introduced arbitrary constants into his products of distributions but in his case, all the constants were unrelated.

Consider now the sequence of functions of variable $x = (x_1, \ldots, x_m)$ in $\mathbf{R}^m$

(5) $\delta_n(x) = n_1\rho(n_1x_1)\ldots n_m\rho(n_mx_m), \qquad n = (n_1, \ldots, n_m) \in \mathbb{N}^m.$

Clearly, it converges in $\mathcal{D}'_m$ (short for $\mathcal{D}'(\mathbf{R}^m)$) to the $\delta$-function, and for any distribution $G$ in $\mathcal{D}'_m$ and all test-functions $\phi$ in $\mathcal{D}_m$ it holds

$$\lim_{n_1\to\infty} \cdots \lim_{n_m\to\infty} \langle G_n(x), \phi(x)\rangle = \langle G, \phi\rangle,$$

$$G_n(x) = (G * \delta_n)(x) = \langle G(x-t), \delta_n(t)\rangle.$$

The next definition of the neutrix product of distributions in $\mathcal{D}'_m$ extends slightly that given in ([5]) with respect to the open sets in $\mathbf{R}^m$. This extension however makes the definition more 'flexible'; it will enable the handling of products of sums of distributions not defined everywhere by using the sheaf properties of the distribution spaces (cf. [2]).

DEFINITION 4. Let $F$ and $G$ be distributions in $\mathcal{D}'_m$ and let $G_n = G * \delta_n$, with $\delta_n$ as in (5) and $n$ in $\mathbb{N}^m$. We say that the neutrix product $F \circ G$ exists and is equal to $H$ on the open set $U$ in $\mathbf{R}^m$ if

$$\mathrm{N}-\lim_{n_1\to\infty} \ldots \mathrm{N}-\lim_{n_m\to\infty} \langle FG_n, \phi\rangle = \langle H, \phi\rangle,$$

or briefly

$$\mathrm{N}-\lim_{n\to\infty} \langle FG_n, \phi\rangle = \langle H, \phi\rangle,$$

for all test-functions $\phi$ in $\mathcal{D}_m(U)$, provided $H$ is independent of the order in which the neutrix limits are taken.

The next theorem now establishes the consistency of the neutrix product of distributions in $\mathcal{D}'_m$ with their pull-back by a diffeomorphic map of the underlying domains.

THEOREM 2. *Let $F$ and $G$ be distributions in $\mathcal{D}'_m$ and let the neutrix product $F \circ G$ exist and is equal to $H$ on the open set $U_2$ in $\mathbf{R}^m$. If $\kappa : U_1 \to U_2$ is a $C^\infty$-diffeomorphic map of open sets in $\mathbf{R}^m$, then the neutrix product $(\kappa^*F) \circ (\kappa^*G)$ exists and*

(6) $(\kappa^*F) \circ (\kappa^*G) = \kappa^*H \qquad$ on the open set $U_1$ in $\mathbf{R}^m$.

Proof. First of all, applying the pull-back $\kappa^*$ on the function $G_n(x) = (G * \delta_n)(x)$ in $\mathbf{R}^m$, we have, in view of Theorem 1,

$$\kappa^*(G_n(x)) = \kappa^*(\langle G(x-t), \delta_n(t)\rangle) = \langle G(\kappa(x)-t), \delta_n(t)\rangle = (\kappa^*G)_n.$$

Now let $\phi$ be an arbitrary test-function in $\mathcal{D}_m$ with support contained in $U_1$. Taking into account the above equation as well as (1) and (2), we obtain

$$\langle(\kappa^*F).(\kappa^*G)_n, \phi\rangle = \langle(\kappa^*F).G_n(\kappa), \phi\rangle = \langle\kappa^*(FG_n), \phi\rangle = \langle FG_n, \psi\rangle,$$

where $\psi = \phi(\kappa^{-1}) \mid \det D\kappa^{-1} \mid$ is in $\mathcal{D}'_m(U_2)$. Thus we have for any $\phi$ in $\mathcal{D}_m(U_1)$

$$\langle(\kappa^* F) \circ (\kappa^* G), \phi\rangle = \mathrm{N}-\lim_{n\to\infty}\langle \kappa^*(F).G_n(\kappa), \phi\rangle = \mathrm{N}-\lim_{n\to\infty}\langle FG_n, \psi\rangle = \langle H, \psi\rangle$$

with the test-function $\psi$ given as above.

On the other hand, for any test-function $\phi$ in $\mathcal{D}_m(U_1)$, equation (1) gives

$$\langle \kappa^* H, \phi\rangle = \langle H, \psi\rangle$$

with a test-function $\psi$ as above. Comparing now the left hand side of the last two equations, we have that the neutrix product in consideration exists and equation (6) holds on the open set $U_1$ in $\mathbf{R}^m$. This completes the proof.

The proposition below will also be needed in the sequel.

THEOREM 3. *If $F$ and $G$ are distributions in $\mathcal{D}'_m$ and the neutrix product $F \circ G$ exists and is equal to $H$ on the open set $V$ in $\mathbf{R}^m$, then the neutrix product $(F\,|_V) \circ (G\,|_V)$ also exists and*

$$(F\,|_V) \circ (G\,|_V) = H\,|_V \qquad \text{on the open set } V \text{ in } \mathbf{R}^m. \tag{7}$$

Proof. In view of equation (3), we have for any test-function $\phi$ in $\mathcal{D}_m$ with support contained in the open set $V$:

$$\langle (F\,|_V).(G\,|_V)_n, \phi\rangle = \langle (F\,|_V)\cdot(G_n\,|_V), \phi\rangle = \langle (F\cdot G_n)\,|_V, \phi\rangle = \langle F\cdot G_n, \bar\phi\rangle,$$

where the test-function $\bar\phi$ in $\mathcal{D}$ coincides with $\phi$ on $V$ and is 0 elsewhere. Taking the neutrix limit as $n \to \infty$, we get

$$\begin{aligned}\langle (F\,|_V)\circ(G\,|_V), \phi\rangle &= \mathrm{N}-\lim_{n\to\infty}\langle (F\,|_V)\cdot(G\,|_V)_n, \phi\rangle \\ &= \mathrm{N}-\lim_{n\to\infty}\langle F\cdot G_n, \bar\phi\rangle \\ &= \langle H, \bar\phi\rangle = \langle H\,|_V, \phi\rangle\end{aligned}$$

for all $\phi$ in $\mathcal{D}_m$ with support contained in $V$. This proves that the neutrix product in consideration exists and obeys equation (7).

**3.** Now we extend the definition of the neutrix distribution product in $\mathcal{D}'_m$ so as to be applicable to the space $\mathcal{D}'(M)$ of distributions on a manifold $M$.

DEFINITION 5. If $M$ is a manifold with atlas $\{\kappa_i\}_{i\in I}$ on it, let $\{F_i\}_{i\in I}$ and $\{G_i\}_{i\in I}$ be distributions in $\mathcal{D}'(M)$ and let $G_{in} = G_i * \delta_n$, with $\delta_n$ as in (5), for all $i$ in $I$. We say that the neutrix product $F \circ G$ exists in $\mathcal{D}'(M)$ and is equal to $H = \{H_i\}_{i\in I}$ on the open set $U$ in $M$ if, for each $i$ in $I$,

$$\mathrm{N}-\lim_{n_1\to\infty}\ldots\mathrm{N}-\lim_{n_m\to\infty}\langle F_i G_{in}, \phi\rangle = \langle H_i, \phi\rangle \tag{8}$$

for all test-functions $\phi$ in $\mathcal{D}'_m(\widetilde{U}_i)$, provided each $H_i$ is independent of the order in which the neutrix limits are taken.

The next theorem now gives a natural sufficient condition for the existence of the neutrix distribution product in the space $\mathcal{D}'(M)$.

THEOREM 4. *Given the distributions* $F = \{F_i\}_{i\in I}$ *and* $G = \{G_i\}_{i\in I}$ *on a manifold* $M$ *with an atlas* $\{\kappa_i, M_i\}_{i\in I}$ *on it, suppose the neutrix product* $F_i \circ G_i$ *exist (in* $\mathcal{D}'_m$*) and is equal to* $H_i$ *on the whole domain* $\widetilde{M}_i$ *for all* $i$ *in* $I$*. Then the neutrix product* $F \circ G$ *exist in* $\mathcal{D}'(M)$ *and is equal to* $H = \{H_i\}_{i\in I}$ *on the whole manifold* $M$.

Proof. Consider the distribution $H$ on $M$ defined by the collection $\{H_i = F_i \circ G_i\}_{i\in I}$ of distributions in $\mathcal{D}'(\widetilde{M}_i)$. Then for each $i$ in $I$ equation (8) holds. Taking the pull-back map of the component $H_i$ by $\kappa_{ij}$, we get for any $i$ in $I$

$$\begin{aligned}\kappa_{ij}^* H_i = \kappa_{ij}^*(F_i \circ G_i) &= \kappa_{ij}^* F_i \circ \kappa_{ij}^* G_i \qquad \text{[by (6)]}\\ &= F_j \circ G_j = H_j. \qquad \text{[by (4)]}\end{aligned}$$

Each equation here holds on the whole domain $\widetilde{M}_i \subseteq \mathbf{R}^m$, except the third one that holds on $\kappa_j(M_{ij}) \subseteq \widetilde{M}_i$. Thus, we get exactly the consistency condition (4) between the components $H_i$ and $H_j$ for arbitrary $i$ and $j$ in the index set $I$. According to the Lemma, we have thus defined a unique distribution $H$ in $\mathcal{D}'(M)$. Clearly, it satisfies Definition 5 with an open set $U$ coinciding with $M$ (and all $\widetilde{U}_i = \widetilde{M}_i$). The proof of the theorem is complete.

We note that the sufficient condition set up by this theorem would apply to a variety of particular neutrix products in $\mathcal{D}'(M)$ since most of the neutrix distribution products proved so far to exist are each equal to some distribution on the whole space (cf. [4]).

A further refinement of this existence theorem is given below. We first introduce the following notation. Any open set $U$ in given manifold $M$ and atlas $\{\kappa_i\}_{i\in I}$ on it can be viewed as submanifold of $M$ with an inclusion map $\mathrm{id}_U : U \to M : x \mapsto x$ and atlas $\{U_i, \kappa_i^U = \kappa_i\,|_{U_i}\}_{i\in I}$. Thus applying Definition 1, we can define the space of distributions on $U$, which we shall denote by $\mathcal{D}'_M(U)$ (with an index '$M$' indicating the parent manifold).

THEOREM 5. *Given the distributions* $F = \{F_i\}_{i\in I}$ *and* $G = \{G_i\}_{i\in I}$ *on a manifold* $M$ *with atlas* $\{\kappa_i\}_{i\in I}$ *and an arbitrary open set* $U$ *in* $M$*, suppose the neutrix product* $F_i \circ G_i$ *exists (in* $\mathcal{D}'_m$*) and is equal to* $H_i$ *on the open set* $\widetilde{U}_i$ *for any* $i$ *in* $I$*. Then there is a unique distribution* $K = \{K_i\}_{i\in I}$ *on the submanifold* $U$ *of* $M$*, such that* $K_i = H_i\,|_{\widetilde{U}_i}$ *for all* $i$ *in* $I$.

Proof. Consider the distribution $K$ on the submanifold $U$ of $M$ defined by the collection $\{K_i = H_i\,|_{\widetilde{U}_i}\}_{i\in I}$ of distributions in $\mathcal{D}'(\widetilde{U}_i)$. We have

$F_i \circ G_i = K_i$ on $\widetilde{U}_i$ for each $i$ in $I$, and therefore equation (8) holds. Now we show that $\{K_i\}_{i\in I}$ is 'well-defined' distribution on $U$. Indeed, for any $i$ and $j$ in $I$, the following chain of equations for the pull-back map by $\kappa_{ij}^U$ can be obtained:

$$\begin{aligned}
\left(\kappa_{ij}^U\right)^* K_i = \left(\kappa_{ij}^U\right)^* \left(H_i \mid_{\widetilde{U}_i}\right) &= \left(\kappa_{ij}^U\right)^* \left((F_i \circ G_i) \mid_{\widetilde{U}_i}\right) \\
&= \left(\kappa_{ij}^U\right)^* \left(\left(F_i \mid_{\widetilde{U}_i}\right) \circ \left(G_i \mid_{\widetilde{U}_i}\right)\right) && \text{[by (7)]} \\
&= \left(\left(\kappa_{ij}^U\right)^* \left(F_i \mid_{\widetilde{U}_i}\right)\right) \circ \left(\left(\kappa_{ij}^U\right)^* \left(G_i \mid_{\widetilde{U}_i}\right)\right) && \text{[by (6)]} \\
&= \left(\left(\left(\kappa_{ij}^U\right)^* F_i\right) \mid_{\widetilde{U}_i}\right) \circ \left(\left(\left(\kappa_{ij}^U\right)^* G_i\right) \mid_{\widetilde{U}_i}\right) && \text{[by (3)]} \\
&= \left(F_j \mid_{\widetilde{U}_j}\right) \circ \left(G_j \mid_{\widetilde{U}_j}\right) && \text{[by (4)]} \\
&= H_j \mid_{\widetilde{U}_j} = K_j.
\end{aligned}$$

Each equation here holds on the whole $\widetilde{U}_j$, except for that obtained by (4) holding on $\kappa_j(M_{ij}) \cap \widetilde{U}_j = \kappa_j(U_{ij})$. We therefore have: $\left(\kappa_{ij}^U\right)^* K_i = K_j$ on the set $\kappa_j(U_{ij})$ for all $i$ and $j$ in $I$, and it follows from the Lemma that the collection $\{K_i\}_{i\in I}$ defines a unique distribution $K$ in the space $\mathcal{D}'_M(U)$. This completes the proof of the theorem.

Finally we shall employ the following canonical definitions. For a given distribution $F = \{F_i\}_{i\in I}$ in $\mathcal{D}'(M)$ and an open set $U$ in $M$ consider the collection $\{G_i = F_i \mid_{\widetilde{U}_i}\}_{i\in I}$ (we can put $G_i = 0$ if $U \cap M_i$ is empty). In view of (3), their elements satisfy the consistency condition (4) and thus they define a unique distribution in $\mathcal{D}'_M(U)$, that can equally be denoted by $F \mid_U$. Further, we have this definition for the equality of distributions on $M$ : $F = G$ on an open set $U$ if $F \mid_U = G \mid_U$ in $\mathcal{D}'_M(U)$.

With these definitions, the following is now an immediate consequence of the last two theorems.

COROLLARY. *Under the hypothesis of Theorem 5, the neutrix product* $(F|_U)\circ(G|_U)$ *exists in* $\mathcal{D}'_M(U)$ *and is equal to the distribution* $K=\{H_i|_{\widetilde{U}_i}\}_{i\in I}$ *on the whole* $U$.

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