DEMONSTRATIO MATHEMATICA
Vol. XXIX No 3 1996

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THE CONTINUOUS APPROXIMATION OF MEASURABLE MAPPINGS

1. Introduction

Let X and Y be metric spaces. Denote by $\mathcal{B}(X)$ and $\mathcal{B}(Y)$ respectively the Borel σ-algebras on these spaces. Let μ be a finite Borel measure on X. By $\mathcal{B}_\mu(X)$ we shall denote the completion in the measure μ of the σ-algebra $\mathcal{B}(X)$. A mapping f from X into Y is called μ-measurable if it is measurable with respect to $(\mathcal{B}_\mu(X), \mathcal{B}(Y))$.

In our previous paper [2] we have considered the following question: does it follow that every μ-measurable mapping from X into Y is the μ-almost everywhere limit of a sequence of continuous functions?

It is easy to see that in general the answer to this question is negative, even under the additional assumption that X and Y are separable and complete metric spaces (see [2]).

In [2] we have also given one of the affirmative answers of the stated problems. Namely we proved that if μ is a finite Borel measure on an arbitrary metric space X and Y is a separable Banach space with the approximation property, then every μ-measurable mapping f from X into Y is a limit of a sequence $\{f_n\}$ of continuous mappings with respect to μ-almost everywhere convergence ([2, Th. 2]).

The purpose of this paper will be to show that this theorem remains true without the assumption that Y has the approximation property, i.e. that it is valid for an arbitrary separable Banach space Y.

1. Main result

THEOREM. *Let μ be a finite Borel measure on a metric space X, and let Y be a separable Banach space. If f is a μ-measurable mapping from X into Y, then there exists a sequence $\{f_n\}$ of continuous mappings from X into Y such that $f_n \to f$ μ-a.e.*

Proof. To prove the theorem it suffices to show that for any $\varepsilon > 0$ and $\varrho > 0$ there exists a continuous mapping $g : X \to Y$ such that

$$\mu\{x : ||f(x) - g(x)|| > \varepsilon\} < \varrho. \tag{1}$$

Indeed, if this is true, then choosing the sequences $\varepsilon \to 0$ and $\varrho \to 0$ we can construct a sequence of continuous mappings from X into Y which is convergent in the measure μ to f, and from this sequence we may choose a subsequence which is convergent to f μ-almost everywhere.

Let $\varepsilon > 0$ and $\varrho > 0$ be fixed. We must construct a continuous mapping $g : X \to Y$ which satisfies (1).

Denote by ν a finite Borel measure on Y given by the formula $\nu(B) = \mu(f^{-1}(B))$ for every Borel subset B of Y. Since each finite Borel measure on Y is tight (see [1, Theorem 1.4]), there exists a compact subset K of Y such that $\nu(Y - K) < \varrho/2$. Put $K' = f^{-1}(K)$. Then

$$\mu(X - K') < \varrho/2. \tag{2}$$

Let $U = \{y \in Y : ||y|| \le \varepsilon/2\}$. Since $K \subset \bigcup_{y \in K}(y + U)$, i.e. $\bigcup_{y \in K}(y + U)$ is a cover of K, and since K is a compact set, from this cover we may choose a finite subcover. This means that there exist $y_1, y_2, \ldots, y_m \in K$ such that $K \subset \bigcup_{k=1}^{m}(y_k + U)$.

Let $V_k = y_k + U$ $(k = 1, 2, \ldots, m)$. Then $K \subset \bigcup_{k=1}^{m} V_k$. If we now denote $A_1 = V_1$, $A_k = V_k - \bigcup_{i=1}^{k-1} V_i$ for $k = 2, \ldots, m$, then $A_1, \ldots, A_m$ are Borel subsets of Y which are pairwise disjoint (i.e. $A_i \cap A_j = \emptyset$ if $i \neq j$), and such that $A_k \subset V_k$ and $\bigcup_{k=1}^{m} A_k = \bigcup_{k=1}^{m} V_k$. As a consequence we have moreover that $K \subset \bigcup_{k=1}^{m} A_k$.

Take $x \in K'$. Then $f(x) \in K$. Therefore there exists k_0 $(1 \le k_0 \le m)$ such that $f(x) \in A_{k_0}$ and $f(x) \notin A_k$ for every $k \neq k_0$. Since $A_{k_0} \subset V_{k_0}$, then $f(x) \in V_{k_0}$, i.e. $f(x) \in y_{k_0} + U$. Hence $f(x) - y_{k_0} \in U$, and consequently $||f(x) - y_{k_0}|| \le \varepsilon/2$. If we denote by I_A the characteristic function of a set A, then $||f(x) - \sum_{k=1}^{m} I_{A_k}(f(x))y_k|| = ||f(x) - y_{k_0}|| \le \varepsilon/2$.

We show that

$$\Big\| f(x) - \sum_{k=1}^{m} I_{A_k}(f(x))y_k \Big\| \le \varepsilon/2 \quad \text{for any } x \in K'.$$

Hence, and from (2) we obtain that

$$\mu\Big\{x : \Big\| f(x) - \sum_{k=1}^{m} I_{A_k}(f(x))y_k \Big\| > \varepsilon/2\Big\} \le \mu(X - K') < \varrho/2. \tag{3}$$

For every $k = 1, \ldots, m$ the function $g_k : X \to R$ defined by the formula $g_k(x) = I_{A_k}(f(x))$ is a μ-measurable mapping from X into R. Then in view of Theorem 1 in [2], we have that for every $k = 1, \ldots, m$ there exists a sequence $\{g_n^{(k)}\}$ of continuous mappings from X into R such that $g_n^{(k)} \to g_k$ (as $n \to \infty$)μ-a.e.

Hence for any $k = 1, \dots, m$, for which $y_k \neq 0$, there exists $n_k > 0$ such that

$$\mu\left\{x : |g_k(x) - g_{n_k}^{(k)}(x)| > \frac{\varepsilon}{2m||y_k||}\right\} < \varrho/2m. \tag{4}$$

Let $g(x) = \sum_{k=1}^m g_{n_k}^{(k)}(x) y_k$. Then g is a continuous mapping from X into Y. Furthermore, without loss of generality we may obviously suppose that for every $k = 1, \dots, m$ $y_k \neq 0$. Moreover, from (3) and (4) we have

$$\mu\{x : ||f(x) - g(x)|| > \varepsilon\} = \mu\left\{x : \left\|f(x) - \sum_{k=1}^m g_{n_k}^{(k)}(x) y_k\right\| > \varepsilon\right\}$$

$$\leq \mu\left\{x : \left\|f(x) - \sum_{k=1}^m I_{A_k}(f(x)) y_k\right\| > \varepsilon/2\right\}$$

$$+ \mu\left\{x : \left\|\sum_{k=1}^m I_{A_k}(f(x) y_k - \sum_{k=1}^m g_{n_k}^{(k)}(x) y_k\right\| > \varepsilon/2\right\}$$

$$\leq \varrho/2 + \mu\left\{x : \left\|\sum_{k=1}^m (g_k(x) - g_{n_k}^{(k)}(x)) y_k\right\| > \varepsilon/2\right\}$$

$$\leq \varrho/2 + \sum_{k=1}^m \mu\{x : |g_k(x) - g_{n_k}^{(k)}(x) \cdot ||y_k|| > \varepsilon/2m\}$$

$$= \varrho/2 + \sum_{k=1}^m \mu\left\{x : |g_k(x) - g_{n_k}^{(k)}(x)| > \frac{\varepsilon}{2m||y_k||}\right\} \leq \varrho/2 + m \cdot \varrho/2m = \varrho.$$

This completes the proof of the theorem.

References

[1] P. Billingsley, *Convergence of Probability Measures.* Wiley, New York, 1968.

[2] A. Wiśniewski, *The structure of measurable mappings on metric spaces*, Proc. Amer. Math. Soc. 122 (1994), 147–150.

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Received December 16, 1994.