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WEAK SOLUTIONS OF SET-VALUED RANDOM DIFFERENTIAL EQUATIONS

1. Preliminaries

The existence of solutions to set-valued differential equations were considered by many authors (see e.g. [2], [3], [8], [9], [10], [12]). In [11] a random set-valued differential equation has been investi gated. In this paper we consider such equation with purely probabilistic initial conditions. The problem has the form

$$\text{(I)}\qquad \begin{aligned} D_H X_t &= F(t, X_t) \quad \text{P.1}, \quad t \in [0,T] - \text{ a.e.} \\ X_0 &\overset{\mathrm{d}}{=} \mu, \end{aligned}$$

where F is a given set-valued mapping with values in the space K^n of all nonempty compact convex subsets of the space R^n and μ is a probability measure on K^n. The initial condition above requires that the solution of (I) has a given distribution μ at the time $t = 0$.

Let $K_c(S)$ be the space of all nonempty compact and convex subsets of a metric space (S, ρ) equipped with the Hausdorff metric H (see e.g. [5], [7]); i.e., $H(A, B) = \max(\bar{H}(A, B), \bar{H}(B, A))$ for $A, B \in K_c(S)$, where $\bar{H}(A, B) = \sup_{a \in A} \inf_{b \in B} \rho(a, b)$

By $\|A\|$ we denote the distance of A to $\{0\}$, i.e., $H(A, \{0\})$. For S being a separable Banach space, $(K_c(S), H)$ is a Polish metric space.

Let $I = [0, T]$, $T > 0$. For a given multifunction $F : I \to K_c(S)$ by $D_H F(t_0)$ we denote its Hukuchara derivative at the point $t_0 \in I$ (see e.g.[5],

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[12]) if there exist limits (in $K_c(S)$)

$$\lim_{h\to 0+}\frac{F(t_0+h)-F(t_0)}{h},\ \lim_{h\to 0+}\frac{F(t_0)-F(t_0-h)}{h},$$

both equal to the same set $D_H F(t_0) \in K_c(S)$.

The following connection between the Aumann integral of set-valued mapping and its Hukuchara derivative are well known (see e.g. [12]):

PROPOSITION 1. *If the set-valued mapping $F : [0,T] \to K_c(S)$ is Aumann integrable and $U_0 \in K_c(S)$, then if $\Phi(t) = U_0 + \int_0^t F(s)ds$ then $D_H\Phi(t) = F(t)-$a.e. int.*

Let μ be a probabability measure on the metric space (S,ρ).

DEFINITION 1. The probabability measure μ is said to be tight if for every $\epsilon > 0$ there exists a compact set $K_\epsilon \subset S$ such that $\mu(K_\epsilon) \geq 1-\epsilon$.

Similarly if (μ_n) is a sequence of distributions on S then we say that it is tight if for any $\epsilon > 0$ there exists a compact set K_ϵ such that $\mu_n(K_\epsilon) \geq 1-\epsilon$ for all $n \geq 1$.

The next definitions are devoted to weak convergence of probability measures (see e.g. [1], [13]).

DEFINITION 2. The sequance (μ_n) of probability measures is weakly convergent to the the distribution μ ($\mu_n \Rightarrow \mu$) if for every continuous and bounded function $f : S \to R$ one has $\int_S f d\mu_n \to \int_S f d\mu$, as $n \to \infty$.

DEFINITION 3. A family Π of probability measures on S is said to be relatively weakly compact if every sequence of elements of Π contains a weakly converget subsequence.

The following Theorems due to Prochorov (see e.g. [1]) will be needed in the sequel:

THEOREM 1. *If the family Π of probability measures on S is tight then Π is relatively weakly compact.*

THEOREM 2. *A relatively weakly compact family Π of probability measures on Polish metric space S is tight.*

2. Tightness condiditions of probability measures on the space of continuous set-valued mappings

Let $S = R^n$ and $K^n = K_c(R^n)$. By $C_I = C(I, K^n)$ we denote the space of all $H-$continuous set-valued mappings. In C_I we introduce a metric ρ of uniform convergence i.e.

$$\rho(F,G) := \sup_{0\leq t\leq T} H(X(t),Y(t)), \quad \text{for } X,Y \in C_I.$$

Then (C_I, ρ) is a Polish metric space. For $X \in C_I$ we define a modulus of continuity $w_X(\delta) = \sup\{H(X(t), X(s)) : | t - s | < \delta, t, s \in I\}$. We can formulate the following version of Ascoli Theorem for the space C_I:

THEOREM 3 ([5]). *Let $A \subset C_I$. Then the set $\bar{A}$ is compact if and only if*:
i) *there exists $M > 0$ such that* $\sup_{X \in A} \sup_{t \in I} \|X(t)\| < M$,
ii) $\lim_{\delta \to 0} \sup_{X \in A} w_X(\delta) = 0$.

We can prove now the following tightness condition of probability measures on C_I.

THEOREM 4. *A sequence (μ_n) of probability measures on C_I is tight if and only if*
i) $\forall \alpha > 0, \exists a > 0, \forall n \geq 1 : \mu_n(X \in C_I : \sup_{t \in I} \|X(t)\| \leq a) \geq 1 - \alpha$,
ii) $\forall \alpha > 0, \forall \epsilon > 0, \exists \delta < 1, \exists n_0, \forall n \geq n_0 :$
$\mu_n(X \in C_I : w_X(\delta) \leq \epsilon) \geq 1 - \alpha$.

Proof. Let (μ_n) be a sequence of thight probability measures on C_I, and let K_α be a compact subset of C_I such that $\mu_n(K_\alpha) \geq 1 - \alpha$ for all $n \geq 1$ and fixed $\alpha > 0$. Then from i) of the theorem stated above we obtain: $\sup_{X \in K_\alpha} \sup_{t \in I} \|X(t)\| < \infty$. Let $a := \sup_{X \in K_\alpha} \sup_{t \in I} \|X(t)\|$. Hence $K_\alpha \subseteq \{X \in C_I : \sup_{t \in I} \|X(t)\| \leq a\}$. Thus for each $n \geq 1$

$$\mu_n(X \in C_I : \sup_{t \in I} \|X(t)\| \leq a) \geq 1 - \alpha.$$

Similarly, using condition ii) of Theorem 3, for every $\epsilon > 0$ there exists a $\delta > 0$ such that $K_\alpha \subseteq \{X \in C_I : w_X(\delta) \leq \epsilon\}$. Consequently we have $\mu_n(X \in C_I : w_X(\delta) \leq \epsilon) \geq 1 - \alpha$, for $n \geq 1$. Conversely, let $\delta_k > 0$ be chosen such that $\mu_n(X \in C_I : w_X(\delta_k) \leq 1/k) \geq 1 - \alpha_k$ for $n \geq 1$, where $\alpha_k = \alpha/2^{k+1}$. Let $A_k := \{X \in C_I : w_X(\delta_k) \leq 1/k\}$. From Ascoli Theorem it follows that the set $A := \{X \in C_I : \sup_{t \in I} \|X(t)\| \leq a\} \cap \bigcap_{k=1}^{\infty} A_k$ has compact closure in C_I.

If we put $K_\alpha := \bar{A}$ then

$$\mu_n(C_I \backslash K_\alpha) \leq \mu_n(C_I \backslash A) \leq \mu_n(X \in C_I : \sup_{t \in I} \|X(t)\| > a) + \sum_{k=1}^{\infty} \mu_n(C_I \backslash A_k)$$

for $n = 1, 2, \ldots$. Thus we get $\mu_n(C_I \backslash K_\alpha) \leq \alpha$ for each $n \geq 1$. The proof is completed.

3. Main result

Let $(\Omega, \mathcal{F}, P)$ be a given complete probability space. The family of set-vauled mappings $X = (X_t)_{t \geq 0}$ is said to be a multivalued stochastic process if for every $t \geq 0$, the mapping $X_t : \Omega \to K^n$ is measurable i.e $X_t^-(U) :=$

$\{\omega : X_t(\omega) \cap U \neq \emptyset\} \in \mathcal{F}$, for every open $U \subset E$ (see e.g. [4, 7]). It can be noticed that U can be also chosen both as closed and Borel subset. We restrict our interest to the case when $0 \leq t \leq T$, $T > 0$. If the mapping $t \to X_t(\omega)$ is continuous ($H-$continuous) with probability one (P.1) then we say that the process X has continuous "paths".

Observe that the set–valued stochastic process X can be thought as a random element $X : \Omega \to C_I$. Indeed,it follows immidiately from [6] and from the fact that the topology of uniform convergence and the compact-open topology in C_I are the same. So we can state:

PROPOSITION 2. *The set-valued stochastic process* $X = (X_t)_{0 \leq t \leq T}$ *has continuous "paths" if and only if the mapping* $X : \Omega \to C_I$ *is measurable.*

DEFINITION 4. A probability measure μ on C_I is a distribution of the set–valued process $X = (X_t)_{0 \leq t \leq T}$ if $\mu(A) = P(X^{-}(A))$ for every Borel subset A from C_I. A distribution of X we will be denoted by P^X.

Let $F : I \times K^n \to K^n$ be an integrably bounded set–valued mapping satisfying Caratheodory type conditions:

1) there exists a measurable function $m : I \to R_+$ such that $\int_0^T m(t)dt < \infty$ and $\|F(t, A)\| \leq m(t)$ $t-$a.e. $A \in K^n$,
2) $F(t, \cdot)$ is $H-$continuous $t-$a.e.,
3) $F(\cdot, A)$ is a measurable multifunction for every $A \in K^n$.

Consider now the multivalued random differential equation mentioned above:

$$\text{(I)} \qquad \begin{aligned} &D_H X_t = F(t, X_t) \quad \text{P.1}, \ t \in [0, T] - \text{a.e.} \\ &X_0 \stackrel{d}{=} \mu, \end{aligned}$$

By a weak solution to (I) we mean a system $(\Omega, \mathcal{F}, P, (X_t)_{t \in I})$ where $(X_t)_{t \in I}$ is a set-valued process on some probability space $(\Omega, \mathcal{F}, P)$ such that (I) is fulfield.

Now we can formulate the following theorem:

THEOREM 5. *Let* $F : I \times K^n \to K^n$ *be a set-valued function satisfying Caratheodory type conditions and let* μ *be a probability measure on the space* K^n. *Then there exists at least one weak solution to (I).*

Proof. Let us observe first that the set $S := \{X \in C_I : \exists A \in K^n, \forall t \in I : X(t) = A\}$ is nonempty and closed in C_I. So the space K^n can be identified with the set S, of all "constant" elements from C_I. Thus the measure μ can be considered as the probability μ' on C_I concentrated on S i.e. $\mu'(B) = \mu(B \cap S)$, where B is a Borel subset of C_I. Then there exist a probability space $(\Omega, \mathcal{F}, P)$ and random element $X_0 : \Omega \to S$ such that $X_0 \stackrel{d}{=} \mu'$.

Define the sequence of set-valued stochastic processes as follows

(1) $$X_t^n = X_0 I(t)_{[0,T)} + [X_0 + \int_0^{t-T/n} F(s, X_s^n)ds] I(t)_{(T/n,T]}.$$

Put $M(t) = \int_0^t m(s)ds$ and let $f(\cdot) = \|\cdot\|$. Because of the continuity of $f: K^n \to R_+$, $f(X_0): \Omega \to R_+$ is measurable and hence we have $f(X_0) \stackrel{d}{=} \mu'^f$, where μ'^f is a distribution of f on R_+. Using a continuity property of probability measures and $R_+ = \bigcup_{n\geq 1}[0,n)$, we obtain $\forall\alpha > 0,\ \exists N: \mu'^f([0,N)) \geq 1-\alpha$. Thus we get

(2) $$\forall\alpha > 0,\ \exists N: P(\|X_0\| \leq N) \geq 1-\alpha.$$

For fixed $\omega \in \Omega$ one has

$$\sup_{t\in I} \|X_t^n\| \leq \|X_0\| + \sup_{t\in I} \int_0^t \|F(s, X_s^n)\| ds \leq \|X_0\| + M(T).$$

If $a := N + M(T)$ then from above we get

$$P(\sup_{t\in I} \|X_t^n\| \leq a) \geq P(\|X_0\| + M(T) \leq a).$$

Hence by (2) we get

(3) $$\forall\alpha > 0,\ \exists a,\ \forall n \geq 1: P^{X^n}(X \in C_I; \sup_{t\in I} \|X(t)\| \leq a) \geq 1-\alpha.$$

Let us define now a sequence (Y^n) of set-valued processes by the formula: $Y_t^n = X_0 + \int_0^t F(s, X_s^n)ds$ for $t \geq 0$ and $n = 1, 2, \ldots$. It can be shown (see e.g [11], proof of Lemma 2.1. II) that $H(Y_t^n, Y_s^n) \leq | M(t) - M(s) |$ with Proposition 1 and

$$H(X_t^n, Y_t^n) \leq w_M(T) \text{ for } n = 1, 2, \ldots, t, s \in I,$$

where $w_M(\delta) = \sup\{| M(t) - M(s) |; | t-s | < \delta, t, s \in I\}$. Hence we have

$$w_{X^n}(\delta) \leq 2w_M(\delta) + w_M(T/n) \quad \text{with Proposition 1.}$$

The continuity property of $w_M(\cdot)$ implies

$$\forall\alpha > 0,\ \forall\epsilon > 0,\ \exists\delta_0 < 1,\ \forall\delta < \delta_0,\ \exists n_0,\ \forall n \geq n_0:$$
$$P^{X^n}(X \in C_I; w_X(\delta) \leq \epsilon) \geq 1-\alpha.$$

This together with Theorem 4 implies tightness of the sequence (P^{X^n}). Thus (Theorem 1) there exists a subsequence $(P^{X^{n_k}})$ weakly convergent to some probability measure ν on C_I. Let X be a random element on $(\Omega, \mathcal{F}, P)$ with values in C_I such that $P^X = \nu$. One has $(X^{n_k}, X_0) \xrightarrow{d} (X, X_0)$ (c.f. Th. 4.4 [1]). Thus there exists a sequence (X'^{n_k}, X_0') and a random element (X', X_0') on some probability space $(\Omega', \mathcal{F}', P')$ with values in $C_I x C_I$ such that

i) $(X'^{n_k}, X'_0) \stackrel{d}{=} (X^{n_k}, X_0)$, for $k = 1, 2, \ldots$
ii) $(X', X'_0) \stackrel{d}{=} (X, X_0)$
iii) $(X'^{n_k}, X'_0) \to (X', X'_0)$ with P'.1

(see e.g. [13]). This implies $X'_t = X'_0 + \int_0^t F(s, X'_s)ds$ with P'.1 for $t \in I$. Hence by Proposition 1 one gets

$$D_H X'_0 = F(t, X'_0) \ \ \mathrm{P}'.1, \ \ t \in [0,T]\text{–a.e.}$$

Since $X'_0 \stackrel{d}{=} X_0$ and $X_0 \stackrel{d}{=} \mu$ then $X'_0 \stackrel{d}{=} \mu$. This completes the proof.

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