DEMONSTRATIO MATHEMATICA
Vol. XXIX No 3 1996

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FURTHER PROPERTIES OF ALMOST CONTINUOUS MULTIFUNCTIONS DEFINED BETWEEN BITOPOLOGICAL SPACES

1. Introduction

Kelly [6] introduced the notion of bitopological spaces as a natural generalization of topological spaces. Thereafter tremendous developement in this direction has been noticed and a vast number of papers have appered extending and generalizing many topological concepts to bitopological situations.

By bitopological space (X, P_1, P_2) we shall always mean a space X endowed with two topologies P_1 and P_2. Throughout this paper spaces always mean bitopological spaces on which no pairwise separation axioms are assumed unless explicitly stated. The bitopological spaces (X, P_1, P_2) and (Y, Q_1, Q_2) will sometimes be abbreviated as X and Y, respectively.

Let S be a subset of a space X. The P_i-closure of S and the P_i-interior of S are denoted by P_i-cl(S) and P_i-int(S), respectively. A subset S of X is called an (i,j)-regular open set ((i,j)-regular closed) [(briefly (i,j)-ro, (((i,j)-rc)] iff $S = P_i\text{-int}(P_j\text{-cl}(S))$ [resp. $S = P_i\text{-cl}(P_j\text{-int}(S))$] [16]. It is easy to see that S is an (i,j)-ro set in X iff X-S is a (i,j)-rc set, $i,j = 1,2$ and $i \neq j$. A space (X, P_1, P_2) is called pairwise semi regular iff P_i has a base consisting of all (i,j)-ro sets of X, for $i,j = 1,2$ and $i \neq j$. A space (X, P_1, P_2) is said to be (i,j)-almost regular iff for each P_i-open set V and $x \in V$, there is a P_i-open set U of X such that $x \in U \subseteq P_j$-cl$(U) \subseteq P_i\text{-int}(P_j\text{-cl}(V))$, $i,j = 1,2$ and $i \neq j$ [16]. A space (X, P_1, P_2) is called P_1P_2-paracompact iff every cover W of X with P_1-open sets has a

Key words and phrases: Almost continuity, weak continuity, semi continuity, P_1P_2-upper(lower) almost continuity with respect to Q_2, P_1P_2-upper(lower)weak continuity with respect to Q_2, multifunction, bitopological space.

1991 *Mathematics Subject Classification*: 54C60-54E55.

refinement V with P_1-open sets which cover X and V is P_2-locally finite, i.e. for each point x, there is a P_2-open neighbourhood U of x intersecting at most finitely many elements of V.

A point x in X will be called an (i,j)-δ-closure point of a subset S of X iff $S \cap U \neq \phi$ for any (i,j)-regular open set U containing x, where $i,j = 1,2$ and $i \neq j$. The set of all (i,j)-δ-closure points of S is called (i,j)-δ-closure of S and it is denoted by (i,j)-δ-cl(S). A subset S of X is called (i,j)-δ-closed if $S = (i,j)$-δ-cl(S). A point x in X will be called an (i,j)-δ-interior point of a subset S of X iff there exists a (i,j)-regular open set U containing x and contained in S. The set of all (i,j)-δ-interior points of S is called (i,j)-δ-interior of S and it is denoted by (i,j)-δ-int(S). A subset S of X is called (i,j)-δ-open iff $S = (i,j)$-δ-int(S). Let S be a subset of X, U is a (i,j)-δ-neighbourhood of S which intersects S, if there exists a (i,j)-δ-open subset V of X such that $V \subset U$ and $V \cap S \neq \phi$.

The family of all (i,j)-δ-open [(i,j)-δ-closed] and (i,j)-regular open [(i,j)-regular closed] sets of X are denoted by (i,j)-δ-$O(X)$[(i,j)-δ-$C(X)$] and (i,j)-RO(X) [(i,j)-RC(X)], respectively, [1].

The net $(x_\alpha)_{\alpha \in I}$ is $(1,2)$-δ-convergent to x_0, if for each $(1,2)$-regular open set U containing x_0, there exists a $\alpha_0 \in I$ such that $\alpha \geq \alpha_0 \Rightarrow x_\alpha \in U$.

The concept of almost continuous functions between topological spaces were first introduced and studied by Singal and Singal [17]. The same idea was further developed by many authors, e.g., we refer to the papers of Noiri [9], Herrington [5] and Long and Carnahan [5]. Also, there has appeared a large number of papers which deal with almost continuous functions in relation to other types of functions. The notion of almost continuous multifunctions was introduced and investigated by Popa [11, 13, 15], in which he generalized the definition and properties of almost continuous functions by introducing the notion of upper and lower almost continuous multifunctions. Boshe and Sinha [3] extended the idea of almost continuous single-valued-functions to bitopological space. Mukherjee and Ganguly [10] also extended the idea of almost continuous multifunctions to bitopological space.

A multifunction F of a set X into Y is a correspondence such that $F(x)$ is a nonempty subset of Y, for each $x \in X$, that is it is a function $F : X \to P(Y) \setminus \{\phi\}$, where $P(Y)$ is the power set of Y. We will denote such a multifunction by $F : X \to Y$. For a multifunction F, the upper and lower inverse of a set B of Y will be denoted by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \phi\}$ [2]. A multifunction $F : (X,\tau) \to (Y,\omega)$ is upper semi continuous (in short, u.s.c.) at a point $x_0 \in X$ if for any open set $V \subseteq Y$ such that $F(x_0) \subseteq V$, there exists an open set $U \subseteq X$ containing x_0 such that $F(U) \subseteq V$. A multifunction $F : (X,\tau) \to (Y,\omega)$ is lower semi-continuous (in short, l.s.c.) at a point $x_0 \in X$ if for any open set $V \subseteq Y$ such that $F(x_0) \cap V \neq \phi$, there exists an open set $U \subseteq X$ containing x_0 such that $U \subseteq F^-(V)$ [6]. A multifunction $F : (X, P_1, P_2) \to (Y, Q_1, Q_2)$ is P_1Q_1-upper almost continuous with respect to Q_2 (in short, P_1Q_1-u.a.c.w.r.t. Q_2) at a point $x_0 \in X$ if for any Q_1-open set $V \subseteq Y$ such that $F(x_0) \subseteq V$, there exists an P_1-open set $U \subseteq X$ containing x_0 such that $F(U) \subseteq Q_1$-int(Q_2-cl(V)). A multifunction $F : (X, P_1, P_2) \to (Y, Q_1, Q_2)$ is P_1Q_1-lower almost continuous with respect to Q_2 (in short, P_1Q_1-l.a.c.w.r.t. Q_2) at a point $x_0 \in X$ if for any Q_1-open set $V \subseteq Y$ such that $F(x_0) \cap V \neq \phi$, there exists a P_1-open set $U \subseteq X$ containing x_0 such that $U \subseteq F^-(Q_1$-int(Q_2-cl(V)) [10]. A multifunction $F : X \to Y$ is P_1Q_1-upper weakly continuous with respect to Q_2 (in short, P_1Q_1-u.w.c.w.r.t. Q_2) at a point $x_0 \in X$ if for any Q_1-open set $V \subseteq Y$ such that $F(x_0) \subseteq V$, there exists a P_1-open set $U \subseteq X$ containing x_0 such that $F(U) \subseteq Q_2$-cl(V). A multifunction $F : X \to Y$ is P_1Q_1-lower weakly continuous with respect to Q_2 (in short, P_1Q_1-l.w.c.w.r.t. Q_2) at a point $x_0 \in X$ if for any Q_1-open set $V \subseteq Y$ such that $F(x_0) \cap V \neq \phi$, there exists a P_1-open set $U \subseteq X$ containing x_0 such that $U \subseteq F^-(Q_2$-cl(V)) [12].

2. Some characterisations of the lower almost continuous multifunctions between bitopological spaces

THEOREM 2.1. *Let (X, P_1, P_2) and (Y, Q_1, Q_2) be bitopological spaces. For any multifunction $F : X \to Y$ the following are equivalent:*

(1) *F is P_1Q_1-l.a.c.w.r.t. Q_2 at a point $x_0 \in X$;*

(2) *For each $y \in F(x_0)$ and for every net $(x_\alpha)_{\alpha\in I}$ convergent to x_0 (with respect to P_1), there exists a subnet $(z_\beta)_{\beta\in J}$ of the net $(x_\alpha)_{\alpha\in I}$ and a net $(y_\beta)_{\beta\in J}$ in Y so that $(y_\beta)_{\beta\in J}$ is $(1,2)-\delta$-convergent to y and $y_\beta \in F(z_\beta)$;*

(3) *$x_0 \in P_1$-cl$(A) \Rightarrow F(x_0) \subset (1,2)$-$\delta$-cl$(F(A))$, for each $A \subset X$;*

(4) *$x_0 \in P_1$-cl$(F^+(N)) \Rightarrow x_0 \in F^+((1,2)$-$\delta$-cl$(N))$, for each $N \subset Y$.*

Proof. (1)⇒(2): Suppose that F is P_1Q_1-l.a.c.w.r.t. Q_2 at a point $x_0 \in X$. Let $(x_\alpha)_{\alpha\in I}$ be a net convergent to x_0 and $y \in F(x_0)$. If $V \in (1,2)$-RO(Y, y) where $(1,2)$-RO(Y, y) denote the family of all $(1,2)$-regular open sets which contains y, then $F(x_0) \cap V \neq \phi$. Since F is P_1Q_1-l.a.c.w.r.t. Q_2 at a point $x_0 \in X$, there exists a P_1-open set U containing x_0 such that $F(z) \cap (Q_1$-int$(Q_2$-cl$(V)) \neq \phi$, $z \in U$ or $F(z) \cap V \neq \phi$, $z \in U$. Since the net $(x_\alpha)_{\alpha\in I}$ is convergent to x_0, for this U, there exists a $\alpha_0 \in I$ such that $\alpha > \alpha_0 \Rightarrow x_\alpha \in U$. Therefore we have the implication $\alpha > \alpha_0 \Rightarrow F(x_\alpha) \cap V \neq \phi$. For any $V \in (1,2)$-RO(Y, y), define $I_V = \{\alpha_0 \in I \mid \alpha > \alpha_0 \Rightarrow F(x_\alpha) \cap V \neq \phi\}$, $J = U\{I_V \times V \mid V \in (1,2)\text{-RO}(Y, y)\} = \{(\alpha, V) \mid \alpha \in I_V, V \in (1,2)\text{-RO}(Y, y)\}$ and introduce an order on J as follows; $(\alpha^*, V^*) > (\alpha, V) \Leftrightarrow \alpha^* >$

α and $V^* \subset V$. This is a direction on J. Define $\Omega : J \to I$ by $\Omega[(\beta, V)] = \beta$. Then Ω is increasing and cofinal. So Ω defines a subnet $(x_{\Omega[(\beta,V)]})_{(\beta,V)\in J}$ of $(x_\alpha)_{\alpha\in I}$. We denote the subnet $(x_{\Omega[(\beta,V)]})_{(\beta,V)\in J}$ by $(z_\beta)_{(\beta,V)\in J}$. On the other hand for any $(\beta^*, V^*) \in J$, $(\beta, V) > (\beta^*, V^*) \Rightarrow F(z_\beta) \cap V \neq \phi$. Pick $y_\beta \in F(z_\beta) \cap V$. Then the net $(y_\beta)_{(\beta,V)\in J}$ is $(1,2)$-δ-convergent to y. To see this notice that for any $V_0 \in (1,2)\text{-RO}(Y, y)$ there exists a $\alpha_0 \in I$ such that $(\alpha_0, V_0) \in J$ and $y_{\alpha_0} \in V_0$. If $(\beta, V) > (\beta_0, V_0)$ then $\beta > \beta_0$ and $V \subset V_0$. Therefore $y_\beta \in F(z_\beta) \cap V \subset F(z_\beta) \cap V_0$, so $y_\beta \in V_0$. Thus $(y_\beta)_{(\beta,V)\in J}$ is $(1,2)$-δ-convergent to y.

(2) $\Rightarrow$ (3): Let $x_0 \in P_1\text{-cl}(A)$, then there exists a net $(x_\alpha)_{\alpha\in I}$ convergent to x_0. Let $y \in F(x_0)$. By the hypothesis, there exists a subnet $(z_\beta)_{\beta\in J}$ of $(x_\alpha)_{\alpha\in I}$ and a net $(y_\beta)_{\beta\in J}$ in Y so that $(y_\beta)_{\beta\in J}$ is $(1,2)$-δ-convergent to y and $y_\beta \in F(z_\beta)$. This implies that $y \in (1,2)\text{-}\delta\text{-cl}(F(A))$ and so $F(x_0) \subset (1,2)\text{-}\delta\text{-cl}(F(A))$.

(3) $\Rightarrow$ (4): For any $N \subset Y$ and $x_0 \in P_1\text{-cl}(F^+(N))$. Replacing A by $F^+(N)$ in (3), we get $F(x_0) \subset (1,2)\text{-}\delta\text{-cl}(F(F^+(N))) \subset (1,2)\text{-}\delta\text{-cl}(N)$. So $x_0 \in F^+((1,2)\text{-}\delta\text{-cl}(N))$.

(4) $\Rightarrow$ (1): Let (1) be not true. Then, there is a Q_1-open set G in Y with $F(x_0) \cap G \neq \phi$ such that for each $U \in P_1(x_0)$, where $P_1(x_0)$ is the family of P_1-open sets containing x_0, there is $x_U \in U$ for which $F(x_U) \cap (Q_1\text{-int}(Q_2\text{-cl}(G)) = \phi$. We set $M = \{x_U \mid U \in P_1(x_0)\}$. Then we have $x_0 \in P_1\text{-cl}(M) \subset P_1\text{-cl}(F^+(F(M)))$. From (4), we obtain $x_0 \in (1,2)\text{-}\delta\text{-cl}(F^+(F(M)))$. So $F(x_0) \subset (1,2)\text{-}\delta\text{-cl}(F(M))$. On the other hand, since $F(x_0) \cap G \neq \phi$, there exists a point z in Y such that $z \in F(x_0) \cap G$, $z \in G \in Q_1$. The fact that $Q_1\text{-int}(Q_2\text{-cl}(G))$ is $(1,2)$-regular open set in Y and $F(M) \cap (Q_1\text{-int}(Q_2\text{-cl}(G))) = \phi$ implies $z \notin (1,2)\text{-}\delta\text{-cl}(F(M))$. This contradicts with $F(x_0) \subset (1,2)\text{-}\delta\text{-cl}(F(M))$ and shows that (1) is true.

THEOREM 2.2. *Let (X, P_1, P_2) and (Y, Q_1, Q_2) be bitopological spaces. For any multifunction $F : X \to Y$ the following statements are equivalent:*

(1) *F is P_1Q_1-l.a.c.w.r.t. Q_2.*

(2) *For each $x \in X$ and for each $(1,2) - \delta$-neighbourhood V which intersects $F(x)$, $F^-(V)$ is a P_1-neighbourhood of x.*

(3) *For each $x \in X$ and for each $(1,2) - \delta$-neighbourhood V which intersects $F(x)$, there is $U \in P_1(x)$ such that, $F(z) \cap V \neq \phi$, for each $z \in U$.*

(4) *$F(P_1\text{-cl}(A)) \subset (1,2)\text{-}\delta\text{-cl}(F(A))$, for each $A \subset X$.*

(5) *$P_1\text{-cl}(F^+(B)) \subset F^+((1,2)\text{-}\delta\text{-cl}(B))$, for each $B \subset Y$.*

(6) *For each $(1,2)$-δ-closed subset B of Y, $F^+(B)$ is a P_1-closed subset of X.*

(7) *For each* $(1,2)$-δ*-open subset* B *of* Y, $F^+(B)$ *is a* P_1*-open subset of* X.

(8) $F^-((1,2)\text{-}\delta\text{-int}(B)) \subset P_1\text{-int}(F^-(B))$, *for each* $B \subset Y$.

Proof. (1) $\Rightarrow$ (2): Let $x \in X$ and $V \subset Y$ be a $(1,2)$-δ-neighbourhood V which intersects $F(x)$. Then there is a $(1,2)$-δ-open set G contained in V which intersects $F(x)$ i.e. $G \subset V$ and $F(x) \cap G \neq \phi$. There exists $y \in Y$ such that $y \in F(x) \cap G$, so $y \in G$. Since G is $(1,2)$-δ-open set, there is a $(1,2)$-regular open set T containing y such that $y \in T \subset G$, but this implies that $F(x) \cap T \neq \phi$. According to the Theorem 2.4 [8], there is a P_1-open set U containing x so that $F(z) \cap T \neq \phi$, for each $z \in U$, so $U \subset F^-(T)$. Since $T \subset G \subset V$, then we have $x \in U \subset F^-(T) \subset F^-(G) \subset F^-(V)$. Thus $F^-(V)$ is a P_1-neighbourhood of x.

(2) $\Rightarrow$ (3): Let $x \in X$ and $V \subset Y$ be a $(1,2) - \delta$-neighbourhood V which intersects $F(x)$. According to the hypothesis, $U = F^-(V)$ is a P_1-neighbourhood of x and $F(z) \cap V \neq \phi$, for each $z \in U$.

(3) $\Rightarrow$ (4): Let $y \in F(P_1 - \text{cl}(A))$. Then there exists a point x in P_1-cl(A) such that $y \in F(x)$. Let $G \in (1,2)\text{-RO}(Y,y)$. Then we have $F(x) \cap G \neq \phi$. The fact that a set G is $(1,2)$-RO implies that G is $(1,2)$-δ-open set. By the hypothesis, there exists a $U \in P_1(x)$ such that $U \subset F^-(G)$. On the other hand since $x \in P_1$-cl(A), we have $U \cap A \neq \phi$. So there exists a point z in X such that $z \in U \cap A$. Therefore we obtain $F(z) \cap G \neq \phi$ and $F(z) \subset F(A)$. Finally, we have $F(A) \cap G \neq \phi$, which implies $y \in (1,2)\text{-}\delta\text{-cl}(F(A))$. Since $y \in F(P_1\text{-cl}(A))$ is arbitrary, we get $F(P_1\text{-cl}(A)) \subset (1,2)\text{-}\delta\text{-cl}(F(A))$.

(4) $\Rightarrow$ (5): Let B be any subset of Y. So we have $F^+(B) \subset X$. By (4), we obtain $F(P_1\text{-cl}(F^+(B))) \subset (1,2)\text{-}\delta\text{-cl}(F(F^+(B))) \subset (1,2)\text{-}\delta\text{-cl}(B)$. Then we have $P_1\text{-cl}(F^+(B)) \subset F^+((1,2)\text{-}\delta\text{-cl}(F(F^+(B)))) \subset F^+((1,2))\text{-}\delta\text{-cl}(B))$.

(5) $\Rightarrow$ (6): Let B be any $(1,2)$-δ-closed subset of Y, so $B = (1,2)\text{-}\delta\text{-cl}(B)$. According to the hypothesis, $P_1\text{-cl}(F^+(B)) \subset F^+((1,2)\text{-}\delta\text{-cl}(B)) = F^+(B)$. Thus $F^+(B)$ is P_1-closed subset of X.

(6) $\Rightarrow$ (7): Let B be any $(1,2)$-δ-open subset of Y. So $Y \setminus B$ is $(1,2)$-δ-closed subset of Y. By (6), $F^+(Y \setminus B) = Y \setminus F^-(B)$ is P_1-closed subset of X. Thus $F^-(B)$ is P_1-open subset of X.

(7) $\Rightarrow$ (8): For any $B \subset Y$, $(1,2)$-δ-int(B) is $(1,2)$-δ-open subset of Y. By (7), $F^-((1,2)\text{-}\delta\text{-int}(B))$ is P_1-open subset of X. Then we obtain $F^-((1,2)\text{-}\delta\text{-int}(B)) \subset P_1\text{-int}(F^-(B))$.

(8) $\Rightarrow$ (1): Let G be any $(1,2)$-regular open subset of Y. Then G is $(1,2)$-δ-open subset of Y and $G = (1,2)\text{-}\delta\text{-int}(G)$. By (8), we have $F^-(G) \subset P_1\text{-int}(F^-(G))$, which shows that $F^-(G)$ is P_1-open subset of X. According to Theorem 2.6 in [10], F is P_1Q_1-l.a.c.w.r.t. Q_2.

Theorems 2.1 and 2.2 generalize Theorems 4,5 in [15].

3. Weak continuity and almost continuity of the multifunction

THEOREM 3.1. *Let* (X, P_1, P_2) *and* (Y, Q_1, Q_2) *be bitopological spaces. If a multifunction* $F : X \to Y$ *is* P_1Q_1*-u.w.c.w.r.t.* Q_2 *and* $F : (X, P_1) \to (Y, Q_1)$ *open then* F *is* P_1Q_1*-u.a.c.w.r.t.* Q_2.

Proof. Suppose that F is P_1Q_1-u.w.c.w.r.t. Q_2 at $x \in X$. Then for any Q_1-open subset V of Y with $F(x) \subset V$, there is a P_1-open set U containing x of X such that $F(U) \subset Q_2\text{-cl}(V)$. As F is P_1Q_1-open, $F(U)$ is Q_1-open, which implies $F(U) \subseteq Q_1\text{-int}(Q_2\text{-cl}(V))$.

Above Theorem generalizes Theorem 2.1 in [13].

COROLLARY 3.2. *Let* (X, P_1, P_2) *and* (Y, Q_1, Q_2) *be bitopological spaces. If a multifunction* $F : X \to Y$ *is* P_1Q_1*-u.w.c.w.r.t.* Q_2 *and* $F : X \to (Y, Q_1)$ *point-open then* F *is* P_1Q_1*-u.a.c.w.r.t.* Q_2.

Proof. By the previous theorem, the proof is clear.

THEOREM 3.3. *Let* (Y, Q_1, Q_2) *be a* $(1,2)$*-almost regular,* Q_1Q_2*-paracompact space and* $F : (X, P_1, P_2) \to (Y, Q_1, Q_2)$ *be a point-*P_1*-closed, then the notion* P_1Q_1*-u.w.c.w.r.t.* Q_2 *coincides with the notion* P_1Q_1*-u.a.c.w.r.t.* Q_2.

Proof. ($\Rightarrow$) Let V be a $(1,2)$-regular open set in Y and $x \in F^+(V)$, namely $F^+(V) \subset X$. Since Y is $(1,2)$-almost regular, for each $y \in F(x)$, there is a Q_1-open set V_y containing y such that $V_y \subset Q_2\text{-cl}(V_y) \subset Q_1\text{-int}(Q_2\text{-cl}(V)) = V$. Thus we have $F(x) \subset \bigcup\{V_y \mid y \in F(x)\} \subset \bigcup\{Q_2\text{-cl}(V_y) \mid y \in F(x)\} \subset V$. Therefore the family $\{Y \setminus F(x)\} \cup \{V_y \mid y \in F(x)\}$ is is a Q_1-open cover of Y. Since Y is Q_1Q_2-paracompact, there exists a point finite Q_2-open refinement G of this cover. So for each $y \in F(x)$, there exists a $G_y \in G$ such that $G_y \subset V_y$ and we have $F(x) \subset \bigcup\{G_y \mid y \in F(x)\} \subset \bigcup\{V_y \mid y \in F(x)\} \subset \bigcup\{Q_2\text{-cl}(V_y) \mid y \in F(x)\}$. If we set $G = \bigcup\{G_y \mid y \in F(x)\}$, then $Q_2\text{-cl}(G) = \bigcup\{Q_2\text{-cl}(G_y) \mid y \in F(x)\}$, so $F(x) \subset G \subset Q_2\text{-cl}(G) \subset V$. Since F is P_1Q_1-u.w.c.w.r.t. Q_2 at x, there is a P_1-open set U containing x such that $F(U) \subset Q_2\text{-cl}(V) \subset V$. Then $x \in U \subset F^+(V)$. So $F^+(V)$ is P_1-open set in X and F is P_1Q_1-u.a.c.w.r.t. Q_2 at x [8, Theorem 2.8].

($\Leftarrow$) By the definitions of continuities, the proof is clear.

This Theorem generalize Theorem 2.3 in [14].

The following example shows that the relative continuity (P_1Q_1-u.s.c. w.r.t. Q_2) of $F : (X, P_1, P_2) \to (Y, Q_1, Q_2)$ does not imply the upper semi continuity (u.s.c.) of $F : (X, P_1) \to (Y, Q_1)$. The reverse implication is always true.

EXAMPLE 3.4. Let $X = \mathbb{R}$ be a bitopological space with the usual *topology* P_1 and the cofinal topology P_2. Let $Y = \{a, b, c\}$ be a bitopological space with $Q_1 = Q_2$ =discreate topology. A multifunction $F : (X, P_1, P_2) \to$

(Y, Q_1, Q_2) defined by

$$F(x) = \begin{cases} \{a\}; & x \in (-\infty, 0] \\ \{b, c\}; & x \in (0, \infty) \end{cases}$$

is relatively P_1Q_1-u.s.c.w.r.t. Q_2, but not P_1Q_1-u.a.c.w.r.t. Q_2. In addition $F : (X, P_1) \to (Y, Q_1)$ is not u.s.c. To show this, take any $x \in (-\infty, 0]$. Then $F(x) = \{a\}$ and $F^+(\{a\}) = F^+(Q_1\text{-Int}(Q_2\text{-Cl}(\{a\}))) = (-\infty, 0] = F^+(Q_1\text{-Int}(Q_2\text{-Cl}(\{a\}))) \cap (-\infty, 1)$. We also take $x \in (0, \infty)$, then $F(x) = \{b, c\}$ and $F^+(\{b, c\}) = F^+(Q_1\text{-Int}(Q_2\text{-Cl}(\{b, c\}))) = (0, \infty) = F^+(Q_1\text{-Int}(Q_2\text{-Cl}(\{b, c\}))) \cap (-1, \infty)$. F is not Q_1-u.w.c.w.r.t. Q_2 at $x = 0$. $F(0) = \{a\} \subset \{a\} \in Q_1$ but for all $\varepsilon > 0$ $F((-\varepsilon, \varepsilon)) = Y \not\subset Q_1\text{-Int}(Q_2\text{-Cl}(\{a\})) = \{a\}$. It can be easily seen that $F : (X, P_1) \to (Y, Q_1)$ is not u.s.c. at $x = 0$.

DEFINITION 3.5. A multifunction $F : (X, P_1, P_2) \to (Y, Q_1, Q_2)$ is relatively P_1Q_1-u.s.c.w.r.t. Q_2 at x iff given a $V \in Q_1$ with $F(x) \subset V$, the set $F^+(V)$ is an open set in the subspace $(F^+(Q_1\text{-int}(Q_2\text{-cl}(V))), (P_1)_{F^+(Q_1\text{-int}(Q_2\text{-cl}(V)))})$ [4].

THEOREM 3.6. *If $F : (X, P_1) \to (Y, Q_1)$ is u.s.c., then $F : (X, P_1, P_2) \to (Y, Q_1, Q_2)$ is relatively P_1Q_1-u.s.c.w.r.t. Q_2.*

Proof. Let $x \in X$, $V \in Q_1$ with $F(x) \subset V$ and $F : (X, P_1) \to (Y, Q_1)$ be a u.s.c. Then $F^+(V) \in P_1$. Since $F^+(V) \subset F^+(Q_1\text{-int}(Q_2\text{-cl}(V)))$ and $F^+(V) \cap F^+(Q_1\text{-int}(Q_2\text{-cl}(V))) = F^+(V)$. $F^+(V)$ is an open set in the subspace $F^+(Q_1\text{-int}(Q_2\text{-cl}(V)))$. So F is relatively P_1Q_1-u.s.c.w.r.t. Q_2.

THEOREM 3.7. *A multifunction $F : (X, P_1) \to (Y, Q_1)$ is u.s.c iff $F : (X, P_1, P_2) \to (Y, Q_1, Q_2)$ is relatively P_1Q_1-u.s.c.w.r.t. Q_2 and P_1Q_1-u.a.c. w.r.t. Q_2.*

Proof. ($\Rightarrow$) This part of the theorem is clear.

($\Leftarrow$) Let $V \in Q_1$. Since F is relatively P_1Q_1-u.s.c.w.r.t. Q_2, then $F^+(V)$ is an open set in the subspace $F^+(Q_1\text{-int}(Q_2\text{-cl}(V)))$. So we have $W \cap F^+(Q_1\text{-int}(Q_2\text{-cl}(V))) = F^+(V)$, where W is P_1-open set in X. To prove the openness of $F^+(V)$ in X, let $x \in F^+(V)$, what gives $F(x) \subset V$ and $x \in W$. Since F is P_1Q_1-u.a.c.w.r.t. Q_2 at x there exists an P_1-open set U in X containing x such that $F(U) \subset Q_1\text{-int}(Q_2\text{-cl}(V))$. Since $x \in W$ and W is P_1-open set in X, we may assume that $U \subset W$. It now follows that $x \in U \subset W \cap F^+(Q_1\text{-int}(Q_2\text{-cl}(V))) = F^+(V)$. This shows that $F^+(V)$ is P_1-open in X. Consequently $F : (X, P_1) \to (Y, Q_1)$ is u.s.c..

DEFINITION 3.8. A multifunction $F : (X, P_1, P_2) \to (Y, Q_1, Q_2)$ has P_1Q_1-upper interior condition with respect to Q_2 (briefly P_1Q_1-u.i.c.w.r.t. Q_2) iff for each Q_1-open set V in Y, a multifunction F satisfies $P_1\text{-Int}(F^+(Q_1\text{-int}(Q_2\text{-cl}(V))) \subset F^+(V)$ [5].

The following example gives that the upper semi continuity of $F : (X, P_1) \to (Y, Q_1)$ does not imply the upper interior condition of $F : (X, P_1, P_2) \to (Y, Q_1, Q_2)$. By considering 3.4. Example, it is easily seen that the reverse implication is not true.

EXAMPLE 3.9. Let $X = \mathbb{N}$ be a bitopological space with cofinal topology P_1 and discrete topology. Let $Y = \mathbb{N}$ be a bitopological space with $Q_1 = Q_2$ =cofinal topology. A multifunction $F : (X, P_1, P_2) \to (Y, Q_1, Q_2)$ defined by,

$$F(x) = \begin{cases} \{1,2,3\}; & x = 1 \\ \{x\}; & x \neq 1. \end{cases}$$

F is surely u.s.c. But if V is any proper nonempty Q_1-open set in Y, then Q_1-Int(Q_2-Cl(V)) $= Y$ and so Int $F^+(Q_1\text{-Int}(Q_2\text{-Cl}(V))) = X$ while $F^+(V) = V$. Hence $F^+(Q_1\text{-Int}(Q_2\text{-Cl}(V))) \not\subset F^+(V)$. So F is not P_1Q_1-u.i.c.w.r.t. Q_2.

THEOREM 3.10. *If a multifunction $F : (X, P_1, P_2) \to (Y, Q_1, Q_2)$ is P_1Q_1-u.a.c.w.r.t. Q_2 and F has a P_1Q_1-u.i.c.w.r.t. Q_2 then the multifunction $F : (X, P_1) \to (Y, Q_1)$ is u.s.c.*

Proof. Let V be a given Q_1-open set in Y. By the first hypothesis, we have $F^+(V) \subset P_1\text{-Int}(F^+(Q_1\text{-int}(Q_2\text{-cl}(V)))$ [10, Theorem 2.8]. By the interior condition, we have $P_1\text{-Int}(F^+(Q_1\text{-int}(Q_2\text{-cl}(V))) \subset F^+(V)$. Hence we have $P_1\text{-Int}(F^+(Q_1\text{-int}(Q_2\text{-cl}(V))) = F^+(V)$. Therefore we obtain $P_1\text{-Int}(F^+(V)) = P_1\text{-Int}(P_1\text{-Int}(F^+(Q_1\text{-int}(Q_2\text{-cl}(V))) = P_1\text{-Int}(F^+(Q_1\text{-int}(Q_2\text{-cl}(V))) = F^+(V)$. So $F^+(V)$ is P_1-open in X. This shows that $F : (X, P_1) \to (Y, Q_1)$ is u.s.c.

THEOREM 3.11. *Let (X, P_1, P_2) be a bitopological space and (Y, Q_1, Q_2) be a pairwise normal bitopological space. If for each pair of different points x_1, x_2 in X, there is a multifunction $F : X \to Y$ which has the following properties:*

(1) *F is Q_i-point closed $(i = 1, 2)$,*
(2) *F is P_1Q_1-u.w.c.w.r.t. Q_2 at x_1,*
(3) *F is P_2Q_2-u.a.c.w.r.t. Q_1 at x_2 and*
(4) *$F(x_1) \cap F(x_2) = \phi$, then (X, P_1, P_2) is a pairwise Hausdorff space.*

Proof. Let x_1, x_2 be different points in X. By the hypothesis, for these points, there is a multifunction F such that $F(x_1)$ is Q_2-closed, $F(x_2)$ is Q_1-Closed and $F(x_1) \cap F(x_2) = \phi$. Since (Y, Q_1, Q_2) is pairwise normal, there are two sets $V_1 \in Q_1$, $V_2 \in Q_2$ such that $F(x_1) \subset V_1$, $F(x_1) \subset V_2$ and $V_1 \cap V_2 = \phi$. So we have $Q_1\text{-int}(Q_2\text{-cl}(V_1)) \cap Q_2\text{-int}(Q_1\text{-cl}(V_2) = \phi$, which implies $Q_2\text{-cl}(V_1) \cap Q_2\text{-int}(Q_1\text{-cl}(V_2) = \phi$. Since F is P_1Q_1-u.w.c.w.r.t. Q_2 at x_1 and F is P_1Q_1-u.a.c.w.r.t. Q_2 at x_2, there are sets $U_1 \in P_1(x_1)$ and

$U_2 \in P_2(x_2)$ such that $F(U_1) \subset Q_2\text{-cl}(V_1)$ and $F(U_2) \subset Q_2\text{-int}(Q_1\text{-cl}(V_2))$. It follows that $F(U_1) \cap F(U_2) = \phi$, what implies that $U_1 \cap U_2 = \phi$. The latter means that is X is a pairwise Hausdorff space.

THEOREM 3.12. *Let (X, P_1, P_2) and (Y, Q_1, Q_2) be bitopological spaces. If multifunction $F : X \to Y$ is P_1Q_1-l.w.c.w.r.t. Q_2 and $F : X \to (Y, Q_1)$ point-open then F is P_1Q_1-l.s.c.*

Proof. Suppose that F is P_1Q_1-l.w.c.w.r.t. Q_2 at $x \in X$. Then for any Q_1-open subset V of Y with $F(x) \cap V \neq \phi$, there is a P_1-open set U containing x such that $F(z) \cap Q_2\text{-cl}(V) \neq \phi$, for each $z \in U$. Since $F(z)$ is Q_1-open, we see that $F(z) \cap V \neq \phi$, for each $z \in U$. Consequently F is P_1Q_1-l.s.c.

DEFINITION 3.13. A multifunction $F : (X, P_1, P_2) \to (Y, Q_1, Q_2)$ is relatively P_1Q_1-l.s.c.w.r.t. Q_2 at x iff given a $V \in Q_1$ with $F(x) \cap V \neq \phi$, the set $F^-(V)$ is an open set in the subspace $(F^-(Q_1\text{-int}(Q_2\text{-cl}(V))), (P_1)_{F^-(Q_1\text{-int}(Q_2\text{-cl}(V)))})$ [4].

THEOREM 3.14. *If $F : (X, P_1) \to (Y, Q_1)$ is l.s.c., then $F : (X, P_1, P_2) \to (Y, Q_1, Q_2)$ is relatively P_1Q_1-l.s.c.w.r.t. Q_2.*

Proof. Let $x \in X, V \in Q_1$ with $F(x) \cap V \neq \phi$ and $F : (X, P_1) \to (Y, Q_1)$ be a l.s.c. Then $F^-(V) \in P_1$. Since $F^-(V) \subset F^-(Q_1\text{-int}(Q_2\text{-cl}(V)))$ and $F^-(V) \cap F^-(Q_1\text{-int}(Q_2\text{-cl}(V))) = F^-(V)$, $F^-(V)$ is an open set in the subspace $F^-(Q_1\text{-int}(Q_2\text{-cl}(V)))$. So F is relatively P_1Q_1-l.s.c.w.r.t. Q_2.

THEOREM 3.15. *A multifunction $F : (X, P_1) \to (Y, Q_1)$ is l.s.c iff $F : (X, P_1, P_2) \to (Y, Q_1, Q_2)$ is relatively P_1Q_1-l.s.c.w.r.t. Q_2 and P_1Q_1-l.a.c. w.r.t. Q_2.*

Proof. ($\Rightarrow$) This part of the theorem is clear;

($\Leftarrow$) Let $V \in Q_1$. Since F is relatively P_1Q_1-l.s.c.w.r.t. Q_2, $F^-(V)$ is an open set in the subspace $F^-(Q_1\text{-int}(Q_2\text{-cl}(V)))$. So we have $W \cap F^-(Q_1\text{-int}(Q_2\text{-cl}(V))) = F^-(V)$, where W is P_1-open set in X. To prove the openness of $F^-(V)$ in X, let $x \in F^-(V)$, so $F(x) \cap V \neq \phi$ and $x \in W$. Since F is P_1Q_1-l.a.c.w.r.t. Q_2 at x there exists an P_1-open set U in X containing x such that $F(x) \cap Q_1\text{-int}(Q_2\text{-cl}(V)) \neq \phi$, for each $x \in U$. Since $x \in W$ and W is P_1-open set in X, we may assume $U \subset W$. It now follows that $x \in U \subset W \cap F^-(Q_1\text{-int}(Q_2\text{-cl}(V))) = F^-(V)$. This shows that $F^-(V)$ is P_1-open in X. Consequently $F : (X, P_1) \to (Y, Q_1)$ is l.s.c.

DEFINITION 3.16. A multifunction $F : (X, P_1, P_2) \to (Y, Q_1, Q_2)$ has P_1Q_1-lower interior condition with respect to Q_2 (briefly P_1Q_1-l.i.c.w.r.t. Q_2) iff for each Q_1-open set V in Y, the multifunction F satisfies $P_1\text{-Int}(F^-(Q_1\text{-int}(Q_2\text{-cl}(V))) \subset F^-(V)$ [5].

THEOREM 3.17. *If a multifunction $F : (X, P_1, P_2) \to (Y, Q_1, Q_2)$ is P_1Q_1-l.a.c.w.r.t. Q_2 and F has a P_1Q_1-l.i.c.w.r.t. Q_2 then the multifunction $F : (X, P_1) \to (Y, Q_1)$ is l.s.c.*

Proof. Let V be a given Q_1-open set in Y. By the first hypothesis, we have $F^-(V) \subset P_1\text{-Int}(F^-(Q_1\text{-int}(Q_2\text{-cl}(V)))$ [10, Theorem 2.6]. By the interior condition, we have $P_1\text{-Int}(F^-(Q_1\text{-int}(Q_2\text{-cl}(V))) \subset F^-(V)$. Hence we have $P_1\text{-Int}(F^-(Q_1\text{-int}(Q_2\text{-cl}(V))) = F^-(V)$. Therefore we obtain $P_1\text{-Int}(F^-(V)) = P_1\text{-Int}(P_1\text{-Int}(F^-(Q_1\text{-int}(Q_2\text{-cl}(V))) = P_1\text{-Int}(F^-(Q_1\text{-int}(Q_2\text{-cl}(V))) = F^-(V)$. So $F^-(V)$ is P_1-open in X. This shows that $F : (X, P_1) \to (Y, Q_1)$ is l.s.c..

THEOREM 3.18. *Let $F : (X, P_1, P_2) \to (Y, Q_1, Q_2)$ be a multifunction. If (Y, Q_1, Q_2) be a $(1,2)$-almost regular space and F is P_1Q_1-l.w.c.w.r.t. Q_2, then F is P_1Q_1-l.a.c.w.r.t. Q_2.*

Proof. Let V be a $(1,2)$-regular open set of Y and $x \in F^-(V)$; that is $F(x) \cap V \neq \phi$. From the hypothesis, there is a Q_1-open set G of Y such that $F(x) \cap G \neq \phi$ and $Q_2\text{-cl}(V) \subset V$. Since F is P_1Q_1-l.w.c.w.r.t. Q_2, so for any Q_1-open set G, there is a P_1-open set U of X such that $x \in U$ and $F(z) \cap Q_2\text{-cl}(G) \neq \phi$, for each $z \in U$. This means that $x \in U \subset F^-(V)$ and shows that $F^-(V)$ is P_1-open and by [10, Theorem 2.6] F is P_1Q_1-l.a.c.w.r.t. Q_2.

COROLLARY 3.19. *Let $F : (X, P_1, P_2) \to (Y, Q_1, Q_2)$ be a multifunction. If (Y, Q_1, Q_2) be a $(1,2)$-almost regular space, then the notion of P_1Q_1-l.w.c.w.r.t. Q_2 coincides with the notion of P_1Q_1-l.a.c.w.r.t. Q_2.*

Proof. By the previous theorem, the proof is clear.

4. Some properties of the almost continuous multifunctions

THEOREM 4.1. *If the multifunction $F : (X, P_1, P_2) \to (Y, Q_1, Q_2)$ is P_1Q_1-u.a.c.w.r.t. Q_2, P_2Q_2-l.a.c.w.r.t. Q_1 and Q_1-point -compact and X is P_1-H-closed with respect to P_2, then Y is Q_1-H-closed with respect to Q_2.*

THEOREM 4.2. *If the multifunction $F : (X, P_1, P_2) \to (Y, Q_1, Q_2)$ is a P_1Q_1-u.a.c.w.r.t. Q_2 and Q_1-point -compact and (X, P_1) is compact, then Y is Q_1-H-closed with respect to Q_2.*

Above theorems generalize Theorems 1, 2 in [15].

THEOREM 4.3. *Let $F : (X, P_1, P_2) \to (Y, Q_1, Q_2)$ be a multifunction and $F_G : (X, P_1, P_2) \to (X \times Y, P_1 \times Q_1, P_2 \times Q_2)$ be a graph-multifunction defined by $F_G(x) = \{x\} \times F(x)$ of F. Then F is P_1Q_1-u.a.c.w.r.t. Q_2 [P_1Q_1-l.a.c.w.r.t. Q_2] iff F_G is $P_1(P_1 \times Q_1)$-u.a.c.w.r.t. $P_2 \times Q_2$ [$P_1(P_1 \times Q_1)$-l.a.c.w.r.t. $P_2 \times Q_2$].*

P r o o f. ($\Rightarrow$) Let $x \in X$ and W be a $(P_1 \times Q_1)$-open set with $F_G(x) = \{x\} \times F(x) \subset W$. There exist subsets $R \in P_1(x)$ and $S \in Q_1$ such that $F_G(x) \subset R \times S \subset W$. Since F is P_1Q_1-u.a.c.w.r.t. Q_2 at x, there exists $U \in P_1(x)$ such that $F(U) \subset Q_1$-int(Q_2-cl(S). On the other hand $U \times Q_1$-int(Q_2-cl(S)) $\subset P_2$-cl(U) $\times Q_2$-cl(S) $= U \times Q_1$-int(Q_2-cl(S)) $\subset P_2 \times Q_2$-cl($U \times S$) or $P_1 \times Q_1$-int($U \times Q_1$-int(Q_2-cl(S)) $\subset P_1 \times Q_1$-int($P_2 \times Q_2$-cl($U \times S$)). As $U \times Q_1$-int(Q_2-cl(S)) is a $(P_1 \times Q_1)$-open set containing $F_G(x)$, we obtain $F_G(U) = U \times F(U) \subset U \times Q_1$-int($Q_2$-cl($S$)) $\subset P_1 \times Q_1$-int($P_2 \times Q_2$-cl($U \times S$)) $\subset W$. Consequently F_G is $P_1(P_1 \times Q_1)$-u.a.c.w.r.t. $P_2 \times Q_2$ at x. Since $x \in X$ is arbitrary, F_G is $P_1(P_1 \times Q_1)$-u.a.c.w.r.t. $P_2 \times Q_2$.

($\Leftarrow$) Let $x \in X$ and V be a Q_1-open set with $F(x) \subset W$. Then we have $F_G(x) = \{x\} \times F(x) \subset X \times V$. Since $X \times V \in P_1 \times Q_1$ and F_G is $P_1(P_1 \times Q_1)$-u.a.c.w.r.t. $P_2 \times Q_2$, there exists $U \in P_1(x)$ such that $F_G(U) \subset P_1 \times Q_1$-int($P_2 \times Q_2$-cl($X \times V$)). On the other hand $P_1 \times Q_1$-int($P_2 \times Q_2$-cl($X \times V$)) $= P_1 \times Q_1$-int($[P_2$-cl(X)$] \times [Q_2$-cl(V)$]$) $= P_1 \times Q_1$-int($X \times [Q_2$-cl(V)$]$) $= [P_1$-int(X)$] \times [Q_1$-int(Q_2-cl(V))$] = X \times [Q_1$-int(Q_2-cl(V))$]$. Therefore we obtain $F_G(U) = U \times F(U) \subset X \times [Q_1$-int($Q_2$-cl($V$))$]$. Consequently we have $F(U) \subset [Q_1$-int(Q_2-cl(V))$]$. Thus F is P_1Q_1-u.a.c.w.r.t. Q_2 at x. Since $x \in X$ is arbitrary, F is P_1Q_1-u.a.c.w.r.t. Q_2.

References

[1] G.K. Banerjee, *On pairwise almost strongly θ-continuous mappings*, Bull. Cal. Math. Soc., 79 (1987), 314–320.

[2] C. Berge, *Espaces Topologiques Fonctions Multivoques.* Paris, Dunod, (1959).

[3] S. Boshe, D. Sinha, *Pairwise almost continuous map and weakly continuous map in bitopological spaces*, Bull. Cal. Math. Soc., 74 (1982), 195.

[4] J. Chew and J. Tong, *Some remarks on weak continuity*, Amer. Math. Montly 98, 10 (1991), 931–934.

[5] L.L. Herrington, *Some properties preserved by almost continuous functions*, Bol. U. M. I. 10, (4), (1974), 556.

[6] J.C. Kelly, *Bitopological spaces*, Proc. London Math. Soc. 13 (1963), 71–89.

[7] P.E. Long and D.A. Carnahan, *Comparing almost continuous functions*, Proc. Amer. Math. Soc. 38 (3) (1973), 71.

[8] E. Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. (1951), 71.

[9] T. Noiri, *Between continuity and weak continuity*, Bol. U.M.I. 9 (4) (1974), 647.

[10] M.N. Mukherjee and S. Ganguly, *Generalization of almost continuous multifunctions to bitopological spaces*, Bull. Cal. Math. Soc. 79 (1987), 274–283.

[11] V. Popa, *Almost continuous multifunctions*, Mat. Vesnik, 6, 19, (34), (!982), 75.

[12] V. Popa, *Weakly continuous multifunctions defined on bitopological spaces* (Romanian), Stud. Cerc. Mat. 34 (1982), 561–567.

[13] V. Popa, *Some properties of almost continuous multifunctions*, Mat. Vesnik, 35, 4, (!983), 425–432.

[14] V. Popa, *On some weakened forms of continuity for multifunctions*, Mat. Vesnik 36 (1984), 339–350.

[15] V. Popa, *On almost continuous multifunctions*, Review of Research University of Novi Sad, Mathematics Series 15, (1985), 1.

[16] A.R. Singal and S.P. Arya, *On pairwise almost regular spaces*, Glasnik Math. 6 (26), (1971), 335.

[17] M.K. Singal and A.R. Singal, *Almost continuous mappings*, Yokohama Math.J. 16 (1968), 63–73.

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Received December 12, 1994; revised version June 7, 1995.