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REFLECTIONS IN EQUIDISTANT HYPERSURFACES II. GEOMETRIC CHARACTERIZATION OF THE GROUP GENERATED BY REFLECTIONS

1. Introduction

The group generated by reflections in equidistant hypersurfaces of degenerate hyperbolic space ($\mathbb{H}_k^n$) was described analitically in [5]. In this paper we give a geometric characterization of this group — first for $k = 1$ and next for $k > 1$. In the paper we shall use the notions and notations of [2], [3], [4], [5].

2. Results

In the family $\mathrm{Aut}(\overline{\mathbb{H}}_k^n)$ we define a group of transformations.

DEFINITION 1. Let

$$G_k = \{f \in \mathrm{Aut}(\overline{\mathbb{H}}_k^n) : (\forall T \in \mathfrak{I}_k^n)[f(T) = T \wedge f_{|T} \text{ — equiaffine}]\}.$$

Note that

PROPOSITION 1. *If $k = 1$ and $T \in \mathfrak{I}_k^n$, then $\dim(T) = 1$ and $h : T \mapsto T$ is equiaffine iff for any $a, b \in T$ we have $ab \equiv_1 h(a)h(b)$, where $ab \equiv_1 cd :\Leftrightarrow (\exists f \in G(\Sigma(\mathbb{H}_k^n)))[f(a) = c \wedge f(b) = d] \wedge ab$ — isotropic.*

Just from the definition of $\equiv_1$ we have

PROPOSITION 2. *If $\dim(T) = 1$, $a, b, c, d \in T$, $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$, $c = (c_1, \ldots, c_n)$ and $d = (d_1, \ldots, d_n)$, then $ab \equiv_1 cd \Leftrightarrow |a_n - b_n| = |c_n - d_n|$.*

PROPOSITION 3. *If $\sigma = \sigma_A^\varrho \in \Sigma(\mathbb{H}_k^n)$, $T \in \mathfrak{I}_k^n$, then $\sigma_{|T} = \sigma_{T \cap A_{|T}}$.*

Hence we have

PROPOSITION 4. *$G(\Sigma(\mathbb{H}_1^n))_{|T}$ is the group consisting of central symmetries and translations of T.*

To give a geometric characterization of the groups $G(\Lambda(\mathbb{H}_1^n))$ and $G(\Lambda(\mathbb{H}_k^n))$ with $k > 1$, we need some more definitions and facts. We begin with the following rigidity condition.

PROPOSITION 5. *If* $h \in G_1$, $E \in \mathfrak{G}$, $q \in \mathbb{C}_1^n$, $q \notin E$, $h(q) = q$, *and* $h_{|E} = \mathrm{id}_E$, *then* $h = \mathrm{id}$.

Easily we get

PROPOSITION 6. $G(\Lambda(\mathbb{H}_k^n)) \subseteq G_k$.

Now from Proposition 5, 6, and Corollaries 2.14, 2.15 in [2], we directly get

THEOREM 1. $G(\Lambda(\mathbb{H}_1^n)) = G_1$.

DEFINITION 2.

Considering G_k for $k > 1$ is more complicated. Let $k > 1$. We define a star of $\mathbb{H}_k^n$ to be a family of equidistant hypersurfaces $\mathbb{E} = \{E_1, \ldots, E_k\}$ satisfying the following conditions

(i) $E_i \in \mathfrak{G}$ for $1 \leq i \leq k$;

(ii) $\bigcap_{i=1}^{k} \mathbb{V}(E_i) = \emptyset$, where $\mathbb{V}(E_i)$ is the top of E_i.

Stars are preserved by mappings from G_k, i.e.

PROPOSITION 7. *If* E *is a star of* $\mathbb{H}_k^n$ *and* $f \in G_k$, *then* $f(E)$ *is a star of* $\mathbb{H}_k^n$.

Stars admit all possible reflections.

PROPOSITION 8. *If* $\{E_1, \ldots, E_k\}$ *is a star of* $\mathbb{H}_k^n$, $1 \leq i < j \leq k$, *and* $q \in \bigcap_{s \neq i,j} \mathbb{V}(E_s) \setminus (\mathbb{V}(E_i) \cup \mathbb{V}(E_j))$, *then there exists* $E \in \mathfrak{G}$ *such that* $\sigma_E^q(E_i) = E_j$ *and* $\sigma_E^q(E_s) = E_s$ *for* $s \neq i, j$.

Moreover, the group $G(\Lambda(\mathbb{H}_k^n))$ acts transitively on stars.

PROPOSITION 9. *If* $\mathbb{F} = \{F_1, \ldots, F_k\}$ *and* $\mathbb{E} = \{E_1, \ldots, E_k\}$ *are stars of* $\mathbb{H}_k^n$, *then there exists a transformation* $g \in G(\Lambda(\mathbb{H}_k^n))$ *such that* $g(F_i) = E_i$ *for* $1 \leq i \leq k$.

Using stars we can formulate for $k > 1$ an analogue of rigidity condition Proposition 5.

PROPOSITION 10. *If* $\{E_1, \ldots, E_k\}$ *is a star of* $\mathbb{H}_k^n$, $k > 1$, $g \in G_k$, $a \notin \bigcup_{i=1}^{k} E_i$, $g(a) = a$, *and* $g(E_i) = E_i$ *for* $1 \leq i \leq k$, *then* $g = \mathrm{id}$.

Now from Theorem 1, Definition 2, Propositions 6, 10, 11, 12, 13 we get

THEOREM 2. $G(\Lambda(\mathbb{H}_k^n)) = G_k$.

3. Proofs and auxiliary lemmas

Proof of Proposition 5. Let $E = E_p[a]$ and $E_i = E_p[q]$. By Corollaries 2.14, 2.15 in [2], $E \cap E_1 = \emptyset$. Moreover $q \in h(E_i)$. Now we prove that $h_{|E_1} = \mathrm{id}_{E_1}$. Let $x \in E_1$, $y \in E$, and let xy be isotropic. From Definition 1 and Proposition 1 we have $xy \equiv_1 h(x)y$. Now, from Proposition 2 follows $h(x) = \sigma_y(x)$ or $h(x) = x$. But if $h(x) = \sigma_y(x)$, then $h(E) \cap h(E_1) \neq \emptyset$. Whence $E \cap E_1 \neq \emptyset$ — contradiction. Thus $h_{|E_1} = \mathrm{id}_{E_1}$.

Let $z \notin E \cap E_1$, $z_1 \in E$, $z_2 \in E_1$, and zz_1, zz_2 be isotropic. Because $h(z_1) = z_1$ and $h(z_2) = z_2$, therefore from Proposition 2, we have the thesis of Proposition 5. ∎

LEMMA 1. *If $E \in \mathfrak{G}$, $P : x_n = 0$ is a base of $E, T \in \mathfrak{I}^n_k$, then $E \cap T$ is a $(k-1)$-dimensional hyperplane of T.*

Proof. From Theorem 2.9 and 2.11 in [2], E has the equation $c^2(-1 + x_1^2 + \ldots + x_{n-k}^2) + x_n^2 = 0$ with $c \geq 0$. Hence T is described by a set of equations $\{x_1 = a_1, x_2 = a_2, \ldots, x_{n-k} = a_{n-k}\}$. Thus $E \cap T$ is defined by a system of conditions: $x_1 = a_1$, $x_2 = a_2, \ldots, x_{n-k} = a_{n-k}$, and $x_n = \varepsilon c \sqrt{1 - \sum_{i=1}^{n=k} a_i^2}$, where $\varepsilon = -1$ for $x_n < 0$ and $\varepsilon = 1$ for $x_n > 0$. Hence we have the thesis. ∎

As a direct consequence of Lemma 2.6, Theorem 2.7 in [2], and Lemma 1 we infer

LEMMA 2. *If $E \in \mathfrak{G}$ and $T \in \mathfrak{I}^n_k$, then $E \cap T$ is a $(k-1)$-dimensional hyperplane of T.*

LEMMA 3. *If $f \in \Lambda$ and $T \in \mathfrak{I}^n_k$, then $f_{|T}$ is a symmetry of T.*

Proof. Let $f = \sigma^x_E$. Now, from Definition 1 in [5] and by the affine definition of symmetry we have $\sigma^x_{E_{|T}} = \sigma^x_{E \cap T}$. Next, from Lemma 2, $\sigma^x_{E\cap T}$ is a symmetry of T. ∎

From Lemma 3 we obtain the following:

COROLLARY 1. *If $f \in \Lambda$ and $T \in \mathfrak{I}^n_k$, then $f_{|T}$ is equiaffine.*

Proof of Proposition 6. Proposition 6 is a direct consequence of Theorem 1,2 in [5] and Corollary 1. ∎

Proof of Theorem 1. From Proposition 6 we have $G(\Lambda(\mathbb{H}^n_1) \subseteq G_1$. Let $f \in G_1$ and $E \in \mathfrak{G}$. Let P be a base of E and let P' be a base of $E' = f(E) \in \mathfrak{G}$. There exists a symmetry $g \in G(\Lambda(\mathbb{H}^n_1))$ such that $g(P') = P$. Let $g(E') = E''$. Let us see that P is a base of E and E''. Let K be an isotropic line of $\mathbb{H}^n_k$, and let $\varrho = K \cap E$, $\varrho'' = K \cap E''$. There exists an equidistant hypersurface H such that P is a base of H and $\sigma^x_H(\varrho'') = \varrho$.

Let $\sigma_H^x(E'') = E_0$. From Corollaries 2.14, 2.15 in [2], $E_0 = E$. Whence $\sigma_H^x(E'') = E$. If $h = \sigma_H^x \circ g \circ f$ then $h(E) = E$ and $h \in G_1$, because $\sigma_H^x, g \in G(\Lambda(\mathbb{H}_1^n)) \subseteq G_1$. Whence $h_{|E} = \mathrm{id}_E$. If there exists a point $q \in \mathbb{C}_1^n$ such that $q \notin E$ and $h(q) = q$, then, from Proposition 5, $h = \mathrm{id}$. Thus $f = g^{-1} \circ \sigma_H^x \circ h \in G(\Lambda(\mathbb{H}_1^n))$. If there does not exist a point $q \in \mathbb{C}_1^n$ such that $q \notin E$ and $h(q) = q$, then we consider a transformation $h' = \sigma_E^x \circ h$. Of course h' satisfies the assumptions of Proposition 5, whence $h' = \mathrm{id}$ and $h = \sigma_E^x \in G(\Lambda(\mathbb{H}_1^n))$. Thus also in this case $f = g^{-1} \circ \sigma_H^x \circ h \in G(\Lambda(\mathbb{H}_1^n))$. Hence $G_1 \subseteq G(\Lambda(\mathbb{H}_1^n))$. ∎

Analysing the proof of Lemma 1 we can see that the following lemma is true.

LEMMA 4. *If $E \in \mathfrak{G}$, $P : x_n = 0$ is a base of E, and $T_1, T_2 \in \mathfrak{I}_k^n$, then $E \cap T_1 || E \cap T_2$.*

Now as a direct consequence of Lemma 2.6, Theorem 2.7 in [2], and Lemma 4 we infer.

LEMMA 5. *If $E \in \mathfrak{G}$ and $T_1, T_2 \in \mathfrak{I}_k^n$, then $E \cap T_1 || E \cap T_2$.*

Note that the following lemma is true.

LEMMA 6. *If X_1, X_2, Y_1, Y_2 are subspaces of the affine space with equal dimensions, $X_1 \cap X_2, Y_1 \cap Y_2 \neq \emptyset$, and $X_1 || Y_1$, $X_2 || Y_2$, then $X_1 \cap X_2 || Y_1 \cap Y_2$.*

LEMMA 7. *If L_1, L_2 are isotropic lines of $\mathbb{H}_k^n$, $T_1, T_2 \in \mathfrak{I}_k^n$, $T_1 \neq T_2$, and $L_1 \subset T_1$, $L_2 \subset T_2$, then the following conditions are equivalent:*

(i) $L_1 || L_2$;

(ii) *there exist $E_1, E_2, \ldots, E_{n-2} \in \mathfrak{G}$ such that $L_i = T_i \cap E_1 \cap E_2 \cap \ldots \cap E_{n-2}$ for $i = 1, 2$.*

Proof. (ii)⇒(i) Let $L_i = T_i \cap E_1 \cap E_2 \cap \ldots \cap E_{n-2}$. Thus $L_i = (T_i \cap E_1) \cap (T_i \cap E_2) \cap \ldots \cap (T_i \cap E_{n-2})$. From Lemma 5, we have that $T_1 \cap E_1 || T_2 \cap E_1, \ldots, T_1 \cap E_{n-2} || T_2 \cap E_{n-2}$. Now, from Lemma 6, we get $L_1 || L_2$. ∎

(i)⇒(ii) Let $L_1 || L_2$. Thus there exists a plane Q generated by lines L_1, L_2. Whence $L_i = Q \cap T_i$ for $i = 1, 2$. Let a_n be a direction vector of L_1, L_2 and $a \in L_1$, $b \in L_2$. Whence the vector $a_1 = \overline{ab}$ is non isotropic and Q is generated by the point a and two vectors a_1 and a_n ($Q = a + \langle a_1, a_n \rangle$). The vectors a_1 and a_n are not parallel, because the vector a_n is isotropic. Let $(a_1, a_2, \ldots, a_n)$ with $a_{n-k+1}, \ldots, a_n$ isotropic be a base of $\mathbb{A}_n$; thus $\mathbb{A}_n = a + \langle a_1, a_2, \ldots, a_n \rangle$. Now we define $(n-2)$ non isotropic hyperplanes such that Q is the intersection of them. Let $P_i = a + \langle a_1, a_2, \ldots, a_{n-k+(i-1)}, a_{n-k+(i+1)}, \ldots, a_n \rangle$ for $i = 1, \ldots, k-1$. Let us see that $a_{n-k+i} \notin P_i$ for $i = 1, \ldots, k-1$. Whence P_i for $i = 1, \ldots, k-1$, is a non isotropic hyperplane. Let $R_i = a + \langle a_1, \ldots, a_i, a_{i+1} - a_{n-1}, a_{i+2}, \ldots, a_{n-2}, a_n \rangle$ for $i = 1, \ldots, n-k-1$. From easy considerations we have that $a_{n-1} \notin R_i$ for

$i = 1, \ldots, n-k-1$ and the set of vectors $(a_1, \ldots, a_i, a_{i+1} - a_{n-1}, a_{i+2}, \ldots, a_{n-2}, a_n)$ is linearly independent for $i = 1, \ldots, n-k-1$. Whence R_i is a non isotropic hyperplane for $i = 1, \ldots, n-k-1$. Note that $P = \bigcap_{i=1}^{k-1} P_i = a + \langle a_1, \ldots, a_{n-k}, a_n \rangle$. Now we have $P \cap \bigcap_{i=1}^{n-k-1} R_i = a + \langle a_1, a_n \rangle = Q$. However, $P_i \in \mathfrak{G}$ for $i = 1, \ldots, k-1$ and $R_i \in \mathfrak{G}$ for $i = 1, \ldots, n-k-1$. Hence the thesis. ■

LEMMA 8. *If K_1, K_2 are isotropic lines of $\mathbb{H}_k^n$ and $f \in G_k$, then $K_1 \| K_2$ iff $f(K_1) \| f(K_2)$.*

Proof. Because G_k is a group we need to prove only that $K_1 \| K_2 \Rightarrow f(K_1) \| f(K_2)$. If there exists $T \in \mathfrak{I}_k^n$ such that $K_1, K_2 \subset T$, then, by Definition 1, we have the thesis. Let $K_1 \subset T_1$, $K_2 \subset T_2$, $T_1, T_2 \in \mathfrak{I}_k^n$, and $T_1 \neq T_2$. Let $K_1 \| K_2$. Thus, by Lemma 7, there exist $E_1, E_2, \ldots, E_{n-2} \in \mathfrak{G}$ such that $K_i = T_i \cap E_1 \cap E_2 \cap \ldots \cap E_{n-2}$ for $i = 1, 2$. Whence $f(K_i) = T_i \cap f(E_1) \cap f(E_2) \cap \ldots \cap f(E_{n-2})$ for $i = 1, 2$. From Definition 1 we obtain $f(E_i) \in \mathfrak{G}$ for $i = 1, \ldots, n-2$ and $f(K_i)$ is an isotropic line for $i = 1, 2$. Now, by Lemma 7, we have $f(K_1) \| f(K_2)$. ■

We observe that any $f \in G_k$ determines the transformation $f^v : \mathbb{V}(\mathbb{C}_k^n) \mapsto \mathbb{V}(\mathbb{C}_k^n)$ defined by the condition:

$f^v(q) = q'$ iff there exists an isotropic line K of $\mathbb{H}_k^n$ such that $q \in \bar{K}$ and $q' \in f(\bar{K})$.

From Definition 1 and Lemma 8 we get that the definition of f^v is correct.

Proof of Proposition 7. Let $\{E_1, E_2, \ldots, E_k\}$ be a star of $\mathbb{H}_k^n$ and let $f \in G_k$. Whence $f(E_i) \in \mathfrak{G}$ for $1 \leq i \leq k$. Let $\bigcap_{i=1}^k \mathbb{V}(f(E_i)) \neq \emptyset$. Thus there exists a point x such that $x \in \mathbb{V}(f(E_i))$ for $1 \leq i \leq k$. However, $f^{v^{-1}}(x) \in \mathbb{V}(E_i)$ for $1 \leq i \leq k$. Whence $\bigcap_{i=1}^k \mathbb{V}(E_i) \neq \emptyset$ — contradiction. Hence $\{f(E_1), f(E_2), \ldots, f(E_k)\}$ is a star of $\mathbb{H}_k^n$. ■

As a direct consequence of Lemma 5 and Definition 1 from [5] we infer.

LEMMA 9. *If $q \in \mathbb{V}(E)$, $q \notin \mathbb{V}(F)$, and $E, F \in \mathfrak{G}$, then $\sigma_F^q(E) = E$.*

LEMMA 10. *If $F_i \in \mathfrak{G}$ for $i = 1, 2$, then for any q such that $q \in \mathbb{V}(\mathbb{C}_k^n) \setminus (\mathbb{V}(F_1) \cup \mathbb{V}(F_2))$ there exists $E \in \mathfrak{G}$ such that $q \notin \mathbb{V}(E)$ and $\sigma_E^q(F_1) = F_2$.*

Proof. For $i = 1, 2$ let Q_i be a base of F_i. We assume that $q \in \mathbb{V}(\mathbb{C}_k^n) \setminus (\mathbb{V}(F_1) \cup \mathbb{V}(F_2))$. (Such a point may exist because $\dim(\mathbb{V}(\mathbb{C}_k^n)) = k-1$ and $\dim(\mathbb{V}(F_i)) = k-2$ for $i = 1, 2$.)

Let K be a isotropic line such that $q \in \bar{K}$ and, if $Q_i \neq Q_2$, then $\bar{K} \cap (\bar{Q}_1 \cap \bar{Q}_2) = \emptyset$. Let $q_i = K \cap Q_i$ for $i = 1, 2$.) We construct now a hyperplane P and an equidistant hypersurface E with the base P.

We assume first that $Q_1 \neq Q_2$, then of course we have $q_1 \neq q_2$. Let a be the affine centre of a segment $q_1 q_2$. Since $\bar{Q}_1 \cap \bar{Q}_2$ is an $(n-2)$-dimensional hyperplane of $\mathbb{P}_n$, then there exists $(n-1)$-dimensional non isotropic hyperplane P such that $a \in P$ and $\bar{Q}_1 \cap \bar{Q}_2 \subset \bar{P}$.

Now we assume that $Q_1 = Q_2$; then $P = Q_1$. Of course $K \subset F_i$ for $i = 1,2$. Thus, by Lemma 1 (i), (ii) in [5], there exist points $d_i = K \cap F_i$ for $i = 1,2$. Let b be a affine centre of a segment d_1d_2. Let E an equidistant hypersurface such that $b \in E$ and P is a base of E. Note that $q \notin \mathbb{V}(E)$. Consider the symmetry σ_E^q. From definition, Lemma 3 in [5], and Corollaries 2.14, 2.15 in [2] we have the thesis. ∎

Proof of Proposition 8. Proposition 8 is a direct consequence of Lemma 9 and 10.

Let us see that for any $E \in \mathfrak{G}$, $\mathbb{V}(E) \subset \mathbb{V}(\mathbb{C}_k^n)$, $\dim(\mathbb{V}(E)) = k-2$, and $\dim(\mathbb{V}(\mathbb{C}_k^n)) = k-1$. Whence $\mathbb{V}(E)$ is a hyperplane of $\mathbb{V}(\mathbb{C}_k^n)$. Hence we get

LEMMA 11. *If $\{E_1, E_2, \ldots, E_k\}$ is a star of $\mathbb{H}_k^n$, then there exists a point x such that $x = \bigcap_{i=2}^k \mathbb{V}(E_i)$.*

LEMMA 12. *If $\{E_1, E_2, \ldots, E_k\}$ and $\{E_1', E_2, \ldots, E_k\}$ are stars of $\mathbb{H}_k^n$, then there exists $f \in G(\Lambda(\mathbb{H}_k^n))$ such that $f(E_1) = E_1'$ and $f(E_i) = E_i$ for $2 < i \leq k$.*

Proof. If $\{E_1', E_2, \ldots, E_k\}$ is a star of $\mathbb{H}_k^n$, then, from Lemma 11, there exists a point x such that $x = \bigcap_{i=2}^k \mathbb{V}(E_i)$ and $x \notin \mathbb{V}(E_1'), \mathbb{V}(E_1)$. Now, from Lemma 10, there exists $E \in \mathfrak{G}$ such that $x \notin \mathbb{V}(E)$ and $\sigma_E^x(E_1) = E_1'$. From Lemma 9 we have $\sigma_E^x(E_i) = E_i$ for $2 < i \leq k$. ∎

LEMMA 13. *If $\{E_1, E_2, \ldots, E_k\}$ and $\{F_1, F_2, \ldots, F_k\}$ are stars of $\mathbb{H}_k^n$, then there exists a permutation σ of the set $\{1, 2, \ldots, k\}$ such that for any $i \{E_{\sigma(1)}, \ldots, E_{\sigma(i)}, F_{i+1}, \ldots, F_k\}$ is a star of $\mathbb{H}_k^n$.*

Proof. A desired permutation is defined by induction on i.

First we consider $i = 1$. Whence we prove that there exists z in the set $\{1, 2, \ldots, k\}$ such that $\{E_z, F_2, \ldots, F_k\}$ is a star of $\mathbb{H}_k^n$.

From Lemma 11, there exists a point q_1 such that $q_1 = \bigcap_{i=2}^k \mathbb{V}(F_i)$. Because $\{E_1, E_2, \ldots, E_k\}$ is a star of $\mathbb{H}_k^n$, thus there exists $z \in \{1, 2, \ldots, k\}$ such that $q_1 \notin \mathbb{V}(E_z)$. We set $\sigma(1) = z$. Hence $\{E_{\sigma(1)}, F_2, F_3, \ldots, F_k\}$ is a star of $\mathbb{H}_k^n$.

Now we prove that if $\{E_{\sigma(1)}, \ldots, E_{\sigma(i)}, F_{i+1}, \ldots, F_k\}$ is a star of $\mathbb{H}_k^n$, then there exists z in the set $\{1, 2, \ldots, k\}$ such that $\{E_{\sigma(1)}, \ldots, E_{\sigma(i)}, E_z, F_{i+2}, \ldots, F_k\}$ is a star of $\mathbb{H}_k^n$.

From Lemma 11, there exists a point q_{i+1} such that

$$q_{i+1} = \bigcap_{m=1}^{i} \mathbb{V}(E_{\sigma(m)}) \cap \bigcap_{m=i+2}^{i} \mathbb{V}(F_m).$$

Because $\{E_1, E_2, \dots, E_k\}$ is a star of $\mathbb{H}_k^n$, thus there exists $z \in \{1, 2, \dots, k\}$ such that $q_{i+1} \notin \mathbb{V}(E_z)$. Of course $z \neq \sigma(1), \sigma(2), \dots, \sigma(i)$. Set $\sigma(i+1) = z$. Thus $\{E_{\sigma(i)}, \dots, E_{\sigma(i+1)}, F_{i+2}, \dots, F_k\}$ is a star of $\mathbb{H}_k^n$.

Hence we have the thesis of Lemma 13. ■

LEMMA 14. *If* $\mathbb{E} = \{E_1, E_2, \dots, E_k\}$ *and* $\mathbb{F} = \{F_1, F_2, \dots, F_k\}$ *are stars of* $\mathbb{H}_k^n$, *then there exists* $f \in G(\Lambda(\mathbb{H}_k^n))$ *such that* $f(\mathbb{F}) = \mathbb{E}$.

Proof. Let $\mathbb{F}, \mathbb{E}$ be stars of $\mathbb{H}_k^n$. Thus, from Lemma 13, there exists a permutation σ of the set $\{1, 2, \dots, k\}$ such that $\mathbb{F}_i = \{E_{\sigma(1)}, \dots, E_{\sigma(i)}, F_{i+1}, \dots, F_k\}$ is a star of $\mathbb{H}_k^n$ for $0 \geq i \geq k$. Note $\mathbb{F}_0 = \mathbb{F}$ and $\mathbb{F}_k = \mathbb{E}$. Now, from Lemma 12, there exists $f_i \in G(\Lambda(\mathbb{H}_k^n))$ such that $f_i : \mathbb{F}_{i-1} \mapsto \mathbb{F}_i$ for $1 \geq i \geq k$. Thus we see that $f = f_k \circ \dots \circ f_2 \circ f_1 \in G(\Lambda(\mathbb{H}_k^n))$ and $f(\mathbb{F}) = \mathbb{E}$. ■

Since every permutation is a superposition of a finite number of transpositions, then as a direct consequence of Lemma 11 and Proposition 8 we infer

LEMMA 15. *If* $\{E_1, E_2, \dots, E_k\}$ *is a star of* $\mathbb{H}_k^n$ *and* σ *is a permutation of the set* $\{1, 2, \dots, k\}$, *then there exists* $f \in G(\Lambda(\mathbb{H}_k^n))$ *such that* $f(E_i) = E_{\sigma(i)}$ *for* $1 \leq i \leq k$.

Proof of Proposition 9. If $\mathbb{F}$ and $\mathbb{E}$ are stars of $\mathbb{H}_k^n$, then, from Lemma 14, there exists $f_1 \in G(\Lambda(\mathbb{H}_k^n))$ such that $f_1(\mathbb{F}) = \mathbb{E}$. Whence $f_1(F_i) = E_{\sigma(i)}$ where σ is some permutation of the set $\{1, 2, \dots, k\}$. Now, from Lemma 15, there exists $f_2 \in G(\Lambda(\mathbb{H}_k^n))$ such that $f_2(E_{\sigma(i)}) = E_i$ for $1 \leq i \leq k$. Hence $g = f_1 \circ f_2 \in G(\Lambda(\mathbb{H}_k^n))$ and $g(F_i) = E_i$ for $1 \leq i \leq k$. ■

Now we define the following relation α:

$$\alpha(E, F) :\Leftrightarrow E, F \in \mathfrak{G} \wedge E \cap F = \emptyset \wedge \mathbb{V}(E) = \mathbb{V}(F) \wedge$$
$$\wedge (\exists x \notin E \cup F)(\exists! G \in \mathfrak{G})[x \in G \wedge G \cap E = \emptyset = G \cap F] \text{ or } E = F.$$

From this definition we get

Remark. 1. Let $E, F \in \mathfrak{G}$. Then $\alpha(E, F)$ iff $E = F$ or $E \neq F$ and there exists a hyperplane P which is a common base of E and F.

Proof of Proposition 10. Let $T = \langle a, \mathbb{V} \rangle \setminus \mathbb{V}$. From the assumptions we get $g(E_i \cap T) = E_i \cap T$ for $1 \leq i \leq k$. From Lemma 2, $E_i \cap T$ is a $(k-1)$-dimensional hyperplane of T, for $1 \leq i \leq k$. Because the set $\{E_1, E_2, \dots, E_k\}$ is a star of $\mathbb{H}_k^n$ therefore $\dim(\bigcap_{i=1}^k (E_i \cap T)) = 0$. Whence we can consider a coordinate system given by hyperplanes $E_i \cap T$ for $1 \leq i \leq k$. From assumptions, g is affine, $a \notin \bigcup_{i=1}^k E_i$, and $g(a) = a$, thus $g_{|T} = \mathrm{id}$. Let $x \in \mathbb{C}_k^n$ and $x \notin T$. Let the set $\{F_1, F_2, \dots, F_k\}$ be a star of $\mathbb{H}_k^n$ such that $x \in F_i$ and E_i, F_i have a common base Q_i for $1 \leq i \leq k$. Of course $\dot{T} \cap F_i \subset T$, $g_{|T} = \mathrm{id}$, thus $g(T \cap F_i) = T \cap F_i$ for $1 \leq i \leq k$. By Remark 1, $\alpha(E_i, F_i)$ for $1 \leq i \leq k$. However, $g \in G_k$, whence $\alpha(E_i, F_i) \Leftrightarrow \alpha(g(E_i), g(F_i))$. Thus Q_i

is a base of $g(F_i)$ for $1 \leq i \leq k$. But we have $g(T \cap F_i) = T \cap F_i$ for $1 \leq i \leq k$, thus $g(F_i) = F_i$ for $1 \leq i \leq k$. Let $T_1 = \langle x, \mathbb{V}\rangle \setminus \mathbb{V}$. Then $g(T_1) = T_1$ and $x = \bigcap_{i=1}^{k}(F_i \cap T_1)$. Hence $g(x) = x$. ∎

Proof of Theorem 2. From Theorem 1 we get the thesis for $k = 1$. Let $k > 1$. From Proposition 6 we have $G(\Lambda(\mathbb{H}_k^n)) \subseteq G_k$.

Let $f \in G_k$, and let $\{E_1, E_2, \ldots, E_k\}$ be a star of $\mathbb{H}_k^n$ such that E_i : $x_{n-k+i} = 0$ for $1 \leq i \leq k$. Set $E_i' = f(E_i)$ for $1 \leq i \leq k$. From Proposition 7, $\{E_1', E_2', \ldots, E_k'\}$ is a star of $\mathbb{H}_k^n$. Now, from Proposition 9, there exists $g \in G(\Lambda(\mathbb{H}_k^n))$ such that $g(E_i') = E_i$ for $1 \leq i \leq k$. Let $f_1 = g \circ f$. Whence $f_1 \in G_k$ and $f_1(E_i) = E_i$ for $1 \leq i \leq k$. Let $q = (0, 0, \ldots, 0, 1, \ldots, 1)$. Note that $q \notin E_i$ for $1 \leq i \leq k$. Set $q' = f_1(q)$.

Assume $q' = q$. Thus, from Proposition 10, $f_1 = \mathrm{id}$. Whence $g \circ f = \mathrm{id}$, thus $f = g^{-1}$, hence $f \in G(\Lambda(\mathbb{H}_k^n))$.

Now we assume that $q' \neq q$. Let $T = \langle q, \mathbb{V}\rangle \setminus \mathbb{V}$. Whence $E_i \cap T$ is described by the set of equations $\{x_1 = 0, x_2 = 0, \ldots, x_{n-k} = 0, x_{n-k+i} = 0\}$ for $1 \leq i \leq k$.

(1) Let $\lambda \in F$, $\lambda \neq 0$, and let $g_1 : T \mapsto T$ be a transformation defined by

$$g_1((x_{n-k+1}, x_{n-k+2}, x_{n-k+3}, x_{n-k+4}, \ldots, x_n)) = \\ = (\lambda x_{n-k+2}, (1/\lambda)x_{n-k+1}, x_{n-k+3}, x_{n-k+4}, \ldots, x_n).$$

Let $\alpha_1, \alpha_2 \in F$, $\alpha_1, \alpha_2 \neq 0$, and let $\varrho = [0, 0, \ldots, 0, \underset{(k)}{\alpha_1}, \alpha_2, 0, \ldots, 0]$. Thus $\varrho \in \bigcap_{i=1}^{k} \mathbb{V}(E_i) \setminus (\mathbb{V}(E_1) \cup \mathbb{V}(E_2))$. From Proposition 8, there exists $E \in \mathfrak{G}$ such that $\sigma_E^{\varrho}(E_1) = E_2$ and $\sigma_E^{\varrho}(E_i) = E_i$ for $3 \leq i \leq k$. Of course we have $\sigma_E^{\varrho} \in G_k$, $\sigma_{E_{|T}}^{\varrho}(E_i \cap T) = E_i \cap T$ for $3 \leq i \leq k$, and $\sigma_{E_{|T}}^{\varrho}(E_1 \cap T) = E_2 \cap T$. Now from this and from Lemma 2, 3 we get that $\sigma_{E_{|T}}^{\varrho}$ is described in T by a $k \times k$ matrix M such that

$$M = \begin{bmatrix} 0 & x_2 & 0 & 0 & \ldots & 0 \\ y_1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{bmatrix}$$

where $\det(M) = -1$. Let L be a line such that $\varrho, q \in \bar{L}$. Thus L is described by the set of equations

$$\{x_{n-k+1} = 1 + \alpha_1 t, x_{n-k+2} = 1 + \alpha_2 t, x_{n-k+3} = 1, x_{n-k+4} = 1, \ldots, x_n = 1)$$

where $t \in F$. Whence $L \cap E_1 = (0, 1 - (\alpha_2/\alpha_1), 1, 1, \ldots, 1)$ and $L \cap E_2 =$

$(1-(\alpha_1/\alpha_2),0,1,1,\dots,1)$. $\sigma^{\varrho}_{E_{|T}}(L\cap E_1)=L\cap E_2$, thus

$$M=\begin{bmatrix} 0 & (-\alpha_1/\alpha_2) & 0 & 0 & \dots & 0\\ (-\alpha_2/\alpha_1) & 0 & 0 & 0 & \dots & 0\\ 0 & 0 & 1 & 0 & \dots & 0\\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & \dots & 1\end{bmatrix}.$$

Let $(-\alpha_1/\alpha_2)=\lambda$. Whence we see that for any g_1 there exists $\sigma^{\varrho}_E\in G(\Lambda(\mathbb{H}^n_k))$ such that $\sigma^{\varrho}_{E_{|T}}=g_1$ and $\sigma^{\varrho}_E(E_i)=\begin{cases}E_{3-i} & \text{for } i\le 2\\ E_i & \text{for } 3\le i\le k.\end{cases}$

(2) Let $\lambda\in F$, $\lambda\neq 0$ and let $h_1:T\mapsto T$ be a transformation defined by

$$h_1((x_{n-k+1},x_{n-k+2},x_{n-k+3},\dots,x_n))=$$
$$=(\lambda x_{n-k+1},(1/\lambda)x_{n-k+2},x_{n-k+3},x_{n-k+4},\dots,x_n).$$

Let g_1,g_2 be transformations such that

$$g_1((x_{n-k+1},x_{n-k+2},x_{n-k+3},x_{n-k+4},\dots,x_n))=$$
$$((1/\lambda)x_{n-k+2},\lambda x_{n-k+1},x_{n-k+3},x_{n-k+4},\dots,x_n),$$
$$g_2((x_{n-k+1},x_{n-k+2},x_{n-k+3},x_{n-k+4},\dots,x_n))=$$
$$=(x_{n-k+2},x_{n-k+1},x_{n-k+3},x_{n-k+4},\dots,x_n).$$

From (1) we get $\bar{g}_1,\bar{g}_2\in G(\Lambda(\mathbb{H}^n_k))$ such that $\bar{g}_{j_{|T}}=g_j$ and $\bar{g}_j(E_i)=\begin{cases}E_{3-i} & \text{for } i\le 2\\ E_i & \text{for } 3\le i\le k\end{cases}$ for $j=1,2$. Let $g=g_2\circ g_1$. Note that $h_1=g_2\circ g_1$. Whence for any h_1 there exists a transformation $\bar{h}_1\in G(\Lambda(\mathbb{H}^n_k))$ such that $\bar{h}_{1_{|T}}=h_1$ and $\bar{h}_1(E_i)=E_i$ for $1\le i\le k$.

(3) At the beginning of this proof we defined the transformation f_1. From this definition we have that $f_{1_{|T}}$ is described in T by a $k\times k$ matrix N such that

$$N=\begin{bmatrix}\beta_1 & 0 & \dots & 0\\ 0 & \beta_2 & \dots & 0\\ \vdots & & & \\ 0 & 0 & \dots & \beta_k\end{bmatrix},$$

where $\prod_{i=1}^k\beta_i=\pm 1$. We remember that $f_1(q)=q'$. Let

$$q'=(0,0,\dots,0,q'_{n-k+1},q'_{n-k+2},\dots,q'_n).$$

Thus $q'_{n-k+i}=\beta_i$ for $1\le i\le k$. Whence $\prod_{i=1}^k q'_{n-k+i}=\pm 1$. Let h_i be a transformation such that

$$h_i(x)_j = \begin{cases} x_j & \text{for } n-k+1 \le j < n-k+i \\ & \text{or } n-k+i+1 < j \le n \\ \lambda_i x_{n-k+i} & \text{for } j = n-k+i \\ (1/\lambda_i) x_{n-k+i+1} & \text{for } j = n-k+i+1, \end{cases}$$

where $\lambda_i \in F$, $\lambda_i \neq 0$, and $1 \le i \le k-1$. From considerations analogous to (1) and (2) for any h_i we get a transformation $\bar{h}_i \in G(\Lambda(\mathbb{H}^n_k))$ such that $\bar{h}_{i_{|T}} = h_i$ and $\bar{h}_i(E_r) = E_r$ for $1 \le r \le k$. Let $\lambda_1 = q'_{n-k+1}$, $\lambda_2 = q'_{n-k+1} q'_{n-k+2}, \ldots, \lambda_{k-1} = \prod_{i=1}^{k-1} q'_{n-k+i}$, and let $h = h_{k-1} \circ h_{k-2} \circ \ldots \circ h_1$.

Let us see that if $\prod_{i=1}^{k} q'_{n-k+i} = 1$, then $h(q) = q'$ and there exists $\bar{h} \in G(\Lambda(\mathbb{H}^n_k))$ such that $\bar{h}_{|T} = h$ and $\bar{h}(E_i) = E_i$ for $1 \le i \le k$. Now going to the beginning of this proof we have $\bar{h}^{-1} f_1(q) = q$, $\bar{h}^{-1} f_1(E_i) = E_i$ for $1 \le i \le k$, and $\bar{h}^{-1} \circ f_1 \in G_k$. From Proposition 10 we have $\bar{h}^{-1} \circ f_1 = \mathrm{id}$. But $f_1 = g \circ f$, where $g \in G(\Lambda(\mathbb{H}^n_k))$. Thus $f = g^{-1} \circ f_1 = g^{-1} \circ \bar{h}$. Hence $f \in G(\Lambda(\mathbb{H}^n_k))$.

Let $\prod_{i=1}^{k} q'_{n-k+i} = -1$. Let $\varrho = \bigcap_{i=1}^{k-1} \mathbb{V}(E_i)$ and let $h^* = \sigma^{\varrho}_{E_{k_{|T}}} \circ h$. Note that $h^*(q) = q'$ and there exists $\bar{h}^* \in G(\Lambda(\mathbb{H}^n_k))$ such that $\bar{h}^*_{|T} = h^*$ and $\bar{h}^*(E_i) = E_i$ for $1 \le i \le k$. Now again coming back to the beginning of this proof, analogously, we have $f \in G(\Lambda(\mathbb{H}^n_k))$. ∎

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