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SOME RESULTS ON COMMON FIXED POINTS OF COMPATIBLE MAPPINGS

1. Introduction

Let (X,u) be a uniform space. A family $D = \{d_\alpha : \alpha \in I, I$ being an indexing$\}$ of pseudometrics on X is called an associated family of pseudometrics for u if the family $\beta = \{V(\alpha,r) : \alpha \in I, r > 0\}$, where $V(\alpha,r) = \{(x,y) : x,y \in X, d_\alpha(x,y) < r\}$ is a subbase for the uniformity u. We may assume β itself to be a base by adjoining finite intersection of members of β. The corresponding family of pseudometrics is called an augmented associated family for u (cf. Thron [17]). We shall denote this family by D^*.

DEFINITION 1.1. Mappings $f,g : X \to X$ will be called compatible if and only if for each $d_\alpha \in D^*$, $\lim_n d_\alpha(fg(x_n), gf(x_n)) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_n g(x_n) = t$ for some $t \in X$.

The notion of compatible mappings in metric spaces was first introduced by Jungck [4] and was extended to probabilistic metric spaces (PM-spaces) by the author in [9]. The above notion of compatible mappings is a generalization of the same. By now, it is well-known that these mappings are more general than commuting mappings and weakly commuting mappings studied by Sessa [14]. For details we refer to Jungck [5, 6].

Uniform spaces are the natural generalization of PM-spaces, where the uniformity is generated by a family of pseudometrics associated with the probabilistic metric, and the Hausdorff topology induced by the probabilistic metric coincides with the uniform topology.

Motivated with this idea, we first prove common fixed point theorems for two pairs of compatible mappings on a uniform space and, subsequently, de-

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rive their analogues in metric and PM-spaces. Finally, we extend our results to 2-metric spaces for the same class of mappings.

2. Common fixed point theorems in uniform spaces

Throughout this section, X will denote a sequentially complete Hausdorff uniform space defined by $D^* = \{d_\alpha : \alpha \in I\}$.

LEMMA 2.1. *If $f, g : X \to X$ are compatible with f continuous and $f(x_n), g(x_n) \to t$ as $n \to \infty$, where $\{x_n\}$ is a sequence in X, then $gf(x_n) \to f(t)$ as $n \to \infty$.*

Proof. We note that if $g(x_n) \to t$, then $fg(x_n) \to f(t)$ since f is continuous. Further, we have for any $d_\alpha \in D^*$,

$$d_\alpha(gf(x_n), f(t)) \leq d_\alpha(gf(x_n), fg(x_n)) + d_\alpha(fg(x_n), f(t)) \to 0 \quad \text{as } n \to \infty$$

since f is compatible. This proves the lemma.

LEMMA 2.2. *Let A, B, S and T be self mappings of X such that $A(X) \subset T(X)$ and $B(X) \subset S(X)$, and $x_0 \in X$. If for each $d_\alpha \in D^*$, there is a constant $k_\alpha \in (0,1)$ such that for all $x, y \in X$, we have*

$$\begin{aligned}(1)\quad & d_\alpha(A(x), B(y)) \\ & \leq k_\alpha \max\{d_\alpha(A(x), S(x)), d_\alpha(B(y), T(y)), d_\alpha(S(x), T(y)), \\ & \qquad \frac{1}{2}[d_\alpha(A(x), T(y)) + d_\alpha(B(y), S(x))]\},\end{aligned}$$

then a sequence $\{y_n\}_{n\in\mathbb{N}}$ beginning at x_0 and defined by

$$(2)\qquad y_{2n-1} = T(x_{2n-1}) = A(x_{2n-2}), y_{2n-2} = S(x_{2n}) = B(x_{2n-1})$$

is a Cauchy one.

Proof. Since $A(X) \subset T(X)$ and $B(X) \subset S(X)$, we may choose x_1 and x_2 in X such that $y_1 = T(x_1) = A(x_0)$ and $y_2 = S(x_2) = B(x_1)$. In general, we may choose x_{2n-1} and x_{2n} in X such that $y_{2n-1} = T(x_{2n-1}) = A(x_{2n-2})$ and $y_{2n} = S(x_{2n}) = B(x_{2n-1})$. Hence the existence of the sequence $\{y_n\}$ as required above is ensured. Further, from (1) and (2) it follows that

$$\begin{aligned}& d_\alpha(T(x_{2n+1}), S(x_{2n+2})) = d_\alpha(A(x_{2n}), B(x_{2n+1})) \\ & \leq k_\alpha \max\{d_\alpha(T(x_{2n+1}), S(x_{2n})), d_\alpha(S(x_{2n+2}), T(x_{2n+1})), \frac{1}{2}d_\alpha(S(x_{2n+2}), S(x_{2n}))\} \\ & \qquad \leq k_\alpha \max\{d_\alpha(S(x_{2n+2}), T(x_{2n+1})), d_\alpha(T(x_{2n+1}), S(x_{2n}))\}\end{aligned}$$

because

$$\begin{aligned}\frac{1}{2}d_\alpha(S(x_{2n+2}), S(x_{2n})) & \leq \frac{1}{2}[d_\alpha(S(x_{2n+2}), T(x_{2n+1})) + d_\alpha(T(x_{2n+1}), S(x_{2n})) \\ & \leq \max\{d_\alpha(S(x_{2n+2}), T(x_{2n+1})), d_\alpha(T(x_{2n+1}), S(x_{2n}))\}.\end{aligned}$$

Since $k_\alpha \in (0,1)$, the relation $d_\alpha(T(x_{2n+1}), S(x_{2n+2})) \leq k_\alpha d_\alpha(T(x_{2n+1}), S(x_{2n+2}))$ is not possible. Therefore we have $d_\alpha(T(x_{2n+1}), S(x_{2n+2})) \leq k_\alpha d_\alpha(T(x_{2n+1}), S(x_{2n}))$. Similarly, $d_\alpha(T(x_{2n+3}), S(x_{2n+2})) \leq k_\alpha d_\alpha (S(x_{2n+2}, T(x_{2n+1}))$. Consequently, $d_\alpha(y_{n+1}, y_n) \leq k_\alpha d_\alpha(y_n, y_{n-1})$ for all n and hence $\{y_n\}$ is a Cauchy sequence.

THEOREM 2.1. *Let A, B, S and T be self mappings of X. Suppose that S and T are continuous, the pairs A, S and B, T are compatible, and $A(X) \subset T(X)$ and $B(X) \subset S(X)$. If for each $d_\alpha \in D^*$, there is a $k_\alpha \in (0,1)$ such that the condition (1) is satisfied for all $x, y \in X$, then A, B, S and T have a unique common fixed point in X.*

Proof. By Lemma 2.2, there is a sequence $\{x_n\}$ in X such that $\{y_n\}$ defined in (2) is a Cauchy sequence. Therefore by the sequential completeness of X, $\{y_n\}$ converges to a point $z \in X$. Consequently, the subsequences $\{A(x_{2n})\}$, $\{S(x_{2n})\}$, $\{B(x_{2n-1})\}$ and $\{T(x_{2n-1})\}$ also converge to z. Hence the continuity of S and T, together with the compatibility of A, S and B, T and Lemma 2.1, implies that

$$SS(x_{2n}) \to S(z), AS(x_{2n}) \to S(z)$$

and

$$TT(x_{2n-1}) \to T(z), BT(x_{2n-1}) \to T(z).$$

Now setting $x = S(x_{2n})$ and $y = T(x_{2n-1})$ in (1) and allowing $n \to \infty$ we get

$$d_\alpha(S(z), T(z)) \leq k_\alpha \max\{0, 0, d_\alpha(S(z), T(z)), d_\alpha(S(z), T(z))\}.$$

Therefore $S(z) = T(z)$.

A similar arguments with $x = z$ and $y = T(x_{2n-1})$ in (1) yields $A(z) = T(z)$.

Finally, taking $x = y = z$ in (1) we get

$$A(z) = B(z) = S(z) = T(z).$$

To prove that z is a common fixed point of A, B, S and T, observe that

$$d_\alpha(A(x_{2n}), B(z)) \leq k_\alpha \max\{d_\alpha(A(x_{2n}), S(x_{2n})), d_\alpha(B(z), T(z)), d_\alpha(S(x_{2n}), T(z)), \frac{1}{2} d_\alpha(A(x_{2n}), T(z)) + d_\alpha(B(z), S(x_{2n}))]\}.$$

Making $n \to \infty$ and using $B(z) = T(z)$, we have $d_\alpha(z, B(z)) \leq k_\alpha d_\alpha(z, B(z))$ proving

$$z = B(z).$$

Hence

$$A(z) = B(z) = S(z) = T(z) = z$$

and thus z is a common fixed point of A, B, S and T.

The uniqueness of z as a common fixed point of A, B, S and T can be easily verified.

THEOREM 2.2. *Let S and T be self mappings of X, and let $A, B : X \to S(X) \cap T(X)$. Suppose that S and T are continuous, and the pairs A, S and B, T are compatible. If for each $d_\alpha \in D^*$, there is a constant $k_\alpha \in (0,1)$ such that the condition (1) is satisfied for all $x, y \in X$, then A, B, S and T have a unique common fixed point in X.*

Proof. Since $A, B : X \to S(X) \cap T(X)$, it follows that $A(X) \subset T(X)$ and $B(X) \subset S(X)$. Hence all the hypotheses of Theorem 2.1 are satisfied. Therefore the result follows.

3. Common fixed point theorems in PM-spaces

A nonnegative real valued function f defined on the reals $\mathbb{R}$ is called a distribution function if it is nondecreasing, left continuous with $\inf f = 0$ and $\sup f = 1$. A PM-space is a pair (X, F), where X is a nonempty set and F is a mapping from $X \times X$ to the set of all distribution functions. The value of F at $(p,q) \in X \times X$ is denoted by $F_{p,q}$, and $F_{p,q}$ are supposed to satisfy the following conditions:

(i) $F_{p,q}(x) = 1$ if $p = q$, $x > 0$ (ii) $F_{p,q}(0) = 0$ (iii) $F_{p,q} = F_{q,p}$,

(iv) If $F_{p,q}(x) = 1$ and $F_{q,r}(y) = 1$, then $F_{p,r}(x+y) = 1$ for all $p, q, r \in X$ and $x, y \geq 0$.

The mapping F is called a probabilistic metric on X. Further, a Menger space is a triplet (X, F, t), where (X, F) is a PM-space and t is a t-norm (cf. Schweizer and Sklar [11]) such that

(v) $F_{p,r}(x + y) \geq t\{F_{p,q}(x), F_{q,r}(y)\}$ for all $p, q, r \in X$ and $x, y \geq 0$.

It is known that the collection $\beta^* = \{U(x, \epsilon, \lambda) : x \in X, \epsilon, \lambda > 0\}$, where $U(x, \epsilon, \lambda) = \{y \in X : F_{x,y}(\epsilon) > 1 - \lambda\}$, is a subbase for the Hausdorff topology induced by the probabilistic metric F (cf. [11]). It is also known that this topology is induced by a uniformity with a countable basis and hence is metrizable (cf. [12]). It was shown by Cain and Kasriel [1] that for each $\alpha \in (0,1)$, there is a pseudometric d_α defined by $d_\alpha(x,y) = \sup\{v : F_{x,y}(v) \leq 1 - \alpha\}$ such that $d_\alpha(x,y) = 0$ iff $x = y$ and d_α is nondecreasing left continuous function of α with $F_{x,y}(d_\alpha(x,y)) \leq 1 - \alpha$. Further, $F_{x,y}(\epsilon) > 1 - \alpha$ iff $d_\alpha(x,y) < \epsilon$, and the topology generated by the family of pseudometrics $\{d_\alpha : \alpha \in (0,1)\}$ associated with the probabilistic metric F coincides with the Hausdorff topology induced by F. Hence the following results are the direct consequences of Theorems 2.1 and 2.2.

THEOREM 3.1. *Let A, B, S and T be self mappings of a compatible Menger space (X, F, t), where t is continuous and $t(x,x) \geq x$ for all $x \in [0,1]$.*

Suppose that S and T are continuous, the pairs A, S and B, T are compatible, and that $A(X) \subset T(X)$ and $B(X) \subset S(X)$. If for each $\alpha \in (0,1)$, there is a constant $k_\alpha \in (0,1)$ such that for all $x, y \in X$ and $v > 0$, we have

$$(3) \quad F_{A(x),B(y)}(k_\alpha v) \geq t(F_{A(x),S(x)}(v), t(F_{B(y),T(y)}(v), t(F_{S(x),T(y)}(v), t(F_{A(x),T(y)}(2v), F_{B(y),S(x)}(2v)))))$$

whenever $F_{x,y}(v) > 1 - \alpha$, then A, B, S and T have a unique common fixed point in X.

THEOREM 3.2. *Let S and T be self mappings of a complete Menger space (X, F, t) where t is continuous and $t(x,x) \geq x$ for all $x \in [0,1]$ and let $A, B : X \to S(X) \cap T(X)$. Suppose that S and T are continuous and the pairs A, S and B, T are compatible. If for each $\alpha \in (0,1)$, there is a constant $k_\alpha \in (0,1)$ such that the condition (3) is satisfied for all $x, y \in X$ and $v > 0$, then A, B, S and T have a unique common fixed point in X.*

Since each metric space (X, d) is a PM-space (cf. [13]) via $F_{p,q}(x) = H(x - d(p,q))$, where H is the distribution function defined by $H(x) = 0$ if $x \leq 0$, $H(x) = 1$ if $x > 0$, we have the following corollaries as consequences of Theorems 3.1 and 3.2.

COROLLARY 3.1 [5, Theorem 3.1]. *Let A, B, S and T be self mappings of a metric space (X, d). Suppose that S and T are continuous, the pairs A, S and B, T are compatible, and that $A(X) \subset T(X)$ and $B(X) \subset S(X)$. If there is a constant $k \in (0,1)$ such that for all $x, y \in X$ we have*

$$(4) \quad d(A(x), B(y)) \leq k \max\{d(A(x), S(x)), d(B(y), T(y)), d(S(x), T(y)), \frac{1}{2}[d(A(x), T(y)) + d(B(y), S(x))]\},$$

then A, B, S and T have a unique common fixed point in X.

COROLLARY 3.2. *Let S and T be self mappings of a metric space (X, d), and let $A, B : X \to S(X) \cap T(X)$. Suppose that S and T are continuous, and the pairs A, S and B, T are compatible. If there is a constant $k \in (0,1)$ such that the condition (4) is satisfied for all $x, y \in X$, then A, B, S and T have a unique common fixed point in X.*

4. Common fixed point theorems in 2-metric spaces

We shall first recall some preliminaries on 2-metric spaces from Gahler [3].

DEFINITION 4.1. Let X be a nonempty set. A nonnegative real valued function d on $X \times X \times X$ is called a 2-metric on X if the following conditions hold:

(i) to each pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x, y, z) \neq 0$.

(ii) $d(x, y, z) = 0$ when at least two of the points x, y and z of X are equal.

(iii) $d(x, y, z) = d(y, z, x) = d(x, z, y)$ for all $x, y, z \in X$.

(iv) $d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all $x, y, z, w \in X$.

The pair (X, d) is called a 2-metric space.

Just as a metric abstracts the properties of the length function, a 2-metric space has its topology given by a real function of point triples which abstracts the properties of the area function for Euclidean triangles. In the above topology we have the following:

DEFINITION 4.2. A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ iff $\lim_n d(x_n, x, a) = 0$ for all $a \in X$. Further, the sequence $\{x_n\}$ is called a Cauchy sequence iff $\lim_{m,n} d(x_m, x_n, a) = 0$ for all $a \in X$. Finally X is said to be complete if every Cauchy sequence in X is convergent.

A 2-metric d on X is said to be continuous if it is continuous in two of its three arguments. If d is continuous in any two arguments, then it is continuous in all the three arguments.

LEMMA 4.1. (Singh [15]). *Let $\{x_n\}$ be a sequence in a 2-metric space (X, d). If there exists a constant $k \in (0,1)$ such that $d(x_n, x_{n+1}, a) \leq kd(x_{n-1}, x_n, a)$ for all naturals n, then $\{x_n\}$ is a Cauchy sequence.*

DEFINITION 4.3. Self mappings f and g of a 2-metric space (X, d) are called compatible if $\lim_n d(fg(x_n), gf(x_n), a) = 0$ for all $a \in X$, whenever $\{x_n\}$ is a sequence in X such that $\lim_n f(x_n) = \lim_n g(x_n) = t$ for some $t \in X$.

The following results are the analogues of the results proved in section 2. We shall outline the main sketch of the proof of these results and omit the routine details.

LEMMA 4.2. *Let f and g be compatible self mappings of a 2-metric space (X, d), and let $\lim_n f(x_n) = \lim_n g(x_n) = t$ for some $t \in X$. If f is continuous, then $\lim_n gf(x_n) = f(t)$.*

LEMMA 4.3. *Let A, B, S and T be self mappings of a 2-metric space (X, d) such that $A(X) \subset T(X)$ and $B(X) \subset S(X)$, and let $x_0 \in X$. If there is a constant $k \in (0,1)$ such that for all $x, y, a \in X$, we have*

$$(5) \quad d(A(x), B(y), a) \leq k \max\{d(S(x), A(x), a), d(T(y), B(y), a), d(S(x), T(y), a), \frac{1}{2}[d(S(x), B(y), a) + d(T(y), A(x), a)]\},$$

then there is a Cauchy sequence $\{y_n\}_{n\in\mathbb{N}}$ in X defined by (2).

Proof. The existence of the sequence $\{y_n\}$ follows as in the case of Lemma 2.2. By condition (5), for all $a \in X$, we have

$$(6)\quad \begin{aligned} d(y_{2n+1}, y_{2n+2}, a) &= d(A(x_{2n}), B(x_{2n+1}), a) \\ &\le k \max\{d(y_{2n}, y_{2n+1}, a), d(y_{2n+1}, y_{2n+2}, a), \frac{1}{2} d(y_{2n}, y_{2n+2}, a)\} \end{aligned}$$

and

$$\begin{aligned} d(y_{2n}, y_{2n+2}, a) &\le d(y_{2n}, y_{2n+2}, y_{2n+1}) + d(y_{2n}, y_{2n+1}, a) + d(y_{2n+1}, y_{2n+2}, a) \\ &= d(y_{2n+1}, y_{2n+2}, y_{2n}) + d(y_{2n}, y_{2n+1}, a) + d(y_{2n+1}, y_{2n+2}, a) \\ &= d(y_{2n}, y_{2n+1}, a) + d(y_{2n+1}, y_{2n+2}, a) \end{aligned}$$

as $d(y_{2n+1}, y_{2n+2}, y_{2n}) = 0$ follows from (5).

Therefore $\frac{1}{2} d(y_{2n}, y_{2n+2}, a) \le \frac{1}{2}[d(y_{2n}, y_{2n+1}, a) + d(y_{2n+1}, y_{2n+2}, a)] \le \max\{d(y_{2n}, y_{2n+1}, a), d(y_{2n+1}, y_{2n+2}, a)\}$. Thus using this fact in (6) and following the arguments of Lemma 2.2, the result follows.

Now we state without proof the following theorems. The proofs can be similarly constructed on the lines of the proofs of Theorems 2.1 and 2.2.

THEOREM 4.1. *Let A, B, S and T be self mappings of a complete 2-metric space (X, d) with d continuous. Suppose that S and T are continuous, the pairs A, S and B, T are compatible, and that $A(X) \subset T(X)$ and $B(X) \subset S(X)$. If there is a constant $k \in (0,1)$ such that the condition (5) holds for all $x, y, a \in X$, then A, B, S and T have a unique common fixed point in X.*

THEOREM 4.2. *Let S and T be self mappings of a complete 2-metric space (X, d) with d continuous, and let $A, B : X \to S(X) \cap T(X)$. Suppose that S and T are continuous and the pairs A, S and B, T are compatible. If there is a constant $k \in (0,1)$ such that the condition (5) holds for all $x, y, a \in X$, then A, B, S and T have a unique common fixed point in X.*

5. Remarks

(i) Theorem 4.1 improves a result of Kubiak [8, Theorem 1] in the sense that the requirement of compatibility is more general than that of commutativity.

(ii) With the proper choice of the mappings A, B, S and T it is easy to see that our results generalize the results of Khan and Fisher [7, Theorem 1], Rhoades [10, Theorem 4] and Singh, Tiwari and Gupta [16, Theorem 1].

(iii) Continuity requirements for the mappings A and B in Ding [2] can be dispensed with.

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