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# INVARIANT SUBMANIFOLDS OF A MANIFOLD ADMITTING $f_\lambda(2\nu+3,-1)$-STRUCTURE

Invariant submanifolds of an almost complex manifold $M^{2n}$ have been studied by Yano and Schouten [6]. Yano and Ishihara also studied invariant submanifolds of almost contact manifolds [4]. The purpose of the present paper is to study the invariant submanifolds of $f_\lambda(2\nu+3,-1)$-structure manifold. Some interesting results have been stated and proved.

## 1. Preliminaries

Let $\widetilde{M}$ be an $m$-dimensional $C^\infty$ Riemannian manifold imbedded in another $n$-dimensional $C^\infty$ Riemannian manifold $M$, $m < n$. We denote the imbedding by $\phi : \widetilde{M} \to M$ and by $B$ the mapping induced by $\phi$ from $T(\widetilde{M})$ to $T(M)$, where $T(\widetilde{M})$ and $T(M)$ denote the tangent bundles of the manifolds $\widetilde{M}$ and $M$, respectively. Let $T(\widetilde{M}, M)$ be the set of all vector fields in $M$ tangent to $\widetilde{M}$. Then the mapping $B : T(\widetilde{M}) \to T(\widetilde{M}, M)$ is an isomorphism [5].

The set of all vectors normal to $\phi(\widetilde{M})$ forms a vector bundle $N(\widetilde{M}, M)$ over $\phi(\widetilde{M})$ and is called the normal bundle of $\widetilde{M}$. The vector bundle induced from $N(\widetilde{M}, M)$ by $\phi$ is denoted by $N(\widetilde{M})$. Let us denote $\psi : N(\widetilde{M}) \to N(\widetilde{M}, M)$, the natural isomorphism.

Throughout this paper, we shall use the following notations and conventions:

(i) $\mathcal{I}^r_s(\widetilde{M})$ denotes the set of all $C^\infty$ tensor fields of type $(r,s)$ associated with $T(\widetilde{M})$.

(ii) $\mathcal{U}^r_s(\widetilde{M})$ denotes the space of all $C^\infty$ tensor fields of type $(r,s)$ normal to $\widetilde{M}$.

An element of $\mathcal{I}^1_0(\widetilde{M})$ is a vector field on $\widetilde{M}$ and an element of $\mathcal{U}^1_0(\widetilde{M})$ is a vector field normal to $\widetilde{M}$.

Let $X$, $Y$ be any vector fields defined along $\phi(\widetilde{M})$ and tangential to

$\phi(\widetilde{M})$. Let $\bar{X}$ and $\bar{Y}$ be local extension of $X$ and $Y$, respectively. Then $[\bar{X}, \bar{Y}]$ is a vector field tangential to $M$ and its restriction $[\bar{X}, \bar{Y}]/\phi(\widetilde{M})$ to $\phi(\widetilde{M})$ can be determined independently from the choice of local extensions $\bar{X}$ and $\bar{Y}$. Thus we can define $[X, Y]$ by

$$[X, Y] = [\bar{X}, \bar{Y}]/\phi(\widetilde{M}). \tag{1.1}$$

Since $B$ is an isomorphism, for all $\widetilde{X}, \widetilde{Y} \in \mathcal{I}_0^1(\widetilde{M})$ we have

$$[B\widetilde{X}, B\widetilde{Y}] = B[\widetilde{X}, \widetilde{Y}]. \tag{1.2}$$

Suppose that on the ambient manifold $M$ there exists a $C^\infty$ tensor field $f$ of type (1.1) satisfying

$$f_\lambda^{2\nu+3} - \lambda^2 f = 0, \tag{1.3}$$

where $\lambda$ is a non-zero complex number. Then we say that the manifols $M$ admits an $f_\lambda(2\nu+3, -1)$-structure [1]. If we put in such manifold

$$s = \left(\frac{f^{\nu+1}}{\lambda}\right)^2, \quad t = I - \left(\frac{f^{\nu+1}}{\lambda}\right)^2, \tag{1.4}$$

where $I$ denotes the identity tensor field, then we have

$$s^2 = s,\ t^2 = t,\ s + t = I,\ st = ts = 0. \tag{1.5}$$

Thus the operators $s$ and $t$ are complementary projection operators. Consequently, there exist complementary distributions $S$ and $T$ corresponding to the projection operators $s$ and $t$, respectively. The projection operators $s$ and $t$ satisfy the following relations

$$\begin{cases} fs = sf = f,\ ft = tf = 0, \\ (f^{\nu+1})^2 s = \lambda^2 s,\ (f^{\nu+1})^2 t = 0. \end{cases} \tag{1.6}$$

Thus $f^{\nu+1}$ acts on $S$ as a $\pi$-structure operator and on $T$ as a null operator [1].

Such a manifold $M$ always admits a Riemannian metric $G$ such that

$$\widetilde{G}(\widetilde{X}, \widetilde{Y}) = \widetilde{G}(f\widetilde{X}, f\widetilde{Y}) + \widetilde{G}(t\widetilde{X}, \widetilde{Y}) \tag{1.7}$$

for all $\widetilde{X}, \widetilde{Y} \in \mathcal{I}_0^1(\widetilde{M})$. Thus, in view of (1.6) and (1.7), we get

$$\widetilde{G}(\widetilde{X}, f\widetilde{Y}) = \widetilde{G}(f\widetilde{X}, f^2\widetilde{Y}) + \widetilde{G}(t\widetilde{X}, f\widetilde{Y}) \tag{1.8}$$

and

$$\widetilde{G}(f\widetilde{X}, \widetilde{Y}) = \widetilde{G}(f^2\widetilde{X}, f\widetilde{Y}). \tag{1.9}$$

Now, let us define $\widetilde{g}$ and $g^*$ as

$$\widetilde{g}(\widetilde{X}, \widetilde{Y}) = \widetilde{G}(B\widetilde{X}, B\widetilde{Y}) \circ \phi \tag{1.10}$$

and

$$g^*(N, N') = G(\psi N, \psi N') \tag{1.11}$$

for all $\widetilde{X}, \widetilde{Y} \in \mathcal{I}_0^1(\widetilde{M})$ and $N, N' \in \mathcal{U}_0^1(\widetilde{M})$, respectively.

It can be easily shown that $\tilde{g}$ is a Riemannian metric tensor on $\widetilde{M}$ called the induced metric tensor on $\widetilde{M}$ and $g^*$ is a tensor field which defines an inner product in $N(\widetilde{M})$. The tensor field $g^*$ is called the induced metric tensor of $N(\widetilde{M})$.

Let $\bar{\nabla}$ be the Riemannian connection induced by the metric tensor $\widetilde{G}$ on $M$. Then $\bar{\nabla}$ induces a connection $\nabla$ in $\phi(\widetilde{M})$ defined by (cf. [4])

$$\nabla_X Y = \bar{\nabla}_X Y/\phi(\widetilde{M}), \tag{1.12}$$

where $X, Y$ are $C^\infty$ vector fields defined along $\phi(\widetilde{M})$ and tangential to $\phi(\widetilde{M})$.

## 2. Invariant submanifolds of $f_\lambda(2\nu+3,-1)$-structure manifold

Let $\widetilde{M}$ be a $C^\infty$ $m$-dimensional manifold imbedded in a $C^\infty$ $f_\lambda(2\nu+3,-1)$-manifold $M$ endowed with an $(1,1)$-tensor field $f$ satisfying the equation (1.3). We say that $\widetilde{M}$ is an invariant submanifold of $M$, if the tangent space $T_p(\phi(\widetilde{M}))$ of $\phi(\widetilde{M})$ is invariant by $f$ at each point $p$ of $\phi(\widetilde{M})$, that is

$$fB\widetilde{X} = BX^\circ, \tag{2.1}$$

where $X^\circ$ is some vector field in $\widetilde{M}$. Thus we define an $(1,1)$-tensor field $\tilde{f}$ in $\widetilde{M}$ as

$$\tilde{f}(\widetilde{X}) = X^\circ. \tag{2.2}$$

Thus from (2.1) and (2.2) we have

$$f(B\widetilde{X}) = B\tilde{f}(\widetilde{X}). \tag{2.3}$$

THEOREM 1. *Let $N$ and $\widetilde{N}$ be Nijenhuis tensors of $M$ and $\widetilde{M}$ formed with $(1,1)$-tensor field $f$ and $\tilde{f}$, respectively. Then $N$ and $\widetilde{N}$ are related by*

$$N(B\widetilde{X}, B\widetilde{Y}) = B\widetilde{N}(\widetilde{X}, \widetilde{Y}). \tag{2.4}$$

Proof. In view of the equations (1.2), (2.3) and of the definition of Nijenhuis tensor, we have

$$\begin{aligned} N(B\widetilde{X}, B\widetilde{Y}) &= [fB\widetilde{X}, fB\widetilde{Y}] - f[B\widetilde{X}, fB\widetilde{Y}] - f[fB\widetilde{X}, B\widetilde{Y}] + f^2[B\widetilde{X}, B\widetilde{Y}] \\ &= B\{[\tilde{f}\widetilde{X}, \tilde{f}\widetilde{Y}] - \tilde{f}[\tilde{f}\widetilde{X}, \widetilde{Y}] - \tilde{f}[\widetilde{X}, \tilde{f}\widetilde{Y}] + \tilde{f}^2[\widetilde{X}, \widetilde{Y}]\} = B\widetilde{N}(\widetilde{X}, \widetilde{Y}) \end{aligned}$$

which proves (2.4).

For the invariant submanifold $\widetilde{M}$ of $f_\lambda(2\nu+3,-1)$-manifold $M$ we shall consider the following two cases.

Case I. The distribution $T$ is nowhere tangential to $\phi(\widetilde{M})$.

Case II. The distribution $T$ is everywhere tangential to $\phi(\widetilde{M})$.

Let us consider the first case in which the distribution $T$ is nowhere tangential to the invariant submanifold $\phi(\widetilde{M})$. In this case, any vector field of type $t\widetilde{X}$ is independent of any vector field of the same frame $B\widetilde{X}$ for $\widetilde{X} \in \mathcal{I}_0^1(\widetilde{M})$. In view of the equation (2.3), applying $f$ further $(2\nu + 1)$ times, we get

$$f^{2\nu+2}(B\widetilde{X}) = B\widetilde{f}^{2\nu+2}(\widetilde{X}). \tag{2.5}$$

Since any vector field tangential to $\phi(\widetilde{M})$ is not contained in the distribution $T$, the vector fields of the type $B\widetilde{X}$ are in the distribution $S$. Thus, in consequence of the equation (1.6), we have $B\widetilde{f}^{2\nu+2}(\widetilde{X}) = \lambda^2 B\widetilde{X}$. Hence, we have

$$\widetilde{f}^{2\nu+2}(\widetilde{X}) = \lambda^2 \widetilde{X}. \tag{2.6}$$

Thus the tensor field $\widetilde{f}^{\nu+1}$ acts as a $\pi$-structure on the invariant submanifold $\widetilde{M}$.

Let us define a tensor field $\bar{S}$ of type (1,2) on $M$ as follows

$$\bar{S}(\bar{X},\bar{Y}) = N(\bar{X},\bar{Y}) + \bar{\nabla}_{\bar{X}}(t\bar{Y}) - \bar{\nabla}_{\bar{Y}}(t\bar{X}) - t[\bar{X},\bar{Y}] \tag{2.7}$$

for any vector fields $\bar{X},\bar{Y} \in \mathcal{I}_0^1(M)$.

THEOREM 2. *Let the distribution $T$ be nowhere tangential to $\phi(\widetilde{M})$. Then the tensor field $\bar{S}$ defined on $M$ by (2.7) satisfies the relation*

$$\bar{S}(B\widetilde{X},B\widetilde{Y}) = N(B\widetilde{X},B\widetilde{Y}) = BN(\widetilde{X},\widetilde{Y}) \tag{2.8}$$

*for $\widetilde{X},\widetilde{Y} \in \mathcal{I}_0^1(\widetilde{M})$.*

Proof. Since any vector field tangential to $\phi(\widetilde{M})$ is not contained in the distribution $T$, hence in consequence of the equation (1.6) we have $t(B\widetilde{X}) = 0$ for $\widetilde{X} \in \mathcal{I}_0^1(\widetilde{M})$. Hence, in view of the equations (2.4) and (2.7), the result follows.

## 3. Some other results

We say that the $f_\lambda(2\nu+3,-1)$-structure is normal, if $\bar{S} = 0$. Now we have the following theorem.

THEOREM 3. *An invariant submanifold $\widetilde{M}$ imbedded in $f_\lambda(2\nu+3,-1)$-structure manifold $M$ such that the distribution $T$ is nowhere tangential to $\phi(\widetilde{M})$ is a $\pi$-manifold with induced $\pi$-structure $\widetilde{f}^{\nu+1}$. If the structure on the ambient manifold $M$ is normal, the $\pi$-structure $\widetilde{f}^{\nu+1}$ is integrable on $\widetilde{M}$.*

Proof. The proof follows easily, by virtue of equation (2.4), (2.6) and (2.7).

Now, consider the case in which the distribution $T$ is everywhere tangential to the invariant submanifold $\phi(\widetilde{M})$. Thus, for $\widetilde{X} \in \mathcal{I}_0^1(\widetilde{M})$, we have

$$tBX = BX^\circ, \tag{3.1}$$

where $X^\circ$ is some vector field in $\widetilde{M}$.

Let us define an (1,1)-tensor field $\widetilde{t}$ in $\widetilde{M}$ such that $\widetilde{t}\widetilde{X} = x^\circ$. Then the equation (3.1) can be expressed as

$$tB\widetilde{X} = B\widetilde{t}\widetilde{X}. \tag{3.2}$$

Also we can define (1,1)-tensor field $\widetilde{s}$ on $\widetilde{M}$ as

$$sB\widetilde{X} = B\widetilde{s}\widetilde{X}. \tag{3.2$'$}$$

Since in $M$ the relation $s + t = I$ holds therefore (1,1)-tensor fields $\widetilde{s}$ and $\widetilde{t}$ on $\widetilde{M}$ are well defined.

THEOREM 4. *The* $(1,1)$*-tensor fields* $\widetilde{s}$ *and* $\widetilde{t}$ *on* $\widetilde{M}$ *defined by the equations* (3.2) *and* (3.2′) *satisfy the following relations*

$$\begin{cases} \widetilde{s} + \widetilde{t} = \widetilde{I}, \ \widetilde{s}\widetilde{t} = \widetilde{t}\widetilde{s} = 0, \\ \widetilde{s}^2 = \widetilde{s}, \widetilde{t}^2 = \widetilde{t}. \end{cases} \tag{3.3}$$

Proof. As for the ambient manifold $M$ there is $s+t = I$. Operating the above equation by $B\widetilde{X}$, we get $sB\widetilde{X} + tB\widetilde{X} = IB\widetilde{X}$. In view of (3.2) and (3.2′), it takes form $B\widetilde{s}\widetilde{X} + B\widetilde{t}\widetilde{X} = B\widetilde{I}\widetilde{X}$, or

$$\widetilde{s} + \widetilde{t} = \widetilde{I}. \tag{3.4}$$

Again operating $st = ts = 0$ by $B\widetilde{X}$ and making use of the same equations (3.2) and (3.2′), we get $B\widetilde{s}\widetilde{t}\widetilde{X} = B\widetilde{t}\widetilde{s}\widetilde{X} = 0$, or

$$\widetilde{s}\widetilde{t} = \widetilde{t}\widetilde{s} = 0. \tag{3.5}$$

Similarly, making use of the same equations (3.2) and (3.2′) in (1.5), we can prove that

$$\widetilde{s}^2 = \widetilde{s}, \quad \widetilde{t}^2 = \widetilde{t}. \tag{3.6}$$

Thus the operators $\widetilde{s}$ and $\widetilde{t}$ given by

$$\widetilde{s} = \left(\frac{\widetilde{f}^{\nu+1}}{\lambda}\right)^2 \quad \text{and} \quad \widetilde{t} = I - \left(\frac{\widetilde{f}^{\nu+1}}{\lambda}\right)^2,$$

respectively, when applied to tangent space of $\widetilde{M}$ at a point, are complementary projection operators on $\widetilde{M}$. This proves Theorem 4.

Now in consequence of the equations (1.3) and (2.3) we have

$$\begin{gathered} B\widetilde{f}^{2\nu+3}(\widetilde{X}) = f^{2\nu+3}(B\widetilde{X}) = \lambda^2 B\widetilde{f}\widetilde{X}, \\ \widetilde{f}^{2\nu+3} - \lambda^2 \widetilde{f} = 0. \end{gathered} \tag{3.7}$$

Hence, $\widetilde{f}$ acts as $f_\lambda(2\nu + 3, -1)$-structure operator on $\widetilde{M}$ and thus the induced structure on $\widetilde{M}$ is $f_\lambda(2\nu + 3, -1)$-structure. The induced Riemannian metric $\widetilde{g}$ is given by (cf. [4])

$$\widetilde{g}(\widetilde{X}, \widetilde{Y}) = \widetilde{g}(\widetilde{f}\widetilde{X}, \widetilde{f}\widetilde{Y}) + \widetilde{g}(\widetilde{t}\widetilde{X}, \widetilde{Y}). \tag{3.8}$$

Let $\tilde{\nabla}$ be the connection induced in $\widetilde{M}$ from the Riemannian connection $\bar{\nabla}$ on $M$. Then

$$\bar{\nabla}_{B\widetilde{X}} B\widetilde{Y} = B\tilde{\nabla}_{\widetilde{X}}\widetilde{Y}. \tag{3.9}$$

It can be easily shown that $\tilde{\nabla}$ is also a Riemannian connection on $\widetilde{M}$.

Let us define a tensor field $\widetilde{S}$ of type (1,2) on the submanifold $\widetilde{M}$ as follows

$$\widetilde{S}(\widetilde{X},\widetilde{Y}) = N(\widetilde{X},\widetilde{Y}) + \tilde{\nabla}_{\widetilde{X}}\tilde{t}\widetilde{X} - \tilde{\nabla}_{\widetilde{Y}}\tilde{t}\widetilde{X} - \tilde{t}[\widetilde{X},\widetilde{Y}], \tag{3.10}$$

when $\widetilde{X},\widetilde{Y} \in \mathcal{I}_0^1(\widetilde{M})$.

It can be easily shown that

$$\bar{S}(B\widetilde{X},B\widetilde{Y}) = B\widetilde{S}(\widetilde{X},\widetilde{Y}). \tag{3.11}$$

THEOREM 5. *An invariant submanifold $\widetilde{M}$ imbedded in a normal $f_\lambda(2\nu+3,-1)$-manifold such that the distribution $T$ is tangential to $\phi(\widetilde{M})$ is a $f_\lambda(2\nu+3,-1)$-structure manifold and the induced structure is normal in $\widetilde{M}$.*

Proof. Proof follows easily, by virtue of equations (2.7), (3.7) and (3.10).

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## References

[1] D. Demetropoulou-Psomopoulou, *Invariant submanifolds of a manifold admitting an $f(2\nu+3,1)$-structure, Tensor*, N.S. 51 (1992), 133–137.

[2] V. C. Gupta, Renu Dubey, *Invariant submanifolds of $f_\lambda$-manifold,* Demonstratio Math. 15 (1992), 333–342.

[3] M. D. Upadhyay, V. C. Gupta, *Integrability conditions of a structure $f_\lambda$ satisfying* $f^3-\lambda^2 f=0$, Publ. Math. Debrečen 24 (1977), 249–255.

[4] K. Yano, S. Ishihara, *Invariant submanifolds of an almost contact manifold,* Kodai Math. Sem. Rep., 21 (1969), 350–367.

[5] K. Yano, S. Ishihara, *On integrability conditions of a structure $f$ satisfying* $f^3+f=0$. Quart. J. Math. Oxford, Ser. 15 (1964), 217–222.

[6] K. Yano, J. A. Schouten, *On invariant subspace in almost complex* $X^{2n}$, Indiana Math. J. 17 (1968), 261–267.

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