DEMONSTRATIO MATHEMATICA
Vol. XXIX No 3 1996

Anna Dębińska-Nagórska, Andrzej Just, Zdzisław Stempień

ON SOME OPTIMISATION PROBLEM IN A SET OF SOLUTIONS OF NONLINEAR OPERATOR EQUATION IN BANACH SPACES. GALERKIN APPROXIMATION

In the paper we shall present the Galerkin approximation of some optimisation problem concerning minimization of convex functional on the set of solutions of a certain nonlinear operator equation with the monotone operator.

1. Introduction

Let V be a real, reflexive separable Banach space with the norm $\|\cdot\|$. By V^* we denote its dual with the duality relation $\langle\cdot,\cdot\rangle$ between V^* and V.

Let us consider an operator $A : V \to V^*$ (see [2], [3]).

DEFINITION 1.1. We say that A is monotone if $\langle Au - Av, u - v\rangle \geq 0$ for all $u, v \in V$ and strictly monotone if $\langle Au - Av, u - v\rangle > 0$ for $u \neq v$.

DEFINITION 1.2. The operator A is said to be coercive if there exists a real function $\gamma : [0,\infty) \to \mathbf{R}$ such that $\lim_{s\to\infty} \gamma(s) = +\infty$ and $\langle Au, u\rangle \geq \gamma(\|u\|)\|u\|$ for every $u \in V$.

DEFINITION 1.3. We say that A is radially continuous if for all $u, v \in V$ the real function $s \to \langle A(u + sv), v\rangle$ is continuous on $[0, 1]$.

DEFINITION 1.4. The operator A has the S-property if for every sequence $\{v_n\}_{n\in N} \subset V$ such that: $v_n \underset{n\to\infty}{\longrightarrow} v$ weakly in V and $\langle Av_n - Av, v_n - v\rangle \underset{n\to\infty}{\longrightarrow} 0$ we have that $v_n \underset{n\to\infty}{\longrightarrow} v$ strongly in V.

Key words and phrases: Optimisation, Galerkin approximation.
1991 *Mathematics Subject Classification*: 49J, 49M15.

Let $J : V \to \mathbf{R}$ be a continuous, strictly convex, and coercive functional (i.e. $\lim_{\|v\|\to\infty} J(v) = +\infty$) and let $A : V \to V^*$ be a monotone, coercive, and radially continuous operator. We shall consider the following optimization problem:

PROBLEM P. *Find $y^0 \in V$ (if it exists) such that*

$$J(y^0) = \min_{y \in V_{ad}} J(y),$$

where V_{ad} is a set of solutions of the equation

$$Ay = f \tag{1.1}$$

for a given $f \in V^*$.

In what follows we shall use the following theorem (see [3]).

BROWDER-MINTY THEOREM. *If $A : V \to V^*$ is a monotone, radially continuous and coercive operator then the set of solutions of the equation (1.1) is a closed, convex and non-empty subset of V.*

It is known that any continuous and strictly convex functional $J : V \to \mathbf{R}$ is weakly lower semi-continuous. Moreover, in a closed, convex set V_{ad} the optimisation problem (P) has a unique solution y^0 (see [1]).

2. Galerkin approximation

Consider a family $\{V_h\}_{h\in G}$ of finite-dimensional subspaces of V, which satisfies following conditions

$$\overline{\bigcup_{h\in G} V_h} = V, \quad \forall h_1, h_2 \in G\ (h_1 > h_2 \Rightarrow V_{h_1} \subset V_{h_2}),$$

where the set $G \subset (0,1]$ of parameters h has an accumulation point at 0 (see [4]). We shall give a variational formulation of the equation (1.1)

$$\langle Ay, v\rangle = \langle f, v\rangle \quad \forall v \in V. \tag{2.1}$$

Any function $y_h \in V_h$ which is a solution of the equation

$$\langle Ay_h, v_h\rangle = \langle f, v_h\rangle \quad \forall v_h \in V_h \tag{2.2}$$

will be called an approximate solution of (2.1).

Let us denote by I_h the embedding operator from V_h to V, and by $I_h^* : V^* \to V_h^*$ the adjoint operator to I_h. Then the equation (2.2) can be presented in the following operator form

$$A_h y_h = f_h, \tag{2.3}$$

where $A_h = I_h^* A I_h$, $f_h = I_h^* f$, so A_h satisfies the assumptions of the Browder-Minty theorem.

Therefore the set of solutions of equations (2.3) is a closed convex non-empty subset of V_h.

We shall consider the following approximative problem (P_h) in the space V_h.

PROBLEM (P_h). *Find $y_h^0 \in V_h$ such that*

$$J(y_h^0) = \min_{y_h \in V_{adh}} J(y_h)$$

where V_{adh} is a set of solutions of equation (2.3).

Similarly we can prove that the problem (P_h) has a unique solution y_h^0.

3. Convergence of the approximation

Let us now consider the problem of the convergence of the approximation.

LEMMA. *Let $A : V \to V^*$ be a monotone radially continuous and coercive operator. Then for all $y \in V_{ad}$ there exists (y_h) such that $y_h \in V_{adh}$ and $y_h \underset{h\to 0}{\longrightarrow} y$ strongly in V.*

Proof. Let $B : V \to V^*$ be any fixed, radially continuous, strictly monotone operator with S-property. Let $g \in V^*$ be any fixed element and (ε_i) a sequence of positive real numbers convergent to zero.

From [3] it follows that the operators $A + \varepsilon_i B$; $i = 1, 2, \ldots$ are strictly monotone, radially continuous, coercive and with S-property. Then the equations

$$\langle (A + \varepsilon_i B)y, v\rangle = \langle f + \varepsilon_i g, v\rangle \quad \forall v \in V$$

have unique solutions y_0^i for $1, 2, \ldots$

Therefore the sequence (y_0^i) is convergent to some $y_0 \in V_{ad}$

$$y_0^i \underset{i\to\infty}{\longrightarrow} y_0 \text{ strongly in } V,$$

and y_0 is a unique solution of the inequality

$$\langle By_0 - g, v - y_0\rangle \geq 0 \quad \forall v \in V_{ad}. \tag{3.1}$$

Analogously, for $i = 1, 2, \ldots$, the equation

$$\langle (A + \varepsilon_i B)y_h, v_h\rangle = \langle f + \varepsilon_i g, v_h\rangle \quad \forall v_h \in V_h \tag{3.2}$$

has a unique solution y_{0h}^i, therefore a sequence (y_{0h}^i) is strongly convergent in V to $y_{0h} \in V_{adh}$.

From [3] it follows that for all fixed $i = 1, 2, \ldots$, a sequence of approximate solutions (y_{0h}^i) of equation (3.2) is convergent to any $y_0^i \in V$

$$y_{0h}^i \underset{h\to\infty}{\longrightarrow} y_0^i \text{ strongly in } V.$$

From this and the inequality

$$0 \le \|y_{oh} - y_0\| \le \|y_{0h} - y_{0h}^i\| + \|y_{0h}^i - y_0^i\| + \|y_0^i - y_0\|$$

it follows that

$$y_{0h} \underset{h\to\infty}{\longrightarrow} y_0 \text{ strongly in } V.$$

We prove that for a certain $y_0 \in V_{ad}$ there exists a sequence (y_{0h}) of solutions of the approximated problem (P_h), $y_{0h} \in V_{adh}$ and that the sequence converges strongly to y_0 in V. From the uniqueness of the solution y_0 of inequality (3.1), where B and g can be selected arbitrarily, we can deduce that a sequence (y_{0h}) exists for every $y_0 \in V_{ad}$. In order to inequality's unique solution is any $y_0 \in V_{ad}$ it is enough to put $g = By_0$.

Now we shall prove that the sequence (y_h^0) of solutions of problem (P_h) is convergent to the solutions of problem (P).

THEOREM 1. *Let A be a monotone, coercive, radially continuous operator from a reflexive Banach space V into V^*, and let J be a continuous strictly convex and coercive functional from V into* **R**. *Then a sequence of optimal solutions (y_h^0) of problem (P_h) is weakly convergent to an optimal solution y^0 of the problem (P)*

$$y_h^0 \underset{h\to 0}{\longrightarrow} y^0 \text{ weakly in } V.$$

Proof. Since J is continuous and coercive, then there exists a positive constant $M_1 < \infty$ such that

$$\|y_h^0\| \le M_1 \quad \forall h. \tag{3.3}$$

Taking into account that V is a reflexive Banach space we can deduce that the sequence (y_h^0) contains a subsequence, which will be also denoted (y_h^0), weakly convergent to $\overline{y}$

$$y_h^0 \underset{h\to 0}{\longrightarrow} \overline{y} \text{ weakly in } V.$$

We prove that $\overline{y} \in V_{ad}$. Since the equation (2.2) is satisfied for $v_h \in V_h$, we can put $v_h = y_h^0$ and get from (3.3) that there exists a positive constant $M_2 < \infty$ such that

$$\langle Ay_h^0, y_h^0\rangle = \langle f, y_h^0\rangle \le \|f\|_{V^*}\|y_h^0\| \le M_2.$$

Then from [3] we conclude that there exists a positive constant $M_3 < \infty$ such that

$$\|Ay_h^0\|_{V^*} \le M_3. \tag{3.4}$$

Putting in the equation (2.2) $v_h = e_i$; $i = 1, 2, \dots, m(h) = \dim V_h$, where $(e_1, e_2, \dots, e_{m(h)})$ is a basis of V_h, we see that

$$\langle Ay_h^0, e_i\rangle = \langle f, e_i\rangle \quad i = 1, 2, \dots, m(h).$$

Consequently

$$\left\langle Ay_h^0, \sum_{i=1}^{m(h)} \alpha_i e_i \right\rangle = \left\langle f, \sum_{i=1}^{m(h)} \alpha_i e_i \right\rangle$$

for any $\alpha_i \in \mathbf{R}$ and therefore $\langle Ay_h^0, e\rangle = \langle f, e\rangle \ \forall e \in V_h$.

Then there exists a limit

$$\lim_{h\to 0}\langle Ay_h^0, e\rangle = \langle f, e\rangle \quad \forall e \in \bigcup_h V_h.$$

Thus by (3.4) we obtain

$$Ay_h^0 \underset{h\to 0}{\longrightarrow} f \text{ weakly in } V^*. \tag{3.5}$$

From $\lim_{h\to 0}\langle Ay_h^0, y_h^0\rangle = \lim_{h\to 0}\langle f, y_h^0\rangle = \langle f, \overline{y}\rangle$ it follows that $A\overline{y} = f$.

Let (y_h), $y_h \in V_{adh}$ be any sequence strongly convergent to y^0 in V. The existence of a sequence (y_h) is ensured by Lemma.

Because J is weakly lower semi-continuous it is easy to prove that

$$J(\overline{y}) \leq \liminf J(y_h^0) \leq \liminf J(y_h) = J(y^0).$$

In fact from the definition of y^0 it follows that

$$\overline{y} = y^0.$$

Due to the uniqueness of the solution not only a subsequence, but the whole sequence (y_h^0) is weakly convergent to y^0, the optimal solution of (P) in V.

THEOREM 2. *Let the assumption of Theorem* 1 *be satisfied. If the operation A has S-property then the sequence of optimal solutions (y_h^0) of problem (P_h) is strongly convergent to an optimal solution y^0 of problem (P)*

$$y_h^0 \underset{h\to 0}{\longrightarrow} y^0 \text{ strongly in } V.$$

Proof. From Theorem 1 we conclude that $y_h^0 \underset{h\to 0}{\longrightarrow} y^0$ weakly in V. Thus with (3.5) we obtain $Ay_h^0 \underset{h\to 0}{\longrightarrow} f$ weakly in V^*. It follows that

$$\lim_{h\to 0}\langle Ay_h^0 - Ay^0, y_h^0 - y^0\rangle$$
$$= \lim_{h\to 0}(\langle Ay_h^0, y_h^0\rangle - \langle Ay^0, y_h^0\rangle + \langle Ay_h^0, y^0\rangle - \langle Ay^0, y^0\rangle) = 0.$$

Due to the S-property of A we have

$$y_h^0 \underset{h\to 0}{\longrightarrow} y^0 \text{ strongly in } V.$$

4. An example

The method of the previous sections has been applied to a variety of problems. We present here one selected result.

A typical functional appearing in optimisation problems is the quadratic functional:

$$J(y) = \|E(y - y_d)\|_H^2$$

where $E \in L(V, H)$, H is a Hilbert space, y_d is a given element of V.

In particular, let $V = W_0^{1,p}(\Omega)$, $H = L^2(\Omega)$, $y_d = 0$, where $\Omega \subset \mathbf{R}^n$ is set of C^0 class (see [3]) with boundary Γ.

Let E be an embedding operator from V into H. The cost functional is equivalent to:

$$J(y) = \int_\Omega E^2(y) d\Omega, \quad \text{where } E^2(y) = (E(y), E(y))_H. \tag{4.1}$$

We introduce the operator $A : V \to V^*$, $V^* = W^{-1,q}(\Omega)$, $p^{-1} + q^{-1} = 1$ and

$$Ay = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i(x, \omega) + a_{n+1}(x, \omega), \quad x \in \Omega,$$

where

$$\omega = \left(\frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \ldots, \frac{\partial y}{\partial x_n}, y\right) = (\omega_1, \omega_2, \ldots, \omega_n, \omega_{n+1}).$$

Let $a_i(x, \omega) = \Phi(x, |\omega|^{p-1})|\omega|^{p-2} \sum\limits_{j=1}^{n+1} b_{ij}\omega_j \quad i = 1, 2, \ldots, n+1$, where $|\omega| = (\sum_{i,j=1}^{n+1} b_{ij}\omega_i\omega_j)^{1/2}$, $b_{ij} \in L^\infty(\Omega)$, $b_{ij} = b_{ji}$; $i, j = 1, 2, \ldots, n+1$; $p \geq 2$,

$$\sum_{i,j=1}^{n+1} b_{ij} d_i d_j \geq b \sum_{i=1}^{n+1} d_i^2, \quad b = const. > 0, \quad d_i \in \mathbf{R} \quad i = 1, 2, \ldots, n+1.$$

We assume that

(a) $\forall s \in [0, \infty)$ a function $x \to \Phi(x, s)$ is measurable in Ω.

(b) for almost all $x \in \Omega$ a function $s \to \Phi(x, s)$ is continuous in $[0, \infty)$.

(c) $\exists M > 0 \; \forall s \in [0, \infty)$ for a.a. $x \in \Omega$ $\Phi(x, s) \leq M$.

THEOREM 3. *If $p \geq 2$, the function Φ fulfils conditions* (a)÷(c), *there exists a positive constant l such that*: $\Phi(x, s) \geq l > 0$ *for a.a.* $x \in \Omega$ *and any* $s \in [0, \infty)$ *and the function* $s \to \Phi(x, s)s$ *is increasing, then the operator* A *is demicontinuous, monotone and coercive* (*see* [3]).

As an optimisation problem (P) we shall consider the following:

Find $y^0 \in V$ which minimizes the functional

$$J(y) = \int_{\Omega} E^2(y) d\Omega,$$

where y is solution of equation

$$(4.2) \qquad -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \Phi(x, |\omega|^{p-1}) |\omega|^{p-2} \sum_{j=1}^{n+1} b_{ij}\omega_j + \Phi(x, |\omega|^{p-1}) |\omega|^{p-2} \sum_{j=1}^{n+1} b_{ij}\omega_j = f(x);$$

$f \in W^{-1,q}(\Omega)$.

As an approximation of problem (P_h) we shall consider the following: *Find $y_h^0 \in V_h$ which minimizes the functional J*

$$J(y_h) = \int_{\Omega} y_h^2 d\Omega, \quad y_h = \sum_{i=1}^{m(h)} \alpha_h^{(i)} e_i,$$

where $\dim V_h = m(h)$ and $(\alpha_h^{(1)}, \dots \alpha_h^{(m(h))})$ is the solution of the system of algebraic equations following to (4.2)

$$\Big\langle -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \Phi(x, |\omega_h|^{p-1}) |\omega_h|^{p-2} \sum_{j=1}^{n+1} b_{ij}\omega_h^{(j)} + \Phi(x, |\omega_h|^{p-1}) |\omega_h|^{p-2} \sum_{j=1}^{n+1} b_{ij}\omega_h^{(j)}, e_j \Big\rangle = \langle f, e_j \rangle$$

for $j = 1, 2, \dots, m(h)$, where

$$\omega_h = (\omega_h^{(1)}, \omega_h^{(2)}, \dots, \omega_h^{(n+1)}), \quad \omega_h^{(i)} = \sum_{j=1}^{m(h)} \alpha_h^{(j)} \frac{\partial e_j}{\partial x_i}$$

for $i = 1, 2, \dots, n$; $\omega_h^{(n+1)} = y_h$, where the elements e_i, $i = 1, 2, \dots, m(h)$ form the basis of subspace V_h.

In this way the optimisation problem (P) has been reduced to a typical problem of mathematical programming (P_h).

References

[1] J. Céa, *Optimisation: théories et algorithmes.* Dunod, Paris, 1971.

[2] K. Deimling, *Nonlinear Functional Analysis.* Springer-Verlag, 1985.

[3] H. Gajewski, K. Gröger, K. Zacharias, *Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen*. Akademie-Verlag, Berlin, 1974.

[4] P. Raviart, J. Thomas, *Introduction à l'analyse numérique des équations aux dérivées partielles*. Masson, Paris, 1983.

INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY OF ŁÓDŹ
Al. Politechniki 11
90-924 ŁÓDŹ, POLAND

Received February 28, 1994.