DEMONSTRATIO MATHEMATICA
Vol. XXIX No 2 1996

T. D. Narang

BEST APPROXIMATION AND BEST SIMULTANEOUS APPROXIMATION IN ULTRAMETRIC SPACES

1. Introduction

There is an extensive literature on best approximation and best simultaneous approximation in normed linear spaces over the real or complex number fields (see e.g. [1], [5] and [9]). In 1956, A. F. Monna [3] initiated the study of best approximation in non-archimedean normed linear spaces over valued fields other than R or C. But not much is known about best approximation in ultrametric spaces, i.e. metric spaces X in which strong triangle inequality viz. $d(x,y) \le \operatorname{Max}\{d(x,z), d(z,y)\}$ is satisfied for all $x, y, z \in X$. Our aim in this paper is to examine the existence questions on best approximation and best simultaneous approximation in ultrametric spaces.

2. Best approximation in ultrametric spaces

Let (X, d) be an ultrametric space with ultrametric d, G a subset of X and $x \in X$. The problem of best approximation amounts to the problem of finding a point $g_0 \in G$ which is nearest to x among all the elements of G, i.e. $d(x, g_0) = \inf\{d(x,g) : g \in G\} \equiv d(x, G)$. Such a g_0, if it exists, is called a best approximation to x in G. We denote by $P_G(x)$ the set of all best approximations of x in G. There are two main problems to be considered, once x and G are given:

(i) When is $P_G(x) \neq \emptyset$, i.e. existence of best approximation.

(ii) When $P_G(x)$ contains no more than one element, i.e. the uniqueness of best approximation. Since

$$P_G(x) = \begin{cases} x & \text{for } x \in G, \\ \emptyset & \text{for } x \in \overline{G} \setminus G, \end{cases}$$

without any loss of generality we may assume that G is closed and $x \in X \setminus G$.

Let us start with the second problem, which in the non-archimedean case has a very simple solution viz. there is no uniqueness. More exactly, we have the following result which in non-archimedean normed linear space was given by A.F. Monna [4].

THEOREM 2.1. *Let G be a closed subset of an ultrametric space (X,d) and $x \in X \setminus G$. If $y \in P_G(x)$ and $g \in G$ such that $d(g,y) < r = d(x,y)$, then $g \in P_G(x)$.*

Proof. Since $x \notin G$ and $y \in G, r = d(x,y) > 0$. Now $d(x,g) \leq \max\{d(x,y), d(y,g)\}$ implies $d(x,g) \leq r$. Since $g \in G, d(x,g) \geq r$. Therefore $d(x,g) = r$, i.e. $g \in P_G(x)$.

Now we consider the first problem i.e. the existence of best approximation. If $P_G(x) \neq \emptyset$ for each x in X (equivalently, for each $x \in X \setminus G$), then the set G is called proximinal. It is well known that a compact set in a metric space is proximinal. The condition of compactness has been weakened to approximative compactness (Efimov and Steckin, see [9]), bounded compactness (V. Klee, see [6]), and spherical compactness (G. Albinus, see [6]). Here in this section we introduce analogous notions in ultrametric spaces and prove certain existence theorems on best approximation. To start with, we recall a few definitions.

DEFINITION 2.1. A sequence $\langle x_n \rangle$ in an ultrametric space (X,d) is said to be a *pseudo Cauchy* or *pC-sequence*, if there exists some n_0 such that whenever $n_0 \leq n_1 < n_2 < n_3$, then $d(x_{n_3}, x_{n_2}) < d(x_{n_2}, x_{n_1})$. An element $x \in X$ is called a *pseudo limit* or *p-limit* of a sequence $\langle x_n \rangle$ if for all n and m sufficiently large $n > m$ implies $d(x_n, x) < d(x_m, x)$, i.e. if ultimately the sequence $\langle d(x_n, x) \rangle$ decreases monotonically. We denote this by $x_n \to_p x$. The space X is said to be *pseudo complete* or *p-complete* if every *pC*-sequence in X has a p-limit.

DEFINITION 2.2. In an ultrametric space (X,d) an element $x \in X$ is called *pseudo limit point* of a subset G of X, if there exists a pseudo Cauchy sequence $\langle g_n \rangle$ in G such that $g_n \to_p x$. We denote by

$$\overline{G}_p = G \cup \{x \in X : x \text{ is a pseudo limit point of } G\}.$$

The set G is said to be *pseudo closed*, if every pseudo limit point of G is a member of G, i.e. if $\overline{G}_p = G$.

The set G is said to be *pseudo compact*, if every non-constant sequence in G has a subsequence with all pseudo limits in G.

Analogous to the notions of approximatively compact, boundedly compact, spherically compact and approximatively complete in archimedean spaces, we introduce corresponding notions in non-archimedean spaces as below.

DEFINITION 2.3. A subset G of an ultrametric space (X, d) is said to be

(i) *approximatively pseudo compact*, if for every $x \in X$ and every non-constant sequence $\langle g_n \rangle$ with $\lim_{n\to\infty} d(x, g_n) = d(x, G)$ there exists a subsequence $\langle g_{n_i} \rangle$ with a pseudo limit in G.

(ii) *boundedly pseudo compact*, if every non-constant bounded sequence in G has a subsequence with a pseudo limit in X.

(iii) *spherically pseudo compact*, if for every $x \in X \setminus G$ there exists a positive real number $r > d(x, G)$ such that the set $\{y \in G : d(x, y) \leq r\}$ is pseudo compact.

(iv) *approximatively pseudo complete*, if for every $x \in X$ and every pseudo Cauchy sequence $\langle g_n \rangle$ in G with $\lim_{n\to\infty} d(x, g_n) = d(x, G)$ the sequence $\langle g_n \rangle$ has a pseudo limit in G.

In archimedean theory, it was proved by Efimov and Steckin (see [9]) that an approximatively compact subset of a metric space is proximinal. In non-archimedean theory we have the following result

THEOREM 2.2. *An approximatively pseudo compact subset of an ultrametric space is proximinal.*

Proof. Let G be an approximatively pseudo compact subset of an ultrametric space (X, d) and $x \in X$ be arbitrary. By the definition of $d(x, G)$, there exists a nonconstant sequence $\langle g_n \rangle$ in G such that $d(x, g_n)$ converges monotonically to $d(x, G)$ from above. Hence for every n

$$d(x, g_n) > d(x, g_{n+1}).$$

By the strong triangle inequality, we then have

$$(1) \qquad d(g_n, g_{n+1}) = max\{d(x, g_n), d(x, g_{n+1})\} = d(x, g_n) \quad \text{for every } n.$$

Since $\lim_{n\to\infty} d(x, g_n) = d(x, G)$ and G is approximatively pseudo compact, there is a subsequence $\langle g_{n_i} \rangle$ with a pseudo limit $g_0 \in G$, i.e. $g_{n_i} \to_p g_0$. So $\langle g_{n_i} \rangle$ is a pseudo Cauchy sequence (see [7], p. 32). Therefore for every n_i

$$\begin{aligned} d(g_0, g_{n_i}) &= d(g_{n_{i+1}}, g_{n_i}) \quad \text{(see [7], p. 32)} \\ &= d(x, g_{n_i}) \quad \text{(by (1))}. \end{aligned}$$

Thus

$$d(x, g_0) \leq \max\{d(x, g_{n_i}), d(g_{n_i}, g_0)\} = d(x, g_{n_i})$$

for every n_i. So

$$d(x, g_0) \leq \lim_{n_i\to\infty} d(x, g_{n_i}) = d(x, G) \leq d(x, g_0),$$

i.e., $d(x, g_0) = d(x, G)$ and hence G is proximinal.

NOTE. Since every proximinal set is closed (for otherwise $P_G(x) = \emptyset$ for $x \in \overline{G} \setminus G$), it follows that in an ultrametric space, every approximatively pseudo compact set is closed.

It was proves by Efimov and Steckin (see [6]) that a boundedly compact closed set in a metric space is approximatively compact and hence proximinal.

THEOREM 2.3. *In an ultrametric space (X,d) a boundedly pseudo compact, pseudo closed set G is approximatively pseudo compact and hence proximinal.*

Proof. Let $x \in X$ and $\langle g_n \rangle$ be a nonconstant sequence in G with

$$\lim_{n\to\infty} d(x, g_n) = d(x, G).$$

Then there exists an integer N such that

$$d(x, g_n) < d(x, G) + 1$$

for all $n \geq N$. Let

$$K = \max\{d(x,g_1), d(x,g_2), \ldots, d(x, g_{N-1}), \quad d(x,G)+1\}.$$

Then $d(x, g_n) \leq K$ for all n and so $\langle g_n \rangle$ is a bounded sequence in G. Since G is boundedly pseudo compact, there is a subsequence $\langle g_{n_i} \rangle$ of $\langle g_n \rangle$ and $g_0 \in X$ such that $g_{n_i} \to_p g_0$. As G is pseudo closed, $g_0 \in G$. Thus G is approximatively pseudo compact and hence proximinal by Theorem 2.2.

It was proved in [6] that a spherically compact set in a metric space is approximatively compact and hence proximinal. In the non-archimedean situation we have the proposition.

THEOREM 2.4. *A spherically pseudo compact set in an ultrametric space is approximatively pseudo compact and hence proximinal.*

Proof. Let G be a spherically pseudo compact subset of an ultrametric space (X,d). Let $x \in X$ and $\langle g_n \rangle$ be a nonconstant sequence in G with

$$\lim_{n\to\infty} d(x, g_n) = d(x, G).$$

Now, if $x \in G$, then $d(x,G) = 0$, i.e. $\lim_{n\to\infty} d(x, g_n) = 0$. Clearly, we may choose integers n_i, $n_1 < n_2 < n_3 < \ldots$, such that $d(x, g_{n_1}) > d(x, g_{n_2}) > \ldots$, and so x is a pseudo limit of $\langle g_{n_i} \rangle$ and consequently G is approximatively pseudo compact.

If $x \notin G$, then $x \in X \setminus G$ and there exists a positive real number $r > d(x, G)$ such that the set

$$S = \{y \in G : d(x,y) \leq r\}$$

is pseudo compact. Since $r > d(x, G) = \lim_{n\to\infty} d(x, g_n)$, only finitely many terms of the sequence $\langle g_n \rangle$ will lie outside S. The pseudocompactness of the set S implies that the new sequence obtained by deleting these finitely many terms will have a subsequence with all pseudo limits in G. So G is approximatively pseudo compact and hence proximinal by Theorem 2.2.

The following theorem gives other condition under which a set is proximinal.

THEOREM 2.5. *An approximatively pseudo complete subset G of an ultrametric space (X, d) is proximinal.*

Proof. Let $x \in X$ be arbitrary. By the definition of $d(x, G)$, there is a nonconstant sequence $\langle g_n \rangle$ in G such that $d(x, g_n)$ converges monotonically to $d(x, G)$ from above, i.e. $d(x, g_n) > d(x, g_{n+1})$ for every n. Then

$$d(g_n, g_{n+1}) = \max\{d(x, g_n), d(x, g_{n+1})\} = d(x, g_n)$$

and so the sequence $\langle d(g_n, g_{n+1}) \rangle$ is strictly decreasing, i.e. $\langle g_n \rangle$ is a pseudo Cauchy sequence in G. Since G is approximatively pseudo complete, $\langle g_n \rangle$ has a pseudo limit $g_0 \in G$. So, proceeding as in Theorem 2.2, we see that $d(x, g_0) = d(x, G)$. Hence G is proximinal.

Remark. Since an ultrametric space is spherically complete if and only if it is pseudo complete (see [7], p. 34) and as every spherically complete (pseudo complete) subset of an ultrametric space is approximatively pseudo complete, from Theorem 2.5 follows the well known result (see [2]), [8]) that a spherically complete (pseudo complete) subset of an ultrametric space is proximinal.

3. Best simultaneous approximation in ultrametric spaces

Let (X, d) be an ultrametric space and $A \subset X$. For each $x \in X$, we define $d(x, A) = \inf_{y\in A} d(x, y)$. If a set of elements of B is given in X, one might like to approximate all the elements of B simultaneously by a single element of A. This type of problem arises when a function being approximated is not known precisely, but is known to belong to a set.

In approximating simultaneously all elements of B by a single element of A, one identifies elements of the set

$$E_A(B) = \{a \in A : \sup_{b\in B} d(a, b) = \inf_{a\in A} \sup_{b\in B} d(a, b)\}.$$

The elements of $E_A(B)$ are solutions to the simultaneous approximation problem and are called simultaneous approximate of B or restricted centres of B.

When $B = \{x\}$, $x \in X$, then $E_A(B)$ is nothing but the set

$$P_A(x) = \{a \in A : d(x,a) = \inf_{y \in A} d(x,y)\}.$$

For ultrametric spaces Theorem 4 of [2] in terms of best simultaneous approximation may be stated as follows.

THEOREM 3.1. *Let (X,d) be an ultrametric space, A a non-empty subset of X and B a non-empty bounded subset of X. Then there exists a best simultaneous approximation to the set B in A if and only if A is proximinal.*

Thus the problem of best simultaneous approximation is solved for all proximinal subsets A of X; in particular, since spherically complete subsets, approximatively pseudo compact subsets and approximatively pseudo complete subsets of an ultrametric space are proximinal, we have the result.

COROLLARY. *Let A be a spherically complete (approximatively pseudo compact, approximatively pseudo complete) subset of an ultrametric space (X,d). Then for every bounded subset B of X there exists a best simultaneous approximation to the set B in A.*

References

[1] G. C. Ahuja, T. D. Narang, *On best simultaneous approximation*, Nieuw. Arch. Wisk., 27 (1978), 255–261.

[2] J. Martinez-Maurice, M. T. Pellon, *Non-archimedean Chebyshev centres*, Proc. Kon. Ned. Akad. Wetensch., A 90 (1987), 417–421.

[3] A. F. Monna, *Sur les espaces normeés non-archimediens I*, Proc. Kon. Ned. Akad. Wetensch., A 59 (1956), 475–483.

[4] A. F. Monna, *Remarks on some problems in linear topological spaces over fields with non-archimedean valuation*, Proc. Kon. Ned. Akad. Wetensch., A 71 (1968), 484–496.

[5] T. D. Narang, *On best simultaneous approximation*, J. Approx. Theory, 39 (1983), 93–96.

[6] T. D. Narang, Swaran Khanna, *Inter-relationship between sets useful in solving best approximation problems*, Math. Education, 15 (1981), 48–50.

[7] L. Narici, E. Beckenstein, G. Bachman, *Functional Analysis and Valuation Theory*, Marcel Dekker Inc., 1971.

[8] J. B. Prolla, *Topics in Functional Analysis over Valued Division Rings*, North Holland Pub. Co., 1982.

[9] I. Singer, *Best Approximation in Normed Linear Spaces by Elements of Linear subspaces*, Springer Verlag, Berlin, 1970.

DEPARTMENT OF MATHEMATICS
GURU NANAK DEV UNIVERSITY,
AMRITSAR-143005 (INDIA)

Received November 21st, 1994.