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ON ORDER CONTINUITY OF QUANTUM STRUCTURES AND THEIR HOMOMORPHISMS

Introduction

Properties of order convergence for partially defined difference and sum in abelian relative inverse posets, as a common generalization of quantum structures and positive cones of partially ordered abelian groups are shown. Conditions for order continuity of homomorphisms and some sufficient conditions for the uniqueness of decompositions of homomorphisms are given. It is also shown that the set of all homomorphisms from an abelian RI–poset into an abelian RI–poset forms the abelian RI–poset with pointwise defined partial order and difference.

1. Order convergence in abelian RI-posets

Abelian RI-posets (introduced in [8]) provide a common axiomatic base for positive cones of partially ordered abelian groups and quantum structures (e.g. orthomodular posets, orthoalgebras, effect algebras and D-posets). Moreover homomorphisms of abelian RI-posets are common generalizations of measures, probabilities, observables and states from classical and also noncompatible probability theory, as well as measures with values in partially ordered spaces. Much more subtle are these facts discussed in [5], [8] and [9].

DEFINITION 1.1 Let X be a partially ordered set with a special element 0 and a partially defined binary operation $\ominus$ on X. We call $(X\,;\,\leq,\ominus,0)$ an abelian RI-poset if $b\ominus a$ is defined iff $a\leq b$ and the following conditions are satisfied:

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(i) $a \ominus 0 = a$ for all $a \in X$.
(ii) $a \ominus a = 0$ for all $a \in X$.
(iii) If $b \ominus a$ is defined then $b \ominus (b \ominus a)$ is defined.
(iv) $(a \ominus b) \ominus c = (a \ominus c) \ominus b$, if one side of the equality is defined.
(v) $c \ominus a = d \ominus a$ implies $c = d$.

In every abelian RI-poset we define a partial binary operation $\oplus$ on X by

(vi) $a \oplus b$ is defined and $a \oplus b = c$ iff $c \ominus b$ is defined and $c \ominus b = a$.

It is easy to verify that the partial operation $\oplus$ on X satisfies the following conditions:

(1) $a \oplus b = b \oplus a$, if one side of the equality is defined.
(2) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$, if one side of the equality is defined.
(3) $a \oplus 0 = a$ for all $a \in X$.
(4) $0 \leq a$ for all $a \in X$.
(5) $a \leq b$ implies $a \oplus c \leq b \oplus c$ whenever $a \oplus c$ and $b \oplus c$ are defined.
(6) If $a \leq b$ then there exists the unique $c \in X$ such that $a \oplus c = b$ (we denote $c = b \ominus a$).

Conversely, a poset $(X; \leq)$ with a special element 0 and a partial binary operation $\oplus$ which satisfies conditions (1)–(6) (we call it an *abelian RI-semigroup*) can be organized into an abelian RI-poset if we define partial binary operation $\ominus$ on X by the condition (vi).

LEMMA 1.2. *Suppose that $(X; \leq, \ominus, 0)$ is an abelian RI-poset and the partial binary operation $\oplus$ is defined by* (vi). *Then for elements $a, b, c, d \in X$ the following conditions are satisfied:*

(i) *$b \ominus a = 0$ implies $b = a$.*
(ii) *If $a \leq b$ then $b \ominus (b \ominus a) = a$ and $b = a \oplus (b \ominus a)$.*
(iii) *$a \leq b \leq c$ implies $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.*
(iv) *If $a \oplus b$ is defined then $a, b \leq a \oplus b$ and $(a \oplus b) \ominus a = b$.*
(v) *$a \ominus b = a \ominus d$ implies $b = d$.*

The reader can consult [8] for the proof.

LEMMA 1.3. *Suppose that $(X; \leq, \ominus, 0)$ is an abelian RI-poset and $\oplus$ is the operation in X defined by condition* (vi). *If for $a, b \in X$ with defined $a \oplus b$ there exist the join $a \vee b$ and the meet $a \wedge b$ then $(a \vee b) \oplus (a \wedge b)$ is defined and $(a \vee b) \oplus (a \wedge b) = a \oplus b$.*

We refer the reader to [13] for the proof.

LEMMA 1.4. *For elements of abelian RI-poset* $(X; \leq, \ominus, 0)$ *the following conditions are satisfied*:

(i) *If* $u \leq x$, $v \leq y$ *and* $x \oplus y$ *is defined then also* $u \oplus v$ *is defined*,
(ii) $x \oplus y \leq z$ *iff* $x \leq z \ominus y$.

The proof is left to the reader.

Suppose that $(P; \leq)$ is a poset. Let a set $\mathcal{E}$ of indices be directed upwards (that means, for each pair $\alpha, \beta \in \mathcal{E}$ there exist $\gamma \in \mathcal{E}$ with $\gamma \geq \alpha$ and $\gamma \geq \beta$). A net $(a_\alpha)_{\alpha \in \mathcal{E}}$ of elements of the poset P is *increasingly directed* if $a_\alpha \leq a_\beta$ whenever $\alpha \leq \beta$, in which case we shall write $a_\alpha \uparrow$. If moreover $a = \bigvee\{a_\alpha \mid \alpha \in \mathcal{E}\}$, then we write $a_\alpha \uparrow a$. The definition of *decreasingly directed* net and of the symbol $a_\alpha \downarrow a$ is dual. We say that a net $(a_\alpha)_{\alpha \in \mathcal{E}}$ *order converges* to a point $a \in P$ and we write $a_\alpha \xrightarrow{(o)} a$ (in P) if there exist nets $(u_\alpha)_{\alpha \in \mathcal{E}}$, $(v_\alpha)_{\alpha \in \mathcal{E}}$ of elements of P such that $u_\alpha \leq a_\alpha \leq v_\alpha$, for all $\alpha \in \mathcal{E}$ and $u_\alpha \uparrow a$, $v_\alpha \downarrow a$.

If P is a complete poset (i.e. $\vee M$ and $\wedge M$ exist for all $M \subseteq P$) then for every net $(a_\alpha)_{\alpha \in \mathcal{E}} \subseteq P$ it holds

$$a_\alpha \xrightarrow{(o)} a \quad \text{iff} \quad \bigwedge_\beta \bigvee_{\alpha \geq \beta} a_\alpha = \bigvee_\beta \bigwedge_{\alpha \geq \beta} a_\alpha = a.$$

A poset $(X; \leq)$ is *Dedekind complete* if every nonempty subset which has an upper bound has also the least upper bound and every subset which has a lower bound has also the greatest lower bound.

PROPOSITION 1.5. *Assume that* $(X; \leq, \ominus, 0)$ *is an abelian RI-poset. Then for elements of* X *the following conditions are satisfied*:

(i) $x_\alpha \uparrow x \Rightarrow x \ominus x_\alpha \downarrow 0$,
(ii) $x \ominus x_\alpha \uparrow x \Rightarrow x_\alpha \downarrow 0$,
(iii) $x_\alpha \downarrow x \Rightarrow x_\alpha \ominus x \downarrow 0$.

Proof. (i) Suppose $x_\alpha \uparrow x$. Then for all $\alpha_1 \leq \alpha_2$ we have $x_{\alpha_1} \leq x_{\alpha_2} \leq x$ and hence $x \ominus x_{\alpha_2} \leq x \ominus x_{\alpha_1}$. If $c \in X$ with $c \leq x \ominus x_\alpha$ for all α then $c \leq x \ominus x_\alpha \leq x$ which implies $x_\alpha = x \ominus (x \ominus x_\alpha) \leq x \ominus c \leq x$. Thus $x \leq x \ominus c \leq x$ which implies $c = 0$.

(ii) If $x \ominus x_\alpha \uparrow x$ then $x_\alpha = x \ominus (x \ominus x_\alpha) \downarrow 0$ by (i).

(iii) If $x_\alpha \downarrow x$ then for $c \in X$ with $c \leq x_\alpha \ominus x$ for all α we have $x = x_\alpha \ominus (x_\alpha \ominus x) \leq x_\alpha \ominus c$ and hence $x \oplus c \leq (x_\alpha \ominus c) \oplus c = x_\alpha$. It follows $x \oplus c \leq x \leq x \oplus c$ which implies $c = 0$.

THEOREM 1.6. *Assume that* $(X; \leq, \ominus, 0)$ *is a Dedekind complete abelian RI-poset. Then for elements of* X *the following conditions are satisfied*:

(i) $x_\alpha \uparrow x$ *iff* $x \ominus x_\alpha \downarrow 0$,

(ii) $x \ominus x_\alpha \uparrow x$ *iff* $x_\alpha \downarrow 0$ *and* $x_\alpha \leq x$ *for all* α,

(iii) $x_\alpha \ominus x \downarrow 0$ *iff* $x_\alpha \downarrow x$,

(iv) *if* $x_\alpha \xrightarrow{(o)} x$, $y_\alpha \xrightarrow{(o)} y$ *and* $(\bigvee_\alpha x_\alpha) \oplus (\bigvee_\alpha y_\alpha)$ *is defined then* $x_\alpha \oplus y_\alpha \xrightarrow{(o)} x \oplus y$,

(v) *if* $x_\alpha \uparrow x$, $y_\alpha \downarrow y$ *and* $x_\alpha \leq y_\alpha$ *for all* α *then* $y_\alpha \ominus x_\alpha \downarrow y \ominus x$.

Proof. (i) By Proposition 1.5 $x_\alpha \uparrow x \Rightarrow x \ominus x_\alpha \downarrow 0$. Suppose conversely that $x \ominus x_\alpha \downarrow 0$. Then for all $\alpha_1 \leq \alpha_2$ we have $0 \leq x \ominus x_{\alpha_2} \leq x \ominus x_{\alpha_1} \leq x$ which implies by Lemma 1.2 (iii) and (ii) that $x_{\alpha_1} \leq x_{\alpha_2} \leq x$. Hence there exists $y \in X$ with $y = \bigvee_\alpha x_\alpha \leq x$. Moreover the inequality $x_\alpha \leq y \leq x$ implies $x \ominus y \leq x \ominus x_\alpha$ which implies $x \ominus y = 0$, hence $x = y$. We conclude $x_\alpha \uparrow x$.

(ii) In view of (i) if $x \ominus x_\alpha \uparrow x$ then $x_\alpha = x \ominus (x \ominus x_\alpha) \downarrow 0$. Conversely, if $x_\alpha \leq x$ for all α and $x_\alpha \downarrow 0$ then $x_\alpha = x \ominus (x \ominus x_\alpha) \downarrow 0$ which implies $x \ominus x_\alpha \uparrow x$ by (i).

(iii) If $x_\alpha \downarrow x$ then $x_\alpha \ominus x \downarrow 0$ by Proposition 1.5 (iii). Assume that $x_\alpha \ominus x \downarrow 0$. Then $x \leq x_\alpha$ for all α which implies that there exists $y = \bigwedge_\alpha x_\alpha$. Moreover $x \leq y \leq x_\alpha$ for all α which implies that $x_\alpha \ominus y \leq x_\alpha \ominus x$ and $(x_\alpha \ominus x) \ominus (x_\alpha \ominus y) = y \ominus x \leq x_\alpha \ominus x$ for all α by Lemma 1. It follows $y \ominus x = 0$ from which $y = x$. We conclude $x_\alpha \downarrow x$.

(iv) Let us first suppose that $x_\alpha \uparrow x$, $y_\alpha \uparrow y$ and $x \oplus y$ is defined. Then $x_\alpha \oplus y_\beta$ is defined for all α and fixed β and $(x \oplus y_\beta) \ominus (x_\alpha \oplus y_\beta) = x \ominus x_\alpha \downarrow 0$. The last implies that $x_\alpha \oplus y_\beta \uparrow x \oplus y_\beta$ by part (i). Similarly $x \oplus y_\beta \uparrow x \oplus y$. Moreover if $x_\alpha \oplus y_\alpha \leq z$ for all α then $x_\alpha \oplus y_\beta \leq z$ for all α and fixed β (since to every α and β there exists $\gamma \geq \alpha, \beta$ which implies $x_\alpha \oplus y_\beta \leq x_\gamma \oplus y_\gamma \leq z$). It follows that $x \oplus y_\beta \leq z$ for all β and hence also $x \oplus y \leq z$. we conclude $x_\alpha \oplus y_\alpha \uparrow x \oplus y$.

If $x_\alpha \downarrow x$, $y_\alpha \downarrow y$ and $(\bigvee_\alpha x_\alpha) \oplus (\bigvee_\alpha y_\alpha)$ is defined then $x \oplus y$ and $x_\alpha \oplus y_\alpha$ exist and $x \oplus y \leq x_\alpha \oplus y_\alpha$ for all α. Moreover, in the same manner as before, $(x_\alpha \oplus y_\beta) \ominus (x \oplus y_\beta) = x_\alpha \ominus x \downarrow 0$ for fixed β, and so $x_\alpha \oplus y_\beta \downarrow x \oplus y_\beta$. Similarly $x \oplus y_\beta \downarrow x \oplus y$. In consequence, if $z \leq x_\alpha \oplus y_\alpha$ for all α then $z \leq x \oplus y_\beta$ for all β and $z \leq x \oplus y$.

Combining these two parts we obtain that if $x_\alpha \xrightarrow{(o)} x$, $y_\alpha \xrightarrow{(o)} y$ and $(\bigvee_\alpha x_\alpha) \oplus (\bigvee_\alpha y_\alpha)$ is defined then $x_\alpha \oplus y_\alpha \xrightarrow{(o)} x \oplus y$. The only point remaining concerns the fact that if $x_\alpha \leq u_\alpha$, $y_\alpha \leq v_\alpha$ for all α and there exists $u \oplus v$, where $u = \bigvee_\alpha x_\alpha$, $v = \bigvee_\alpha y_\alpha$, then $x_\alpha \leq u_\alpha \wedge u \leq u$, $y_\alpha \leq v_\alpha \wedge v \leq v$ for all α.

(v) Assume that $x_\alpha \uparrow x$, $y_\alpha \downarrow y$ and $x_\alpha \le y_\alpha$ for all α. Suppose that β is fixed. Then for every α there exists $\gamma \ge \alpha, \beta$ and hence $x_\alpha \le x_\gamma \le y_\gamma \le y_\beta$ which implies that $x \le y_\beta$. Therefore $y_\alpha \downarrow y$, we conclude that $x \le y$. Now inequalities $x_\alpha \le x \le y \le y_\alpha$ imply $y \ominus x \le y_\alpha \ominus x_\alpha$ for all α. Suppose that $c \le y_\alpha \ominus x_\alpha$ for all α. Then $c \oplus x_\alpha \le y_\alpha$ for all α. For every α and fixed β there exists $\gamma \ge \alpha, \beta$ and hence $c \oplus x_\alpha \le c \oplus x_\gamma \le y_\gamma \le y_\beta$. We conclude that $c \oplus x_\alpha \le y$ for every α which implies that $c \le y \ominus x_\alpha = (y \ominus x) \oplus (x \ominus x_\alpha) \downarrow y \ominus x$. The last statement follows by part (iv) since $x \ominus x_\alpha \downarrow 0$. Thus $c \le y \ominus x$. This proves that $y_\alpha \ominus x_\alpha \downarrow y \ominus x$.

Suppose that $(X; \le)$ is a poset. We say that $\emptyset \ne Y \subseteq X$ is a *full sub-poset* of X if the following two conditions are satisfied:

(i) For every $M \subseteq Y$ with existing $\vee M = x$ in X it holds $x \in Y$.
(ii) For every $Q \subseteq Y$ with existing $\wedge Q = y$ in X it holds $y \in Y$.

Evidently a full sub-poset of a complete poset is complete.

If $(X; \le, \ominus, 0, 1)$ is a D-poset and $\emptyset \ne Y \subseteq X$ is a full sub-poset of X such that $1 \in Y$ and $b \ominus a \in Y$ for all $a, b \in Y$ with $b \ominus a$ existing in X, then we call Y a *full sub-D-poset* of X.

THEOREM 1.7. *Suppose that $(X; \le, \ominus, 0, 1)$ is a complete lattice D-poset. Let $\emptyset \ne Y \subseteq X$ be such that for all $a, b \in Y$ we have:* (a) $a \wedge b \in Y$, (b) *if $b \ominus a$ exists in X then $b \ominus a \in Y$,* (c) $1 \in Y$. *The following conditions are equivalent:*

(i) *For all $(x_\alpha)_\alpha \subseteq Y$ $x_\alpha \xrightarrow{(o)} x$ in X iff $x \in Y$ and $x_\alpha \xrightarrow{(o)} x$ in Y.*
(ii) *For every $M \subseteq Y$ with $\vee M = x$ in X it holds $x \in Y$.*
(iii) *For every $Q \subseteq Y$ with $\wedge Q = y$ in X it holds $y \in Y$.*
(iv) *Y is a full sub-D-poset of X.*
(v) *Y is a closed set in order topology τ_{01} on X.*

Each of these conditions implies that $\tau_{01} \cap Y = \tau_{02}$, where τ_{02} is an order topology on Y.

Proof. (iv)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii): It follows by de Morgan laws, i.e. for every $P \subseteq X$

$$1 \ominus (\vee\{p \mid p \in P\}) = \wedge\{1 \ominus p \mid p \in P\},$$

if one side of the equality is defined, and

$$1 \ominus (\wedge\{p \mid p \in P\}) = \vee\{1 \ominus p \mid p \in P\},$$

if one side of the equality is defined.

(iv)$\Rightarrow$(i): It follows from the fact that in every complete lattice $x_\alpha \xrightarrow{(o)} x$ if and only if $\bigvee_\beta \bigwedge_{\alpha \ge \beta} x_\alpha = \bigwedge_\beta \bigvee_{\alpha \ge \beta} x_\alpha = x$.

(i)⇔(v): The proof follows from the statement that a set is closed in order topology on a poset if and only if it contains order limits of all its order convergent nets.

(i)⇒(ii): Suppose that $M \subseteq Y$ with $\vee M = x$ in X. Let $\mathcal{E} = \{\alpha \subseteq M \mid \alpha$ is a finite set$\}$ be directed by set inclusion. Let us put $x_\alpha = \vee\alpha$ for all $\alpha \in \mathcal{E}$. Then $x = \vee M = \vee\{x_\alpha \mid \alpha \in \mathcal{E}\}$. It follows that $x \in Y$, since $x_\alpha \in Y$ for all α and $x_\alpha \uparrow x$ in X.

Finally let us show that each of the conditions (i)–(v) implies $\tau_{01} \cap Y = \tau_{02}$. Suppose that $F \subseteq Y$ is a closed set in τ_{01}. Then for any net $(x_\alpha)_\alpha \subseteq F$ and any $x \in Y$ with $x_\alpha \xrightarrow{(o)} x$ in Y we have, applying (i), $x_\alpha \xrightarrow{(o)} x$ in X. Thus $x \in F$ and F is a closed set in τ_{02}. Conversely, let $D \subseteq Y$ is a closed set in τ_{02} and $(y_\alpha)_\alpha \subseteq D$ be such that $y_\alpha \xrightarrow{(o)} y$ in X. In view of (i) resp. (v) we have $y \in Y$ and hence $y \in D$. Thus D is a closed set in τ_{01}. In view of (v), Y is a closed set in τ_{01}. We conclude that $\tau_{01} \cap Y = \tau_{02}$.

2. Homomorphisms of abelian RI-posets, order continuity

DEFINITION 2.1. Suppose that $(X_1; \leq_1, \ominus_1, 0_1)$, $(X_2; \leq_2, \ominus_2, 0_2)$ are abelian RI-posets. A map $h : X_1 \to X_2$ is called a homomorphism if for all $a, b \in X_1$ with defined $b \ominus_1 a$ also $h(b) \ominus_2 h(a)$ is defined in which case $h(b \ominus_1 a) = h(b) \ominus_2 h(a)$.

It follows easily from the definition 2.1 that every homomorphism $h : X_1 \to X_2$ is an increasing map (i.e. $a \leq_1 b \Rightarrow h(a) \leq_2 h(b)$) and $h(0_1) = 0_2$. Moreover a map $h : X_1 \to X_2$ is a homomorphism iff

$$h(a \oplus_1 b) = h(a) \oplus_2 h(b)$$

for all $a, b \in X_1$ with defined $a \oplus_1 b$ (here $\oplus_1$, $\oplus_2$ are partial operations in X_1 and X_2, respectively, defined by condition (vi) of Definition 1.1). If for elements $a, b \in X_1$ there exist elements $a \oplus_1 b$, $a \vee b$ and $a \wedge b$ then $(a \oplus_1 b) = (a \vee b) \oplus_1 (a \wedge b)$ (see Lemma 1.3) and by previous statement

$$h(a \oplus_1 b) = h(a \vee b) \oplus_2 h(a \wedge b).$$

The last statement implies the "orthogonal additivity" of every homomorphism. That means, if $a, b \in X_1$ are such that $a \oplus_1 b$ is defined and $a \wedge b = 0$ (we call elements a, b orthogonal) and $a \vee b$ exists then

$$h(a \vee b) = h(a) \oplus_2 h(b).$$

It is worth to note that important examples of abelian RI-posets is the set R^+ of nonnegative real numbers and the interval $\langle 0, 1\rangle \subseteq R^+$ with $b \ominus a$ replacing $b - a$ defined for all $a \leq b$. Then homomorphism $h : X_1 \to X_2 = \langle 0, 1\rangle$ or R^+ is a state or a finitely additive measure on X_1.

PROPOSITION 2.2. *Suppose that* $(X_1; \leq_1, \ominus_1, 0_1)$, $(X_2; \leq_2, \ominus_2, 0_2)$ *are abelian RI-posets. If for homomorphisms* $h : X_1 \to X_2$ *and* $g : X_1 \to X_2$ *it holds* $g(x) \leq_2 h(x)$ *for all* $x \in X_1$, *then a map* $k : X_1 \to X_2$ *defined by* $k(x) = h(x) \ominus_2 g(x)$ *for all* $x \in X_1$ *is a homomorphism.*

Proof. Assume that $a, b \in X_1$ with $a \leq_1 b$. Then $g(a) \leq_2 g(b) \leq_2 h(b)$ and $g(a) \leq_2 h(a) \leq_2 h(b)$. It follows by Lemma 1.2

$$h(b) \ominus_2 h(a) = [h(b) \ominus_2 g(a)] \ominus_2 [h(a) \ominus_2 g(a)]$$

and

$$h(b) \ominus_2 g(b) = [h(b) \ominus_2 g(a)] \ominus_2 [g(b) \ominus_2 g(a)].$$

From which it follows that

$$\begin{aligned} k(b \ominus_1 a) &= h(b \ominus_1 a) \ominus_2 g(b \ominus_1 a) = [h(b) \ominus_2 h(a)] \ominus_2 [g(b) \ominus_2 g(a)] = \\ &= \{[h(b) \ominus_2 g(a) \ominus_2 [h(a) \ominus_2 g(a)]\} \ominus_2 [g(b) \ominus_2 g(a)] = \\ &= \{[h(b) \ominus_2 g(a)] \ominus_2 [g(b) \ominus_2 g(a)]\} \ominus_2 [h(a) \ominus_2 g(a)] = \\ &= [h(b) \ominus_2 g(b)] \ominus_2 [h(a) \ominus_2 g(a)] = k(b) \ominus_2 k(a). \end{aligned}$$

Hence k is a homomorphism.

Proposition 2.2 allows us to show that the set $\mathcal{H}(X_1, X_2)$ of all homomorphisms from an abelian RI-poset X_1 into an abelian RI-poset X_2 is again an abelian RI-poset.

THEOREM 2.3. *Suppose that* $(X_1; \leq_1, \ominus_1, 0_1)$, $(X_2; \leq_2, \ominus_2, 0_2)$ *are abelian RI-posets. Let* $\mathcal{H}(X_1, X_2) = \{h : X_1 \to X_2 \mid h \text{ is a homomorphism }\}$. *If we define in* $\mathcal{H}(X_1, X_2)$ *the partial order* $\leq$ *and partial binary operation* $\ominus$ *"pointwise" (i.e. for any* $f, g \in \mathcal{H}(X_1, X_2)$ $f \leq g$ *iff* $f(x) \leq_2 g(x)$ *for all* $x \in X_1$ *in which case* $(g \ominus f)(x) = g(x) \ominus_2 f(x)$) *then* $(\mathcal{H}(X_1, X_2); \leq, \ominus, 0)$ *is an abelian RI-poset in which* $0(x) = 0_2$ *for all* $x \in X_1$.

Proof. Pointwise definition of partial order $\leq$ and $\ominus$ in the set $\mathcal{H}(X_1, X_2)$ guarantees that the axioms (i)–(viii) from Definition 1.1 are satisfied. The statement now follows by Proposition 2.2.

THEOREM 2.4. *For Dedekind complete abelian RI-posets* $(X_1; \leq_1, \ominus_1, 0_1)$, $(X_2; \leq_2, \ominus_2, 0_2)$ *and a homomorphism* $h : X_1 \to X_2$ *the following conditions are equivalent:*

(i) $x_\alpha \uparrow x \Rightarrow h(x_\alpha) \uparrow h(x)$,

(ii) $x_\alpha \downarrow x \Rightarrow h(x_\alpha) \downarrow h(x)$,

(iii) $x_\alpha \xrightarrow{(o)} x \Rightarrow h(x_\alpha) \xrightarrow{(o)} h(x)$,

(iv) $x_\alpha \downarrow 0_1 \Rightarrow h(x_\alpha) \downarrow 0_2$.

Proof. We shall use Theorem 1.6 in all parts of the proof.

(i)⇒(iv): Suppose that $x_\alpha \downarrow 0_1$ ($\alpha \in \mathcal{E}$). Let $\beta \in \mathcal{E}$ be fixed and let $\mathcal{E}' = \{\gamma(\alpha) \in \mathcal{E} \mid \alpha \in \mathcal{E} \text{ and } \gamma(\alpha) \geq \alpha, \beta\}$. Then $\mathcal{E}'$ is cofinal in $\mathcal{E}$ and hence $x_{\gamma(\alpha)} \downarrow 0_1$. Therefore $x_{\gamma(\alpha)} \leq x_\beta$ for every $\gamma(\alpha) \in \mathcal{E}'$, application of Proposition 1.6 (ii) gives $x_\beta \ominus x_{\gamma(\alpha)} \uparrow x_\beta$. Now using (i) we obtain $h(x_\beta) \ominus_2 h(x_{\gamma(\alpha)}) \uparrow h(x_\beta)$. Repeated application of Proposition 1.6 (ii) gives $h(x_{\gamma(\alpha)}) \downarrow 0_2$. We conclude $h(x_\alpha) \downarrow 0_2$.

(iv)⇒(ii): If $x_\alpha \downarrow x$ then $x_\alpha \ominus_1 x \downarrow 0_1$ which implies $h(x_\alpha \ominus_1 x) \downarrow 0_2$. Hence $h(x_\alpha) \downarrow h(x)$.

(ii)⇒(i): Suppose $x_\alpha \uparrow x$. Then $x \ominus_1 x_\alpha \downarrow 0$ and hence $h(x \ominus_1 x_\alpha) \downarrow h(0_1) = 0_2$. It follows $h(x) \ominus_2 h(x_\alpha) \downarrow 0_2$, which implies $h(x_\alpha) \uparrow h(x)$.

(iii)⇒(i): This is obvious.

(i)⇒(iii): If $x_\alpha \xrightarrow{(o)} x$ then there exist nets $(u_\alpha)_\alpha$, $(v_\alpha)_\alpha$ of elements of X_1 with $u_\alpha \leq_1 x_\alpha \leq_1 v_\alpha$ and such that $u_\alpha \uparrow x$, $v_\alpha \downarrow x$. Since h is increasing we have $h(u_\alpha) \leq_2 h(x_\alpha) \leq_2 h(v_\alpha)$. Moreover $h(u_\alpha) \uparrow h(x)$ and $h(v_\alpha) \downarrow h(x)$ by (i) and (ii). We conclude $h(x_\alpha) \xrightarrow{(o)} h(x)$.

A homomorphism $h : X_1 \to X_2$ where $(X_1; \leq_1, \ominus_1, 0_1)$, $(X_2; \leq_2, \ominus_2, 0_2)$ are abelian RI-posets is called *order continuous* if $x_\alpha \xrightarrow{(o)} x$ in X_1 implies $h(x_\alpha) \xrightarrow{(o)} h(x)$ in X_2, for all $x_\alpha, x \in X_1$, a homomorphism h is called *bounded* if there exists $y_0 \in X_2$ with property $h(x) \leq_2 y_0$ for all $x \in X_1$. If for a homomorphism h there exists a set $\emptyset \neq C \subseteq X_1$ such that for every $x \in X_1$

$$h(x) = \bigvee \{h(y) \mid y \in C \text{ with } y \leq x\}$$

then we say that h is *C-regular.*

We shall say that a poset $(X; \leq)$ is *compactly generated* by a set $\emptyset \neq C \subseteq X$ if the following two conditions are satisfied:

(i) For every $x \in X$ there exists a set $M \subseteq C$ with $\vee M = x$.

(ii) For every $P \subseteq C$ with $\vee P \in X$ and every $p \in C$ with $p \leq \vee P$ there exists a finite set $F \subseteq P$ with $p \leq \vee F$.

LEMMA 2.5. *Assume that a poset $(X_1; \leq_1)$ is compactly generated by a set $\emptyset \neq C \subseteq X_1$ with property: $x, y \in C$ implies $x \vee y \in C$. Let $(X_2; \leq_2)$ be a Dedekind complete poset. Then for every increasing map $f : X_1 \to X_2$ (i.e. $x \leq_1 y \Rightarrow f(x) \leq_2 f(y)$) the following two conditions are equivalent:*

(i) *f is C-regular*

(ii) *$x_\alpha \uparrow x$ implies $f(x_\alpha) \uparrow f(x)$ for all $x_\alpha, x \in X$.*

Proof. (i)⇒(ii): Suppose that $x_\alpha \uparrow x$, $x_\alpha, x \in X$, $\alpha \in \mathcal{E}$ and put $M = \{y \in C \mid y \le x_\alpha,\ \alpha \in \mathcal{E}\}$. Evidently $x = \bigvee M = \bigvee_\alpha x_\alpha$. Since X_1 is compactly generated by C, for every $q \in C$, $q \le x$ there exists a finite set $Q \subseteq M$ with $q \le \vee Q$. Since $\mathcal{E}$ is directed upwards and Q is finite, there exists $\alpha_0 \in \mathcal{E}$ with $q \le \vee Q \le x_{\alpha_0}$. We obtain $f(x) = \vee\{f(y) \mid y \in C$ with $y \le x\} \le \vee\{f(x_\alpha) \mid \alpha \in \mathcal{E}\} \le f(x)$. Hence, $f(x_\alpha) \uparrow f(x)$.

(ii)⇒(i): For $x \in X_1$ let $C_x = \{y \in C \mid y \le x\}$. Let the set $\mathcal{E} = \{F \subseteq C_x \mid F$ is a finite set$\}$ be directed by set inclusion. For every $F \in \mathcal{E}$ we put $x_F = \vee F$. Then $x_F \in C$ and $x_F \uparrow x$. It follows $f(x_F) \uparrow f(x)$ and hence $f(x) = \vee\{f(y) \mid y \in C$ with $y \le x\}$.

THEOREM 2.6. *Let $(X_1, \le_1, 0_1)$, $(X_2, \le_2, 0_2)$ be Dedekind complete abelian RI-poset. Suppose that X_1 is compactly generated by a set $\emptyset \ne C \subseteq X_1$ with the properties:*

(i) *If $x, y \in C$ then $x \vee y \in C$.*

(ii) *If $x, y \in C$ and $x \oplus_1 y$ is defined then $x \oplus_1 y \in C$.*

Moreover, assume that there exist $y_0 \in X_2$ and a map $\nu : C \to X_2$ such that $\nu(x) \le_2 y_0$ for every $x \in X_1$, $\nu(0_1) = 0_2$ and $\nu(x \oplus_1 y) = \nu(x) \oplus_2 \nu(y)$ for all $x, y \in C$ with defined $x \oplus_1 y$.

Then there exists a unique order-continuous homomorphism $g : X_1 \to X_2$ such that $g(x) = \nu(x)$ for every $x \in C$. In such case g is C-regular.

Proof. Define for every $0_1 \ne x \in X_1$ $g(x) = \vee\{\nu(y) \mid y \in C$ with $y \le x\}$. Evidently $x_1 \le x_2$ implies $g(x_1) \le g(x_2)$ for every $x_1, x_2 \in X_1$ and hence by Lemma 2.5 $x_\alpha \uparrow x$ implies $g(x_\alpha) \uparrow g(x)$ for every net $(x_\alpha)_\alpha \subseteq X_1$ and every $x \in X_1$. It remains to prove that g is a homomorphism. Suppose that $x, y \in X_1$ with defined $x \oplus_1 y$ and let $A_x = \{z \in C \mid z \le x\}$, $A_y = \{z \in C \mid z \le y\}$. Assume that the set $\mathcal{F} = \{F \subseteq Ax \cup Ay \mid F$ is a finite set such that $\emptyset \ne F \cap A_x$ and $\emptyset \ne F \cap A_y\}$ is directed by set inclusion. For every $F \in \mathcal{F}$ we put $x_F = \vee(F \cap A_x)$, $y_F = \vee(F \cap A_y)$. Clearly $x_F, y_F \in C$ for every $F \in \mathcal{F}$ and $x_F \uparrow x$, $y_F \uparrow y$. Thus using our assumption and the previously proved properties of g, we have by Theorem 1.6(iv) that $x_F \oplus_1 y_F \uparrow x \oplus_1 y$, which implies

$$g(x_F \oplus_1 y_F) \uparrow g(x \oplus_1 y)$$

and also

$$g(x_F \oplus_1 y_F) = \nu(x_F \oplus_1 y_F) = \nu(x_F) \oplus_2 \nu(y_F) = \\ = g(x_F) \oplus_2 g(y_F) \uparrow g(x) \oplus_2 g(y).$$

We conclude that $g(x \oplus_1 y) = g(x) \oplus_2 g(y)$, which proves that g is a homomorphism. The order continuity of g follows by Theorem 2.4 and the C-regularity of g is evident from the definition of g. Finally, the uniqueness of g follows from the fact that every nonzero element of X_1 is the supremum of an increasingly directed net of elements of C.

In what follows we use the notations of Theorem 2.3

COROLLARY 2.7. *Assume that abelian RI-posets* $(X_1; \leq_1, \ominus_1, 0_1)$, $(X_2; \leq_2, \ominus_2, 0_2)$ *are Dedekind complete and* X_1 *is compactly generated by a set* $\emptyset \neq C \subseteq X_1$ *with the properties* (i) *and* (ii) *stated in Theorem* 2.6. *Then for every bounded homomorphism* $h \in \mathcal{H}(X_1, X_2)$ *there exist a unique order continuous homomorphism* $g \in \mathcal{H}(X_1, X_2)$ *and a unique* $k \in \mathcal{H}(X_1, X_2)$ *such that* $h = g \oplus k$ *and* $\ell = 0$ *for every order continuous* $\ell \in \mathcal{H}(X_1, X_2)$ *with* $\ell \leq k$.

Proof. Since the restriction $\nu = h|_C$ of any bounded homomorphism $h \in \mathcal{H}(X_1, X_2)$ to the set C satisfies assumptions of Theorem 2.6, there exists a unique order continuous homomorphism $g \in \mathcal{H}(X_1, X_2)$ such that $g(x) = \nu(x) = h(x)$ for every $x \in C$. In view of Proposition 2.2 $k = h \ominus g \in \mathcal{H}(X_1, X_2)$ and $k(x) = 0_2$ for all $x \in C$. On the other hand an order continuous homomorphism $\ell \in \mathcal{H}(X_1, X_2)$ is trivial, i.e. $\ell = 0$ iff $\ell(x) = 0_2$ for every $x \in C$. This completes the proof, since $h = g \oplus (h \ominus g) = g \oplus k$.

3. Applications, examples and remarks

Recall that an abelian RI-poset $(X; \leq, \ominus 0)$ (resp. an abelian RI-semigroup $(X; \leq, \oplus 0))$ is a *D-poset* (resp. an *effect algebra*) if and only if X contains a greatest element denoted by 1. An effect algebra is an *orthoalgebra* if and only if $x \oplus x$ is defined implies $x = 0$, for every $x \in X$. An orthoalgebra can be organized into an orthomodular poset (with orthocomplementation $x' = 1 \ominus x$, $x \in X$) if and only if $x \vee y$ exists whenever $x \oplus y$ is defined for $x, y \in X$. An orthomodular poset is a *Boolean algebra* if and only if it is a distributive lattice. (for definitions and proofs of the statements see [5], [8], [9]). Moreover, a positive cone of a partially ordered abelian group with $+$ replaced by $\oplus$ is an abelian RI-semigroup. In particular a positive cone of a Riesz space (see [6]) or a set R^+ of all nonnegative real numbers and also the interval $\langle 0, 1\rangle$ (in which case $a \oplus b = a + b$ for all $a, b \in \langle 0, 1\rangle$ such that $a + b \in \langle 0, 1\rangle$ (see [10]) are examples of abelian RI-semigroups.

For all these examples of abelian RI-posets (RI-semigroups) and homomorphisms as measures, probabilities, observables and states on these structures theorems of Sections 1 and 2 can be applied. Let us mention at least papers [1], [2], [3] and [4] which deal with existence of decompositions of finitely additive measures defined on orthomodular posets, orthoalgebras and difference posets with values in positive cones of Dedekind complete normed spaces or more general in Dedekind complete lattice ordered abelian groups. Corollary 2.7 contains some sufficient conditions under which such decompositions are unique.

A particular case of Theorem 2.6 for compactly atomistic orthomodular lattices and real valued measures is proved in [12].

The notion of observable as a morphism of quantum logics from the σ-algebra $\mathcal{B}(H)$ of Borel sets of a separable Banach space H into a given quantum logic L is defined and studied in [11].

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