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SOME FIXED POINT THEOREMS

1. Introduction

In recent years, a number of generalizations of a well-known contraction mapping principles due to Banach have appeared in the literature (see [5]).

In our theorems we extend the results of M. R. Tasković and those of M. Ohta and G. Nikaido for multi-valued mappings (see also e.g. [7], [2] or others).

Let (X,d) be a metric space and let T be a correspondens (i.e., mapping from points to sets) from X to $CB(X)$ which is not necessarily continuous.

We shall denote by $CB(X)$ the set of all non-empty closed and bounded subsets of X, by $\delta(A,B)$ the diameter of A and B, i.e., $\delta(A,B) = \sup\{d(a,b) : a \in A,\ b \in B\}$, and the Hausdorff distance of $A, B \in CB(X)$ will be denote by $H(A,B) = \max\{\sup_{a\in A}\inf_{b\in B} d(a,b), \sup_{b\in B}\inf_{a\in A} d(a,b)\}$. Also, for any $x \in X$, we denote by $O(x)$ the orbit of x, i.e.,

$$O(x) = \Big\{\bigcup T^n x : n = 0,1,2,\dots\Big\}.$$

A point $x \in X$ will be called a fixed point of T, if $x \in Tx$.

We will say that $T : X \to CB(X)$ is upper semicontinuous (u.s.c) at $\overline{x} \in X$, if for every neighbourhood U of $T(\overline{x})$, there exists a neighbourhood V of $\overline{x}$ such that $T(x) \subset U$ for every $x \in V$. A set valued function $T : X \to CB(X)$ is called u.s.c. on X, if it is u.s.c. at every $x \in X$.

2. Main results

THEOREM 1. *Let (X,d) be a bounded complete metric space and let T be an upper semicontinuous mapping from X into $CB(X)$ such that for any $x, y \in X$, there is*

(1) $$\delta(Tx,Ty) \le \alpha\delta[O(x) \cup O(y)], \quad \alpha \in [0,1).$$

Then T has a unique fixed point ξ such that $\{\xi\} = T\xi$.

Proof. Let $x_0 \in X$. Define $x_i \in Tx_{i-1}$, $i = 1, 2, 3, \ldots$ From (1) we have

(2) $$d(x_1, x_2) \leq \delta(Tx_0, Tx_1) \leq \alpha\delta[O(x_0) \cup O(x_1)] \leq \alpha\delta[O(x_0)]$$

Now, for non-negative n, we prove that

(3) $$\delta[O(T^n x_0)] \leq \alpha\delta[O(T^{n-1}x_0)] \ .$$

By

(4) $$\delta[O(Tx_0)] = \sup_{i,j}\{\delta(T^i x_0, T^j x_0)\}, \qquad i \geq 1,\ i < j,$$

(5) $$\delta(T^i x_0, T^j x_0) = \sup\{d(x_{i+1}, x_{j+1}) : x_{i+1} \in T^i x_0, x_{j+1} \in T^j x_0\},$$

and from (1) we have,

$$d(x_{i+1}, x_{j+1}) \leq \delta(Tx_i, Tx_j) \leq$$
$$\leq \alpha\delta[O(x_i) \cup O(x_j)] \leq \alpha\delta[O(x_i)] \leq \alpha\delta[O(x_0)].$$

From (5) and (4) we have $\delta[O(Tx_0)] \leq \alpha\delta[O(x_0)]$. Similarly,

(6) $$\delta[O(T^2 x_0)] = \sup_{i,j}\{\delta(T^i x_0, T^j x_0)\}, \qquad i \geq 2,\ i < j,$$

(7) $$\delta(T^i x_0, T^j x_0) = \sup\{d(x_{i+1}, x_{j+1}) : x_{i+1} \in T^i x_0, x_{j+1} \in T^j x_0\},$$

$$d(x_{i+1}, x_{j+1}) \leq \delta(Tx_i, Tx_j) \leq \alpha\delta[O(x_i) \cup O(x_j)]$$
$$\leq \alpha\delta[O(x_i)] \leq \alpha\delta[O(Tx_0)],$$

because $x_i \in Tx_{i-1}$, $x_j \in Tx_{j-1} \subset O(Tx_0)$.

From (7) and (6) we have $\delta[O(T^2 x_0)] \leq \alpha\delta[O(Tx_0)]$.

By induction, we obtain for each $n = 1, 2, 3, \ldots$

(8) $$\delta[O(T^n x_0)] \leq \alpha\delta[O(T^{n-1}x_0)].$$

Now

$$d(x_2, x_3) \leq \delta(Tx_1, Tx_2) \leq \alpha\delta[O(x_1) \cup O(x_2)]$$
$$\leq \alpha\delta[O(x_1)] \leq \alpha\delta[O(Tx_0)] \leq \alpha^2\delta[O(x_0)].$$

So

(9) $$d(x_2, x_3) \leq \alpha^2\delta[O(x_0)].$$

Repeating the above argument, we obtain for each $n = 1, 2, \ldots$

(10) $$d(x_{n+1}, x_n) \leq \alpha^n\delta[O(x_0)].$$

Now,

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m)$$
$$\leq \alpha^n\delta[O(x_0)] + \alpha^{n+1}\delta[O(x_0)] + \ldots + \alpha^{m-1}\delta[O(x_0)] + \ldots$$
$$\leq \alpha^n\delta[O(x_0)][1 + \alpha + \alpha^2 + \ldots] \leq \frac{\alpha^n}{1-\alpha}\delta[O(x_0)].$$

So $\{x_n\}$ is a Cauchy sequence. By the completeness of (X,d), the sequence $\{x_n\}$ converges to some point ξ in X.

We shall now prove that ξ is a fixed point of T. As T is upper semicontinuous, $x_n \to \xi$ and $x_n \in Tx_{n-1}$ implies that $\xi \in T\xi$. Now, by (1),

$$0 \le \delta(\xi, T\xi) \le \delta(T\xi, T\xi) \le \alpha\delta[O(\xi)]$$
$$\le \alpha\delta[O(T\xi)] \le \alpha^2\delta[O(\xi)] \le \ldots \le \alpha^n\delta[O(\xi)]$$

for any $n \ge 1$. So, as $0 \le \alpha < 1$, there is $\delta(\xi, T\xi) = 0$ which means that $\{\xi\} = T\xi$.

Now, let η be a fixed point of T such that $\{\eta\} = T\eta$ and $\xi \ne \eta$. Then

$$d(\xi,\eta) \le \delta(\xi, T\eta) \le \delta(T\xi, T\eta) \le \alpha\delta[O(\xi)\cup O(\eta)] \le \alpha d(\xi,\eta)$$

because $O(\xi) = \{\xi\}$ and $O(\eta) = \{\eta\}$. Hence $d(\xi,\eta) \le \alpha d(\xi,\eta)$, $\alpha \in [0,1)$, which is a contradiction. Thus $\xi = \eta$, and therefore, ξ is a unique fixed point of T such that $\{\xi\} = T\xi$. The proof is complete.

We will use the following lemma.

LEMMA. *Let E be a topological space and let*

$$f : E \to [0,\infty), \; g : E \longrightarrow [0,\infty).$$

If f is upper semicontinuous and g is lower semicontinuous, then f/g is upper semicontinuous.

THEOREM 2. *Let (X,d) be a compact metric space and assume that T is a continuous mapping from X into $(CB(X),H)$ such that for any $x,y \in X$, $x \ne y$, there is*

$$\delta(Tx,Ty) < \delta[O(x)\cup O(y)]. \tag{11}$$

Then T has a unique fixed point ξ such that $\{\xi\} = T\xi$.

Proof. Let $x \in X$. Since $O(T^n x) \supset O(T^{n+1}x) \supset \ldots$, by the compactness of $T^n x$ (see: proposition 2.3. [1], p. 31) and by Cantor theorem, we have $\bigcap_{n=1}^{\infty} O(T^n x) \ne \emptyset$. There exists $\xi \in X$ such that $\xi \in \bigcap_{n=1}^{\infty} O(T^n x)$. So, $O(\xi) \subset O(T^n x)$. The sequence $\{\delta[O(T^n x)]\}$ is non-negative and non-increasing, so it is convergent.

Now $0 \le \delta[O(\xi)] \le \delta[O(T^n x)]$. If $\delta[O(T^n x)] \to 0$, we have $\delta[O(\xi)] = 0$. And hence ξ is a fixed point of T such that $\{\xi\} = T\xi$.

Now we prove that $\delta[O(T^n x)] \to 0$. Let $\gamma = \lim_{n\to\infty} \delta[O(T^n x)] > 0$. Since (X,d) is compact and T is continuous on X, it is uniformly continuous on X, and so

$$\exists_{\varepsilon_0>0}\forall_{x,y\in X}\, d(x,y) < \varepsilon_0 : \delta(Tx,Ty) \le \frac{\gamma}{2}. \tag{12}$$

Let $K = \{(x,y) \in X \times X : d(x,y) \geq \varepsilon_0\}$ and consider the following mappings $f, g, g_n : K \to R$, $n \in N$, defined by

$$f(x,y) = \delta(Tx,Ty); \quad g(x,y) = \delta[O(x) \cup O(y)];$$
$$g_n(x,y) = \delta[\{x,Tx,\ldots,T^n x, y, Ty, \ldots, T^n y\}].$$

Since each g_n is continuous and $g = \sup\{g_n : n \in N\}$, g is lower semi-continuous and hence f/g is upper semi-continuous by the Lemma.

Moreover, we note that $0 \leq f/g < 1$, by (11). It follows from the compactness of K that f/g attains its maximum $\beta_0 \in [0,1)$. Hence, for each $x, y \in X$ with $d(x,y) \geq \varepsilon_0$, we have

$$\delta(Tx,Ty) \leq \beta_0 \delta[O(x) \cup O(y)]. \tag{13}$$

Now let $\beta = \max\{\beta_0, \frac{1}{2}\}$. Then for each non-negtive integer n, p, q, we claim that

$$\delta(T^{n+p+1}x, T^{n+q+1}x) \leq \beta\delta[O(T^n x)]. \tag{14}$$

In fact, if $\delta(T^{n+p}x, T^{n+q}x) < \varepsilon_0$, it follows from (12) that

$$\delta(T^{n+p+1}x, T^{n+q+1}x) \leq \frac{\gamma}{2} \leq \frac{1}{2}\delta[O(T^n x)] \leq \beta\delta[O(T^n x)].$$

On the other hand, if $\delta(T^{n+p}x, T^{n+q}x) \geq \varepsilon_0$, we have

$$\delta(T^{n+p+1}x, T^{n+q+1}x) \leq \beta_0\delta[O(T^n x)] \leq \beta\delta[O(T^n x)],$$

by (13), which proves (14). Therefore, we have

$$\delta[O(T^{n+1}x)] \leq \beta\delta[O(T^n x)]$$

for each non-negative integer n, and so we obtain successively

$$\delta[O(T^n x)] \leq \beta\delta[O(T^{n-1}x)] \leq \ldots \leq \beta^n \delta[O(x)],$$

for each $n \in N$. Since $\beta \in [0,1)$ it follows that

$$\gamma = \lim_{n\to\infty} \delta[O(T^n x)] = 0$$

which is a contradiction to $\gamma > 0$.

Now let η be a fixed point of T such that $\{\eta\} = T\eta$ and $\xi \neq \eta$.

Then

$$d(\xi,\eta) < \delta(\xi, T\eta) < \delta(T\xi, T\eta) < \alpha\delta[O(\xi) \cup O(\eta)] < \alpha d(\xi,\eta), \qquad \alpha \in [0,1)$$

which is a contradiction. Thus, $\xi = \eta$ and therefore ξ is the unique fixed point of T such that $\{\xi\} = T\xi$ and so the proof is complete.

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