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ALMOST SEMI-INVARIANT SUBMANIFOLDS OF AN ε-FRAMED METRIC MANIFOLD

1. Introduction

Study of CR-submanifolds, as a generalization of invariant and anti-invariant submanifolds, of a Kaehler manifold was initiated by Bejancu [5] and was followed by several geometers (see [5, 34] and references cited therein). This concept was further generalized by Chen [10] who introduced generic submanifolds. Later, several authors [1-6, 8, 12–18, 21, 25, 27, 30, 31, 34] defined and studied semi-invariant and almost semi-invariant submanifolds, analogous to these CR and or generic submanifolds, of manifolds possessing structures different from Kaehler viz. almost contact [7], framed metric [33] or almost r-contact [32], almost paracontact [23], almost r-paracontact [9], and almost product Riemannian structures [33].

Recently generic submanifolds of a Kaehler manifold were introduced by Ronsse [22] which imply the generic submanifold given by Chen. Motivated by this, in the present paper we define and study almost semi-invariant submanifolds (section 4) of a manifold with an ε-framed metric structure [28] which reduces to all aforementioned structures in special cases.

The paper is organized as follows. Section 2 is devoted to preliminaries. In section 3 some basic results are given. The definition of an almost semi-invariant submanifold of an ε-framed metric manifold along with an example is given in section 4. In section 5 we establish some necessary and sufficient conditions for a submanifold to be an almost semi-invariant submanifold. Later in this section, an interesting set of twenty two necessary and sufficient conditions for a submanifold to be semi-invariant have been obtained. Section 6 deals with parallelism of certain operators arising

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naturally in our study. Integrability conditions for certain natural distributions on almost semi-invariant and semi-invariant submanifolds have been discussed in section 7. In the last section it has been shown that an almost semi-invariant submanifold, with non-trivial invariant distribution of a normal framed metric manifold [33], is a CR-manifold [11].

2. Preliminaries

Let $\overline{M}$ be an m-dimensional framed metric $(J(3,\varepsilon),g)$ manifold (for brevity ε-framed metric manifold) [28] with a framed metric $(J(3,\varepsilon),g)$ structure (for brevity ε-framed metric structure) of rank $m-r$, $r<m$; i.e., $\varepsilon^2=1$; $J\neq 0, I$ (I is the identity operator) is a tensor field of type $(1,1)$ with $\mathrm{Rank}(J)=m-r$; $\xi_1,\dots,\xi_r$ are vector fields; $\eta^1,\dots,\eta^r$ are 1-forms and g is an associated Riemannian metric such that

$$
(2.1)\qquad \begin{cases}
\text{(i) } J^3=\varepsilon J,\\
\text{(ii) } J^2=\varepsilon(I-\eta^\alpha\otimes\xi_\alpha),\\
\text{(iii) } J(\xi_\alpha)=0,\\
\text{(iv) } \eta^\alpha\circ J=0,\\
\text{(v) } \eta^\alpha(\xi_\beta)=\delta^\alpha_\beta,\\
\text{(vi) } g(JX,JY)=g(X,Y)-\sum_{\alpha=1}^{r}\eta^\alpha(X)\eta^\alpha(Y),\\
\text{(vii) } g(X,JY)=\varepsilon g(JX,Y),\\
\text{(viii) } g(\xi_\alpha,X)=\eta^\alpha(X),\\
\text{(ix) } g((\overline{\nabla}_X J)Y,Z)=\varepsilon g(Y,(\overline{\nabla}_X J)Z),
\end{cases}
$$

for all $X,Y,Z\in T\overline{M}$, where $\alpha,\beta\in\{1,\dots,r\}$ and $\overline{\nabla}$ is the Riemannian connection on $\overline{M}$.

This structure (resp. manifold) is a very general structure (resp. manifold) which in special cases reduces to several known structures (resp. manifolds) given below which are widely studied in recent past.

Structure/Manifold	r	ε	Reference
framed metric		-1	[33]
almost r-contact metric		-1	[32]
almost contact metric	1	-1	[7]
almost r-paracontact metric		1	[9]
almost paracontact metric	1	1	[23]
$(J(2,\varepsilon), g)$	0		[24]
almost Hermitian	0	-1	[33]
almost product Riemannian	0	1	[33]

Let M be a submanifold of a Riemannian manifold $\overline{M}$ with a Riemannian metric g. Then Gauss and Wiengarten formulas are given, respectively, by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \quad \text{and} \quad \overline{\nabla}_X N = -A_N X + \nabla^{\perp}_X N,$$

for all $X, Y \in TM$ and $N \in T^{\perp}M$, where $\overline{\nabla}, \nabla$ and $\nabla^{\perp}$ are the Riemannian, induced Riemannian and induced normal connections in $\overline{M}$, M and the normal bundle $T^{\perp}M$ of M, respectively, and h is the second fundamental form related to A by $g(h(X,Y),N) = g(A_N X, Y)$. Moreover, let J be a (1,1) tensor field on $\overline{M}$. For $X, Y \in TM$ and $N \in T^{\perp}M$ we put

$$JX = PX + FX, \quad PX \in TM, \ FX \in T^{\perp}M, \tag{2.2}$$

$$JN = tN + fN, \quad tN \in TM, \ fN \in T^{\perp}M, \tag{2.3}$$

$$\begin{cases} \text{(i) } (\nabla_X F)Y = \nabla^{\perp}_X FY - F\nabla_X Y, \\ \text{(ii) } (\nabla_X t)N = \nabla_X tN - t\nabla^{\perp}_X N, \\ \text{(iii) } (\nabla_X f)N = \nabla^{\perp}_X fN - f\nabla^{\perp}_X N, \\ \text{(iv) } (\nabla_X tF)Y = \nabla_X tFY - tF\nabla_X Y, \\ \text{(v) } (\nabla_X Ft)N = \nabla^{\perp}_X FtN - Ft\nabla^{\perp}_X N, \\ \text{(vi) } (dF)(X,Y) = \nabla^{\perp}_X FY - \nabla^{\perp}_Y FX - F[X,Y]. \end{cases} \tag{2.4}$$

3. Some basic results

We first state the following two lemmas, whose proofs are straightforward and hence omitted.

LEMMA 3.1. *Let* M *be a submanifold of an* ε*-framed metric manifold* $\overline{M}$ *such that* $\xi_\alpha \in TM$, $\alpha = 1, \ldots, r$. *Then*

$$\begin{cases} \text{(i) } P(\xi_\alpha) = 0 = F(\xi_\alpha), \\ \text{(ii) } \eta^\alpha \circ P = 0 = \eta^\alpha \circ F, \\ \text{(iii) } \varepsilon(I - \eta^\alpha \otimes \xi_\alpha) - P^2 = tF, \\ \text{(iv) } FP + fF = 0, \\ \text{(v) } tf + Pt = 0, \\ \text{(vi) } \varepsilon I - f^2 = Ft, \\ \text{(vii) } g(P^2X,Y) = \varepsilon g(PX,PY) = g(X,P^2Y), \\ \text{(viii) } g(tFX,Y) = \varepsilon g(FX,FY) = g(X,tFY), \\ \text{(ix) } g(FtN,V) = \varepsilon g(tN,tV) = g(N,FtV), \\ \text{(x) } g(f^2N,V) = \varepsilon g(fN,fV) = g(N,f^2V). \end{cases} \tag{3.1}$$

LEMMA 3.2. *For a submanifold M of a Riemannian manifold $\overline{M}$ with a $(1,1)$ tensor field J on $\overline{M}$, we have*

$$(3.2)\qquad \begin{cases} \text{(i) } (\overline{\nabla}_X J)Y = ((\nabla_X P)Y - A_{FY}X - th(X,Y)) + \\ \qquad\qquad + ((\nabla_X F)Y + h(X,PY) - fh(X,Y)), \\ \text{(ii) } (\overline{\nabla}_X J)N = ((\nabla_X t)N - A_{fN}X + PA_N X) + \\ \qquad\qquad + ((\nabla_X f)N + h(X,tN) + FA_N X). \end{cases}$$

Moreover, if $\overline{M}$ possesses an ε-framed metric structure, then

$$(3.3)\qquad \begin{aligned} &\text{(i) } g((\nabla_X P)Y, Z) = \varepsilon g(Y, (\nabla_X P)Z), \\ &\text{(ii) } g((\nabla_X t)N, Y) = \varepsilon g(N, (\nabla_X F)Y), \\ &\text{(iii) } g((\nabla_X f)N, V) = \varepsilon g(N, (\nabla_X f)V). \end{aligned}$$

Now let $\xi_1, \ldots, \xi_r \in TM$, and let $TM = E \oplus L$, where E denotes the distribution in M spanned by $\xi_1, \ldots, \xi_r$ and L is the complementary orthogonal distribution to E in M. Then the Lemma 3.1 leads to the following result.

PROPOSITION 3.3. *If* M *is a submanifold of $\overline{M}$ such that $\xi_1, \ldots, \xi_r \in TM$, then*

$$(3.4)\qquad \begin{cases} \text{(i) } \operatorname{Ker} P = \operatorname{Ker} P^2 = \operatorname{Ker}(tF - \varepsilon(I - \eta^a \otimes \xi_\alpha)), \\ \text{(ii) } \operatorname{Ker} F = \operatorname{Ker} tF = \operatorname{Ker}(P^2 - \varepsilon(I - \eta^a \otimes \xi_\alpha)), \\ \text{(iii) } \operatorname{Ker} t = \operatorname{Ker} Ft = \operatorname{Ker}(f^2 - \varepsilon I), \\ \text{(iv) } \operatorname{Ker} f = \operatorname{Ker} f^2 = \operatorname{Ker}(Ft - \varepsilon I). \end{cases}$$

Consequently, on L

$$(3.5)\qquad \begin{cases} \text{(i) } \operatorname{Ker} P|_L = \operatorname{Ker} P^2|_L = \operatorname{Ker}(tF|_L - \varepsilon I), \\ \text{(ii) } \operatorname{Ker} F|_L = \operatorname{Ker} tF|_L = \operatorname{Ker}(P^2||_L - \varepsilon I). \end{cases}$$

Proof. (3.4) follows from (3.1) (vii)–(x) and (3.1)(iii), (vi). Since $\eta^\alpha(X) = 0$ for $X \in L$, the relations (3.5) are implied by (3.4)(i), (ii).

4. Almost semi-invariant submanifolds

Let M be a submanifold of an ε-framed metric manifold $\overline{M}$. Then from (3.1)(vii) it follows that $(P^2)_x$ is symmetric on T_xM and therefore its eigenvalues are real and it is diagonalizable. If $X_x \in T_xM$ is an eigenvector corresponding to an eigenvalue $\mu(x)$ of $(P^2)_x$, then

$$\mu(x)\|X_x\|^2 = \mu(x)g(X_x, X_x) = g(P^2X_x, X_x) = \varepsilon g(PX_x, PX_x) = \varepsilon\|PX_x\|^2$$

which implies that $\mu(x)/\varepsilon \geq 0$. Moreover from (2.1)(vi) for all $Z \in T\overline{M}$ one has $\|JZ\| \leq \|Z\|$ and therefore

$$\frac{1}{\varepsilon}\mu(x)\|JX_x\|^2 \leq \frac{1}{\varepsilon}\mu(x)\|X_x\|^2 = \|PX_x\|^2.$$

Since decomposition of JX, by (2.2), is orthogonal, $\mu(x)$ is bounded by 0 and ε.

Now let $\xi_1, \ldots, \xi_r \in TM = E \oplus L$. For each $x \in M$ we may set

$$D_x^\lambda = \mathrm{Ker}(P^2|_L - \varepsilon\lambda^2(x)I)_x,$$

where $\lambda(x) \in [0,1]$ is such that $\varepsilon\lambda^2(x)$ is an eigenvalue of $(P^2|_L)_x$. Since $(P^2|_L)_x$ is symmetric and diagonalizable, there is some integer q such that $\varepsilon\lambda_1^2(x), \ldots, \varepsilon\lambda_q^2(x)$ are distinct eigenvalues of $(P^2|_L)_x$ and L_x can be decomposed as the direct sum of the mutually orthogonal P-invariant eigenspaces, i,e., $L_x = D_x^{\lambda_1} \oplus \ldots \oplus D_x^{\lambda_q}$.

If $\varepsilon = -1$ and $\lambda_i(x) > 0$, then $D_x^{\lambda_i}$ is even-dimensional. We note that

$$D_x^1 = \mathrm{Ker}(F|_L)_x = \{X_x \in L_x : \|X_x\| = \|PX_x\|\},$$
$$D_x^0 = \mathrm{Ker}(P|_L)_x = \{X_x \in L_x : \|X_x\| = \|FX_x\|\}.$$

Here D_x^1 is the maximal J-invariant, while D_x^0 is the maximal anti-J-invariant subspace of L_x.

Now, we introduce a notion analogous to generic and skew CR-submanifolds of an almost Hermitian manifold defined in [22].

DEFINITION 4.1. A submanifold M of an ε-framed metric manifold $\overline{M}$ with all $\xi_1, \ldots, \xi_r \in TM$ is said to be an *almost semi-invariant submanifold* of $\overline{M}$, if there exist k functions $\lambda_1, \ldots, \lambda_k$, definded on M with values on (0,1), such that

(i) $\varepsilon\lambda_1^2(x), \ldots, \varepsilon\lambda_k^2(x)$ are distinct eigenvalues of $(P^2|_L)_x$ at $x \in M$ with

$$T_xM = D_x^1 \oplus D_x^0 \oplus D_x^{\lambda_1} \oplus \ldots \oplus D_x^{\lambda_k} \oplus E_x,$$

(ii) the dimensions of $D_x^1, D_x^0, D_x^{\lambda_1}, \ldots, D_x^{\lambda_k}$ are independent of $x \in M$.

If in addition each λ_i is constant, then M is called an *almost semi-invariant* submanifold.* If $k = 0$, then M is called *semi-invariant submanifold.* In fact, if $k = 0$ in Definition 4.1, then (i)→(ii) and M becomes a semi-invariant submanifold (see Proposition 5.3). If $k = 0$ and $D_x^1 = \{0\}$, (resp. $D_x^0 = \{0\}$), then M becomes an *anti-invariant* (resp. *invariant*) *submanifold.*

Condition (ii) in Definition 1.4 enables us to define P-invariant mutually orthogonal distributions

$$D^\lambda = \bigcup_{x\in M} D^\lambda_x, \quad \lambda \in \{0, \lambda_1, \dots, \lambda_k, 1\},$$

on M such that $TM = D^1 \oplus D^0 \oplus D^{\lambda_1} \oplus \dots \oplus D^{\lambda_k} \oplus E$. The differentiability of these distributions follows from the fact that their dimensions are constant [19].

For $X \in TM$ we may write

$$X = U^1X + U^0X + U^{\lambda_1}X + \dots + U^{\lambda_k}X + \eta^\alpha(X)\xi_\alpha, \tag{4.1}$$

where $U^1, U^0, U^{\lambda_1}, \dots$ and U^{λ_k} are orthogonal projection operators of TM on $D^1, D^0, D^{\lambda_1}, \dots$ and D^{λ_k}, respectively.

Example 4.2. We consider the Euclidean space $\Re^{8+r}$ and denote its points by $x = (x^i)$. Let $(e_j), j = 1, \dots, 8+r$, be the natural basis defined by $e_j = \partial/\partial x^j$. We put $\varepsilon^2 = 1$ and define vector fields ξ_α by $\xi_\alpha = e_{8+\alpha}, \alpha = 1, \dots, r$; 1-forms η^α by $\eta^\alpha = \varepsilon dx^{8+\alpha}, \alpha = 1, \dots, r$; and a (1,1) tensor field J by

$$Je_1 = \varepsilon e_2, \quad Je_2 = e_1, \quad Je_3 = \varepsilon e_8, \quad Je_8 = e_3, \quad Je_{8+\alpha} = 0, \ \alpha = 1, \dots, r,$$
$$Je_4 = \varepsilon \cos\nu(x)e_5 - \varepsilon \sin\nu(x)e_6, \quad Je_5 = \cos\nu(x)e_4 + \sin\nu(x)e_7,$$
$$Je_6 = -\sin\nu(x)e_4 + \cos\nu(x)e_7, \quad Je_7 = \varepsilon \ \sin\nu(x)e_5 + \varepsilon\cos\nu(x)e_6,$$

where $\nu : \Re^{8+r} \to (-\pi/2, \pi/2)$ is some function. Then it is easy to verify that $\Re^{8+r}$ possesses an ε-framed metric structure $(J, \xi_\alpha, \eta^\alpha, g)$, where g is the canonical metric on $\Re^{8+r}$ given by $g(e_i, e_j) = \delta_{ij}$; $i, \ j = 1, \dots, 8+r$.

The submanifold

$$\Re^{5+r} = \{(x^1, \dots, x^8, x^9, \dots, x^{8+r}) \in \Re^{8+r} | x^6, x^7, x^8 = 0\}$$

of $\Re^{8+r}$ is an almost semi-invariant submanifold with

$$D^1 = \mathrm{Span}\{e_1, e_2\}, \quad D^0 = \mathrm{Span}\{e_3\},$$
$$D^\lambda = \mathrm{Span}\{e_4, e_5\}, \quad E = \mathrm{Span}\{e_9, \dots, e_{8+r}\},$$

where $\lambda(x) = \cos\nu(x)$ for $x \in \Re^{5+r}$.

From now an almost semi-invariant, almost semi-invariant* and semi-invariant will be denoted by ASI, ASI* and SI, respectively, and we denote by M a submanifold of an ε-framed metric manifold $\overline{M}$ such that $\xi_1, \dots, \xi_r \in TM$ unless otherwise stated.

5. Some characterizations of almost semi-invariant submanifolds

Like P^2, it can be seen that the operators tF, Ft and f^2 are symmetric and their eigenvalues are bounded by 0 and ε. Let $\varepsilon\lambda^2(x)$, $0 \le \lambda(x) \le 1$, be an eigenvalue of $(f^2)_x$ at $x \in M$ and let $\underline{D}_x^\lambda$ denote the corresponding eigenspace $\underline{D}_x^\lambda = \mathrm{Ker}(f^2 - \varepsilon\lambda^2(x)I)_x$.

In particular, we note that

$$\underline{D}_x^1 = \mathrm{Ker}\, t_x = \{N_x \in T_x^\perp M : \|N_x\| = \|fN_x\|\},$$
$$\underline{D}_x^0 = \mathrm{Ker}\, f_x = \{N_x \in T_x^\perp M : \|N_x\| = \|tN_x\|\}.$$

For $\lambda \neq 1$ we have $FD_x^\lambda = \underline{D}_x^\lambda$ and $t\underline{D}_x^\lambda = D_x^\lambda$. Equivalently, at $x \in M$, X_x (resp. N_x) is an eigenvector of $(P^2|_L)_x$ (resp. $(f^2)_x$) corresponding to an eigenvalue $\varepsilon\lambda^2(x)$ iff FX_x (resp. tN_x) is an eigenvector of $(f^2)_x$ (resp. $(P^2|_L)_x$) corresponding to the same eigenvalue $\varepsilon\lambda^2(x)$. Consequently, $\mathrm{Dim}(D_x^\lambda) = \mathrm{Dim}(\underline{D}_x^\lambda)$. Thus, for a submanifold M of $\overline{M}$ with $\xi_1, \ldots, \xi_r \in TM$ the statements

(1) $T_xM = D_x^1 \oplus D_x^0 \oplus D_x^{\lambda_1} \oplus \ldots \oplus D_x^{\lambda_k} \oplus E_x$,
(2) $T_x^\perp M = \underline{D}_x^1 \oplus \underline{D}_x^0 \oplus \underline{D}_x^{\lambda_1} \oplus \ldots \oplus \underline{D}_x^{\lambda_k}$

hold equivalently.

In view of the above discussion, we immediately have the following result.

PROPOSITION 5.1. M *is an ASI-submanifold of* $\overline{M}$ *iff there are k functions* $\lambda, \ldots, \lambda_k$ *defined on M with values in* $(0,1)$ *such that*

(1) $\varepsilon\lambda_1^2(x), \ldots, \varepsilon\lambda_k^2(x)$ *are distinct eigenvalues of* $(f^2)_x$ *with* $T_x^\perp M = \underline{D}_x^1 \oplus \underline{D}_x^0 \oplus \underline{D}_x^{\lambda_1} \oplus \ldots \oplus \underline{D}_x^{\lambda_k}$ *at* $x \in M$,

(2) *the dimensions of* $\underline{D}_x^1$, $\underline{D}_x^0$, $\underline{D}_x^{\lambda_1}, \ldots, \underline{D}_x^{\lambda_k}$ *are independent of* $x \in M$.

Let $\varepsilon(1 - \lambda^2(x)), 0 \leq \lambda(x) \leq 1$, be an eigenvalue of $(tF|_L)_x$ (resp. $(Ft)_x$) and C_x^λ (resp. $\underline{C}_x^\lambda$) be denoted by

$$C_x^\lambda = \mathrm{Ker}(tF|_L - \varepsilon(1 - \lambda^2(x))I)_x \quad (\text{resp. } \underline{C}_x^\lambda = \mathrm{Ker}(Ft - \varepsilon(1 - \lambda^2(x))I)_x).$$

Then X_x (resp. N_x) is an eigenvector of $(P^2|_L)_x$ (resp. $(f^2)_x$) corresponding to an eigenvalue $\varepsilon\lambda^2(x)$ iff X_x (resp. N_x) is an eigenvector of $(tF|_L)_x$ (resp. $(Ft)_x$) corresponding to the eigenvalue $\varepsilon(1-\lambda^2(x))$. Consequently, $D_x^\lambda = C_x^\lambda$ and $\underline{D}_x^\lambda = \underline{C}_x^\lambda$, and hence we have the following result.

PROPOSITION 5.2. M *in an ASI-submanifold of* $\overline{M}$ *iff there are k functions* $\lambda_1, \ldots, \lambda_k$ *defined on M with values in* $(0,1)$ *such that*

(1) $\varepsilon(1-\lambda_1^2(x)), \ldots, \varepsilon(1-\lambda_k^2(x))$ *are distinct eigenvalues of* $(tF|_L)_x$ *(resp.* $(Ft)_x$*) with* $T_xM = C_x^1 \oplus C_x^0 \oplus C_x^{\lambda_1} \oplus \ldots \oplus C_x^{\lambda_k} \oplus E_x$ *(resp.* $T_x^\perp M = \underline{C}_x^1 \oplus \underline{C}_x^0 \oplus \underline{C}_x^{\lambda_1} \oplus \ldots \oplus \underline{C}_x^{\lambda_k}$*) at* $x \in M$,

(2) *the dimensions of* $C_x^1, C_x^0, C_x^{\lambda_1}, \ldots, C_x^{\lambda_k}$ *(resp.* $\underline{C}_x^1, \underline{C}_x^0, \underline{C}_x^{\lambda_1}, \ldots, \underline{C}_x^{\lambda_k}$*) are independent of* $x \in M$.

Last two propositions give characterizations of ASI-submanifolds. Characaterizations of SI-submanifolds are given as follows.

PROPOSITION 5.3. M *is an SI-submanifold of* $\overline{M}$ *iff one of the following equuivalent conditions holds*:

(1) $T_xM = D_x^1 \oplus D_x^0 \oplus E_x$, $x \in M$, (2) $T_x^\perp M = \underline{D}_x^1 \oplus \underline{D}_x^0$, $x \in M$,
(3) $FP = 0$, (4) $fF = 0$, (5) $tf = 0$, (6) $Pt = 0$,
(7) $tFP = 0$, (8) $tfF = 0$, (9) $Ptf = 0$, (10) $P^3 = \varepsilon P$,
(11) $f^2F = 0$, (12) $fFP = 0$, (13) $FP^2 = 0$, (14) $FtF = \varepsilon F$,
(15) $Ftf = 0$, (16) $FPt = 0$, (17) $fFt = 0$, (18) $f^3 = \varepsilon f$,
(19) $P^2t = 0$, (20) $Ptf = 0$, (21) $tf^2 = 0$, (22) $tFt = \varepsilon t$.

Proof. The statements (1), (2) are obviously equivalent and the equivalence of the statements (3)–(22) can be easily verified. Now, we show equivalence of (1) and (3). Since $\mathrm{Ker}(FP)_x = D_x^1 \oplus D_x^0 \oplus E_x$, then (1)⇒(3). Conversely, if (3) holds, then $J(PX_x) = P^2X_x$ for $X_x \in T_xM$. Consequently, defining $D_x = P(T_xM)$, we get $\mathrm{J}(D_x) \subset D_x$. Since $g(PX_x, \xi_\alpha) = 0$, D_x is orthogonal to E_x and therefore, in view of $JX_x = PX_x$ for $X_x \in D_x$, we get $\varepsilon X_x = J^2X_x = JP(X_x)$, i.e., $D_x \subset J(D_x)$. Thus $J(D_x) = P(D_x) = D_x$, which shows that $D_x = D_x^1$. Now, let $D_x^\perp$ denote the orthogonal complement to $D_x^1 \oplus E_x$ in T_xM. Then for $X_x \in D_x^\perp$ and $Y_x \in T_xM$ we have $g(JX_x, Y_x) = \varepsilon g(X_x, JY_x) = \varepsilon g(X_x, PY_x) = 0$ which yields $D_x^\perp = D_x^0$. Hence (3) implies (1). Finally, if M is SI-submanifold, then (1) obviously holds. Conversely, if (1) is true then (3) holds which is equivalent to (10), i.e., $P^3 = \varepsilon P$ and hence $\mathrm{Dim}(D_x^1) = \mathrm{Rank}(P_x)$ is independent of $x \in M$ [29] and so is that of D_x^0. This completes the proof.

6. The parallelism of certain operators

The main purpose of this section is to prove Theorem 6.3 which in special case, when $\overline{M}$ is almost Hermitian manifold, implies Propositions 2.1 and 2.2 of [20] and Theorem 4.3 of [22] as corollaries.

THEOREM 6.1. *If* M *is a submanifold of* $\overline{M}$ *with* $\xi_1, \ldots, \xi_r \in TM$, *then* $\nabla P^2 = 0$ *iff the following conditions hold*:

(A) M *is an* ASI^**-submanifold*,

(B) *each of the distributions* D^1, D^0, $D^{\lambda_1}, \ldots, D^{\lambda_k}$, E *is parallel and, consequently*, M *is locally the product of leaves of these distributions.*

Proof. Let $\nabla P^2 = 0$. We fix $x \in M$. For any $Y_x \in D_x^\lambda$ and any vector field $X \in TM$, let Γ be the integral curve of X passing through x, and let Y be the parallel transport of Y_x along Γ. Since $\nabla P^2 = 0$, we get

$$(6.1)\qquad \nabla_X(P^2Y - \varepsilon\lambda^2(x)Y) = P^2\nabla_XY - \varepsilon\lambda^2(x)\nabla_XY = 0,$$

i.e., $(P^2Y - \varepsilon\,\lambda^2(x)Y)$ is parallel along Γ.

Since parallel transport along a curve is an isometry, from (6.1) we get:

(i) since $P^2Y - \varepsilon\lambda^2(x)Y = 0$ at x, it is identically zero on Γ and hence on M,

(ii) eigenvalues of P^2 are constant,

(iii) $\mathrm{Dim}(D_x^\lambda)$ is independent of x,

which proves (A).

Now, if $Y \in D^\lambda$, then $P^2Y = \varepsilon\lambda^2 Y$ (λ is constant). Operating by ∇_X, we get $P^2\nabla_X Y = \varepsilon\,\lambda^2\nabla_X Y$ which shows that D^λ is parallel. Thus $D^1\oplus D^0\oplus D^{\lambda_1}\oplus\ldots\oplus D^{\lambda_k}$ is parallel and, consequently, E is parallel which proves (B).

Conversely, if (A) and (B) hold, then for $X, Y \in TM$ we have

$$\begin{aligned}\nabla_X P^2Y &= \nabla_X P^2(U^1Y + U^0Y + U^{\lambda_1}Y + \ldots + U^{\lambda_k}Y + \eta^\alpha(Y)\xi_\alpha)\\ &= \nabla_X\varepsilon U^1Y + 0 + \nabla_X\varepsilon\,\lambda_1^2U^{\lambda_1}Y + \ldots + \nabla_X\varepsilon\,\lambda_k^2U^{\lambda_k}Y + 0\\ &= \varepsilon\,\nabla_X U^1Y + \varepsilon\lambda_1^2\nabla_X U^{\lambda_1}Y + \ldots + \varepsilon\lambda_k^2\nabla_X U^{\lambda_k}Y = P^2\nabla_X Y.\end{aligned}$$

Hence $\nabla P^2 = 0$.

THEOREM 6.2. *If* M *is a submanifold of* $\overline{M}$ *with* $\xi_1\ldots,\xi_r \in TM$, *then* $\nabla f^2 = 0$ *iff the following conditions hold*:

(A) M *is an* ASI^**-submanifold*,

(B)′ *each of the subbundles* $\underline{D}^1, \underline{D}^0, \underline{D}^{\lambda_1}, \ldots, \underline{D}^{\lambda_k}$ *of* $T^\perp M$ *is parallel with respect to* $\nabla^\perp$.

Proof. Assume $\nabla f^2 = 0$ and fix $x \in M$. For any $N_x \in \underline{D}_x^\lambda$ and any vector field $X \in TM$ let N be the parallel transport of N_x in the normal bundle $T^\perp M$ along the integral curve of X passing through $x \in M$, i.e., $\nabla_X^\perp N = 0$. Since $\nabla f^2 = 0$, we get

$$\nabla_X^\perp(f^2N - \varepsilon\lambda^2(x)N) = f^2\nabla_X^\perp N - \varepsilon\lambda^2(x)\nabla_X^\perp N = 0, \tag{6.2}$$

i.e., $f^2N - \varepsilon\lambda^2(x)N$ is parallel along the integral curve of X.

Rest of the proof is similar to that of Theorem 6.1.

THEOREM 6.3. *For a submanifold* M *of* $\overline{M}$ *with* $\xi_1,\ldots,\xi_r \in TM$ *we have*

$$\begin{array}{ccccccccc}\nabla t = 0 & \longrightarrow & \nabla tF = 0 & \longleftrightarrow & \{(A),(B)\} & \longleftrightarrow & \nabla P^2 = 0 & \longleftarrow & \nabla P = 0,\\ \updownarrow & & & & & & & & \\ \nabla F = 0 & \longrightarrow & \nabla Ft = 0 & \longleftrightarrow & \{(A),(B)'\} & \longleftrightarrow & \nabla F^2 = 0 & \longleftarrow & \nabla F = 0.\end{array}$$

Proof. The relation (3.3)(ii) implies equivalence of $\nabla t = 0$ and $\nabla F = 0$. The proof of equivalence of $\nabla tF = 0$ and statements (A), (B) together is similar to that of Theorem 6.1. Next, $\nabla f^2 = 0$ is equivalent to $\nabla Ft = 0$, in view of (3.1)(vi). Lastly, taking account of Theorems 6.1 and 6.2, the proof is completed.

7. Integrability conditions

Throughout this section superscripts T and N in a term will denote its tangential and normal parts, respectively.

PROPOSITION 7.1. *For a submanifold* M *of* $\overline{M}$, *with* $\xi_1, \ldots, \xi_r \in TM$, *we have*

$$(7.1)\quad P[X,Y] = \nabla_X PY - \nabla_Y PX + A_{FX}Y - A_{FY}X - ((\overline{\nabla}_X J)Y - (\overline{\nabla}_Y J)X)^{\mathsf{T}},$$

$$(7.2)\quad F[X,Y] = \nabla^{\perp}_X FY - \nabla^{\perp}_Y FX + h(X,PY) - h(PX,Y) - ((\overline{\nabla}_X J)Y - (\overline{\nabla}_Y J)X)^{N},$$

$$(7.3)\quad ([J,J](X,Y))^{\mathsf{T}} = [PX,PY] - P([X,PY] + [PX,Y]) + \varepsilon(U^1 + U^0 + U^{\lambda_1} + \ldots + U^{\lambda_k})[X,Y], \quad X,Y \in D^1 \oplus E,$$

$$(7.4)\quad ([J,J](X,Y))^{N} = -F([X,PY] + [PX,Y]), \quad X,Y \in D^1 \oplus E).$$

The proof of (7.1) and (7.2) follows from (3.2)(i), (ii) and (2.4)(i), while using (2.1)(ii), (4.1) and (2.2), for $X,Y \in D^1 \oplus E$, we get (7.3) and (7.4).

THEOREM 7.2. *The distrbution* $D^{\lambda_i} \oplus E$ *is integrable iff for* $X,Y \in D^{\lambda_i} \oplus E$ *the following conditions hold*:

(1) $P[X,Y] \in D^{\lambda_i}$,
(2) $F[X,Y] \in \underline{D}^{\lambda_i}$.

The proof follows from the equivalence of $Z \in D^{\lambda_i} \oplus E$ and ($PZ \in D^{\lambda_i}$ and $FZ \in \underline{D}^{\lambda_i}$).

THEOREM 7.3. *The distribution* $D^1 \oplus D^0 \oplus E$ *is integrable iff for* $X,Y \in D^1 \oplus D^0 \oplus E$ *one of the following conditions holds*:

(1) $P[X,Y] \in D^1$,
(2) $F[X,Y] \in \underline{D}^0$,
(3) $FP[X,Y] = -fF[X,Y] = 0$.

The proof follows from equivalence of the following statements: $PZ \in D^1$, $FZ \in \underline{D}^0$, $Z \in D^1 \oplus D^0 \oplus E$ and $FPZ = -fFZ = 0$.

THEOREM 7.4. *E is integrable iff* $[J,J](\xi_\alpha, \xi_\beta) = 0$.

The proof follows from (7.3) and (7.4) in view of $P\,\xi_\alpha = 0$.

The ε-framed metric structure is said to be *normal* [26] if the Nijenhuis tensor [J,J] of J satisfies $[J,J] = \varepsilon d\eta^\alpha \otimes \xi_\alpha$. Since normality of the structure implies $[\xi_\alpha, \xi_\beta] = 0$, from the above Theorem we have the following corollary.

COROLLARY 7.5. *If the ε-framed metric structure is normal, then E is integrable.*

THEOREM 7.6. *The distribution $D^0 \oplus E$ is integrable iff for $X, Y \in D^0 \oplus E$ one of the following conditions holds:*

(1) $A_{FX}Y - A_{FY}X = ((\overline{\nabla}_X J)Y - (\overline{\nabla}_Y J)X)^{\top}$,
(2) $[P,P](X,Y) = 0$.

Proof. Since $D^0 \oplus E = \operatorname{Ker} P$, in view of (7.1), the condition (1) is equivalent to $D^0 \oplus E$ being integrable. Next, for $X, Y \in D^0 \oplus E$ we get $[P,P](X,Y) = P^2[X,Y]$, and hence $D^0 \oplus E$ is integrable iff (2) holds.

THEOREM 7.7. *The distribution $D^1 \oplus E$ is integrable iff for $X, Y \in D^1 \oplus E$ one of the following conditions holds:*

(1) $([J,J](X,Y))^{\top} = [P,P](X,Y)$,
(2) $h(X,PY) - h(PX,Y) = ((\overline{\nabla}_X J)Y - (\overline{\nabla}_Y J)X)^N$,
(3) $(dF)(X,Y) = 0$.

Proof. Since $Z \in D^1 \oplus E$ iff $P^2 Z = \varepsilon U^1 Z$, in view of (7.3), the condition (1) is necessary and sufficient for $D^1 \oplus E$ to be integrable. Since $D^1 \oplus E = \operatorname{Ker} F$, in view of (7.2), the condition (2) holds iff $D^1 \oplus E$ is integrable. Finally, for $X, Y \in D^1 \oplus E$ the relation (2.4)(vi) reduces to $dF(X,Y) = F[Y,X]$, and hence (3) holds iff $D^1 \oplus E$ is integrable.

As a consequence of Theorem 7.7 we have the following corollaries.

COROLLARY 7.8. *If $D^1 \oplus E$ is integrable, then for $X, Y \in D^1 \oplus E$ we have*

(1) $([J,J](X,Y))^N = 0$,
(2) $[P,P](X,Y) \in D^1 \oplus E$.

COROLLARY 7.9. *If the ε-framed metric structure is normal, then $D^1 \oplus E$ is integrable iff for $X, Y \in D^1 \oplus E$ we have $[J,J](X,Y) = [P,P](X,Y) = \varepsilon d\eta^{\alpha}(X,Y))\xi_{\alpha}$.*

THEOREM 7.10. *If M is an SI-submanifold of $\overline{M}$, then $D^1 \oplus E$ is integrable iff for $X, Y \in D^1 \oplus E$ the following conditions hold:*

(1) $([J,J](X,Y))^N = 0$,
(2) $U^0[P,P](X,Y) = 0$,
(3) $U^0[\xi_\alpha, \xi_\beta] = 0$.

The proof is similar to that of Theorem 3.2 of [30].

The $(J(2,\ \varepsilon), g)$ structure is said to be *integrable* [24], if $[J,J] = 0$.

THEOREM 7.11. *If M is an SI-submanifold of a $(J(2, \varepsilon), g)$ manifold, then D^1 is integrable iff for $X, Y \in D^1$ one of the following conditions holds.*

(1) $[P,P](X,Y) \in D^1$,
(2) $U^0[P,P](X,Y) = 0$,
(3) $F[P,P](X,Y) = 0$.

The proof is similar to that of Theorem 2.1 of [15].

8. CR-structure on almost semi-invariant submanifolds

In [8], it was proved that a CR-submanifold of an Hermitian manifold is a CR-manifold [11]. This was followed by analogous results for CR-submanifolds of a normal almost contact metric manifold [14], for almost CR-submanifolds of an almost cosymplectic f-manifold [16] and for generic submanifolds (in the sense of Chen) of an Hermitian manifold [5, 20]. Here we prove the following theorem from which the results of [5], [14], [16] and [20] mentioned above can be obtained as special cases.

THEOREM 8.1. *If M is an ASI-submanifold of a normal framed metric manifold $\overline{M}$, with non-trivial invariant distribution, then M is a CR-manifold.*

Proof. Let M be an ASI-submanifold of a normal framed metric manifold $\overline{M}$ [33]. Then for $X, Y \in D^1$ we get $P^2X = -X$ and, in view of [J,J] $= -d\eta^\alpha \otimes \xi_\alpha$, (7.3) and (7.4), we get the relation

$$0 = [J,J](X,Y) + d\eta^\alpha(X,Y)\xi_\alpha = [P,P](X,Y) - F([X,PY] + [PX,Y])$$

from which it follows that $[PX,PY] - [X,Y] = P([PX,Y] + [X,PY]) \in D^1$. Hence, in view of Theorem 1.1 from [5] (pp. 128–129), (D^1, P) is a CR-structure on M.

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