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ON SOME SUFFICIENT CONDITIONS FOR UNIVALENCE IN C^n

1. Introduction

The very recent article of Sh. Gong [2] presented many results obtained by the authors and relevant researches in the past few years, in domain of geometric function theory of several complex variables.

The complete characterization of starlikeness and convexity for holomorphic mappings in Banach space was given in '70 by T. J. Suffridge [11], [12]. In the same time K. R. Gurganus [3] generalized Brickman's concept [1] of Φ-like holomorphic functions, and J. A. Pfaltzgraff and T. J. Suffridge in [10] extended Kaplan's and Lewandowski's [4], [6] ideas of close-to-convex functions to locally biholomorphic mappings in C^n with an arbitrary norm.

Very recently P.Liczberski [7],[8] obtained also some sufficient conditions for starlikeness, Φ-likeness and close-to-starlikeness in the Euclidean unit ball in C^n.

In this paper we shall continue this step to give sufficient conditions for starlikeness, Φ-likeness and close-to-starlikeness for holomorphic mappings defined on the unit ball in C^n with an arbitrary norm.

2. Preliminaries

We let C^n denote the space of n-complex tuples $z = (z_1, \ldots, z_n)$ with a norm $\|\cdot\|$. The open ball $\{z \in C^n : \|z\| < r\}$ is denoted by B_r the unit ball is abbreviated by $B_1 = B$. As usual by $L(C^n, C^m)$ we denote the space of all continuous linear operators from C^n into C^m with the standard operator norm $\|\cdot\|$. The letter I will always represent the identity operator in $L(C^n, C^n)$. The class of holomorphic mappings from a region $G \subset C^n$ into

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C^n is denoted by $H(G)$. A function $f \in H(G)$ is said to be locally biholomorphic in G if its Frechet derivative $Df(z)$, as an element of $L(C^n, C^n)$, is nonsingular at each point $z \in G$ (or, equivalently, if it has a locall holomorphic inverse at any point $z \in G$). A mapping $f \in H(G)$ is called biholomorphic if the inverse mapping f^{-1} exists and is holomorphic on an domain Ω and $f^{-1}(\Omega) = G$.

If $D^2 f(z)$ means the Frechet derivative of the second order of $f \in H(G)$ at the point z, then, of course, $D^2 f(z)$ is a continuous symmetric bilinear operator from $C^n \times C^n$ into C^n and its restriction $D^2 f(z)(u, \cdot)$ to C^n belongs to $L(C^n, C^n)$.

For our purpose we define the following classes of mappings :

$$P = \{h \in H(B) : h(0) = 0,\ \Re x^\star(h(z)) \geq 0 \text{ for } z \in B \setminus \{0\} \text{ and } x^\star \in T(z)\},$$

$$N = \{h \in P :\ \Re x^\star(h(z)) > 0 \text{ for } z \in B \setminus \{0\},\ x^\star \in T(z)\},$$

$$M = \{h \in N : Dh(0) = I\},$$

where

$$T(z) = \{x^\star \in L(C^n, C) : \|x^\star\| = 1,\ x^\star(z) = \|z\|\}.$$

The Hahn–Banach Theorem guarantees that $T(z)$ is nonempty for every $z \in C^n \setminus \{0\}$.

Remark 1. (Compare [11, Lemma 3]) If $h \in P$, $z \in B \setminus \{0\}$ and $x^\star \in T(z)$, then $x^\star(h(z)) = 0$ if and only if $x^\star(Dh(0)(z)) = 0$.

Also we shall use the next notations and results:

DEFINITION 1. A mapping $f \in H(B)$ is said to be starlike if f is biholomorphic in B, $f(0) = 0$ and $tf(B) \subset f(B)$ for t, $0 \leq t \leq 1$, (thus $f(B)$ is starlike with respect to the origin in the usual sense).

Remark 2. T. J. Suffridge [11] proved that a locally biholomorphic mapping $f \in H(B)$, with $f(0) = 0$ is starlike iff there exists a mapping $h \in N$ such that

$$f(z) = Df(z)(h(z)), \quad z \in B.$$

Similar equations gave K.R.Gurganus [3] and J.A.Pfaltzgraff with T. J. Suffridge [10] in definitions of Φ-like and close-to-starlike mappings, respectively.

DEFINITION 2. Let f be a locally biholomorphic mapping in B with $f(0) = 0$, $Df(0) = I$ and let Φ, $\Phi(0) = 0$ be a mapping belonging to $H(f(B))$. The above mapping f is said to be the Φ-like if there exists a function $h \in N$ such that

$$\Phi(f(z)) = Df(z)h(z), \quad z \in B.$$

DEFINITION 3. A mapping $f \in H(B)$, $f(0) = 0$, $Df(0) = I$, will be called the close-to-starlike if there exists a starlike mapping g, $Dg(0) = I$ and function $h \in M$ such that

$$g(z) = Df(z)h(z), \quad z \in B.$$

Remark 3. From [3, Theorem 5] it follows that every Φ-like mapping f, (with $D\Phi(0) = I$) biholomorphically maps the unit ball B onto the region $f(B)$.

Remark 4. From [10, Corrolary 1] it follows that any close-to-starlike mapping f biholomorphically maps the unit ball B onto the region $f(B)$ and for each $r, 0 < r < 1$ the complement (in C^n) of $f(B_r)$ is the union of nonintersecting rays.

We close this section proving the following result which is an extension of Jack–Miller–Mocanu Lemma [9] to higher dimensions. Our main theorem is based on this result.

LEMMA 1. *Let r be a real number from the open interval* $(0,1)$ *and let* $f \in H(B)$ *with* $f(0) = 0$. *If for an* $a \in \overline{B}_r$ *we have*

$$\|f(a)\| = \max\{\|f(z)\| : z \in \overline{B}_r\}, \tag{1}$$

then there is a real number s, $s \geq 1$, such that

$$\|Df(a)(a)\| = s\|f(a)\|. \tag{2}$$

Proof. If $f(z) \equiv 0$, then the equality (2) holds. Let $f(z) \not\equiv 0$, then according to (1), $a \neq 0$ and $f(a) \neq 0$. Let $x^\star$ be an arbitrarily fixed functional from $T(f(a))$ and F be the complex function, defined on the unit disc U by the formula

$$F(\lambda) = x^\star(f(\lambda a\|a\|^{-1})).$$

Then F is holomorphic in U and at $\lambda_0 = \|a\|$ we have

$$\|f(a)\| = |F(\lambda_0)| = \max\{|F(\lambda)| : |\lambda| \leq |\lambda_0|\}.$$

So, from Jack-Miller-Mocanu Lemma [9], there is $m \geq 1$ such that $\lambda_0 F'(\lambda_0) = mF(\lambda_0)$. But this implies that $x^\star(Df(a)(a)) = m\|f(a)\|$. Hence using the inequality $|x^\star(y)| \leq \|y\|$, for all $y \in C^n$, we can find s, $s \geq m \geq 1$ such that the equality (2) holds. This completes the proof of Lemma 1.

3. Main results

We shall now give some sufficient conditions for starlikeness, close-to-starlikeness and Φ-likeness. The main results are formulated in Theorem 1 and Theorem 2.

THEOREM 1. *Let f be a locally biholomorphic mapping in the ball B and $f(0) = 0$, $Df(0) = I$. If there exists a mapping $\Phi \in H(f(B))$, $\Phi(0) = 0$, $D\Phi(0) = I$, such that for $z \in B$ the inequality*

$$(3) \quad \|I - (Df(z))^{-1} D\Phi(f(z)) Df(z)\| + \\ + (1 + \|z\|) \|(Df(z))^{-1} D^2 f(z)(z, \cdot)\| < 1$$

holds, then f is Φ-like, hence biholomorphic, mapping in B.

Proof. Define the mapping

$$h(z) = (Df(z))^{-1} \Phi(f(z)), \quad z \in B.$$

Then, in view of Definition 2 and Remark 3, it is sufficient to prove that $h \in N$.

Let $q(z) = z - h(z)$, $z \in B$. If we show that

$$\|q(z)\| \le \|z\|, \quad z \in B,$$

then

$$\Re x^{\star}(h(z)) = \|z\| - \Re x^{\star}(q(z)) \ge \|z\| - \|q(z)\| \ge 0,$$

for all nonzero $z \in B$ and $x^{\star} \in T(z)$. But $Dh(0) = I$ and from Remark 1 we deduce that $\Re x^{\star}(h(z)) > 0$, for $z \in B \setminus \{0\}$, so $h \in N$. From n-dimensional Schwarz Lemma is sufficient to prove that $\|q(z)\| < 1, z \in B$. Assuming opposite we can choose $a \in B$ such that the assumptions of Lemma 1 are satisfied (with $\|q(a)\| = 1$). From Lemma 1 we obtain

$$(4) \quad \|Dq(a)(a)\| \ge 1.$$

On the other hand we have

$$Dq(z)(v) = [I - (Df(z))^{-1} D\Phi(f(z)) Df(z)](v) + \\ + [(Df(z))^{-1} D^2 f(z)(v, \cdot)](z - q(z)),$$

for all $z \in B$ and $v \in C^n$.

Taking $z = a = v$ we then obtain

$$\|Dq(a)(a)\| \le \|I - (Df(a))^{-1} D\Phi(f(a)) Df(a)\| + \\ + (1 + \|a\|) \|(Df(a))^{-1} D^2 f(a)(a, \cdot)\|.$$

From (3) we deduce that $\|Dq(a)(a)\| < 1$ which is a contradiction with (4). So $\|q(z)\| < 1$, $z \in B$ and f is Φ-like.

On the other hand $D\Phi(0) = I$ so, using Remark 3 , we have that f is biholomorphic in the ball B.

This completes the proof of Theorem 1.

THEOREM 2. *Let f be a locally biholomorphic mapping in B, with $f(0) = 0$ and $Df(0) = I$ and assume that there exists a mapping g starlike in B*

and normalized by $Dg(0) = I$, such that

$$\|I - (Df(z))^{-1}Dg(z)\| + (1 + \|z\|)\|(Df(z))^{-1}D^2f(z)(z,\cdot)\| < 1, \quad z \in B.$$

Then f is the close-to-starlike relative to g, hence biholomorphic mapping in B.

The proof of Theorem 2 is quite similar to the proof of Theorem 1. It is sufficient to replace the mapping h from the proof of Theorem 1 by the mapping

$$h(z) = (Df(z))^{-1}g(z)$$

and use Definition 3 and Remark 4.

COROLLARY 1. *Let f be a locally biholomorphic mapping in the ball B and $f(0) = 0$, $Df(0) = I$. If for all $z \in B$ we have*

$$(1 + \|z\|)\|(Df(z))^{-1}D^2f(z)(z,\cdot)\| < 1,$$

then f is Φ-like with $\Phi(w) = w$, starlike in B and close-to-starlike relative to $g = f$.

Proof. From the assumptions, by Theorem 1, it follows that f is *Phi*-like with $\Phi(w) = w$, so in view of Definition 2 and Remark 2, f is starlike. Consequently, from Theorem 2 we obtain that f is close-to-starlike, relative to $g = f$.

This completes the proof.

The above starlikeness criterion was deduced very recently by P. Liczberski in [7] and by the authors in [5] when C^n is normed with the Euclidean and maximum norm, respectively.

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