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Nurettin Ergun, Takashi Noiri

FURTHER RESULTS ON IDEALS

Among other results we prove that a κ-extension of a compatible ideal in a topological space is compatible for any infinite cardinal number κ. The case $\kappa = \omega_0$ is known as the generalized Banach category theorem.

1. Introduction: A short survey on compatible ideals

Ideals in topological spaces have been considered since 1930. This topic has won its importance by the paper of Vaidyanathaswamy [V] in 1945. A nonempty collection of subsets of X with heredity and finite additivity conditions is called as an *ideal* or a *dual filter* on X. Namely a nonempty family $\mathcal{I} \subset \mathcal{P}(X)$ is called an ideal if and only if i) $A \in \mathcal{I}$ gives $\mathcal{P}(A) \subseteq \mathcal{I}$ and ii) A, $B \in \mathcal{I}$ gives $A \cup B \in \mathcal{I}$. $\{\phi\}$, $\mathcal{P}(X)$, the family of all finite subsets, the family of all countable subsets $\mathcal{I}_c$ are ideals. If we additionally have a topology τ defined on X, then the following ideals can also be defined among others: The families of all nowhere dense subset $\mathcal{I}_n$, all meager subsets $\mathcal{I}_m$, all scattered subsets $\mathcal{I}_s$ and all closed-discrete subsets $\mathcal{I}_{cd}$. In this introduction we give a short and independent survey on some important concepts and results of this topic, which will be used in this paper. cl(A) (resp. int(A)) denotes the closure (resp. interior) of A in (X,τ). κ^+ denotes the immediate succesesor of the infinite cardinal number κ.

The local function [V] of an ideal $\mathcal{I}$ with respect to τ is defined on $\mathcal{P}(X)$ as the closed set

$$A^*(\tau,\mathcal{I}) = X - \cup\{G \in \tau : G \cap A \in \mathcal{I}\} \ (A \subseteq X)$$

or equivalently

$$A^*(\tau,\mathcal{I}) = \{x \in X : \forall G_x \in \tau_x, G_x \cap A \notin \mathcal{I}\}, \text{ where } \tau_x = \{G \in \tau : x \in G\}.$$

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This closed set is written simply by $A^*(\mathcal{I})$ or even by A^* if there is no possibility of confusion. $A^* \subseteq \mathrm{cl}(A)$ is evident. It is quite easy to see that $A^*(\tau_2,\mathcal{I}_2) \subseteq A^*(\tau_1,\mathcal{I}_1)$ whenever $\tau_1 \subseteq \tau_2$ and $\mathcal{I}_1 \subseteq \mathcal{I}_2$. It is also not difficult to prove that local function operator * has the following basic properties: $(A \cup B)^* = A^* \cup B^*$, $I^* = \phi$, $(G - I) \cap A^* \subseteq ((G - I) \cap A)^*$ and $A^{**} \subseteq A^*$ for any A, $B \subseteq X$, $G \in \tau$ and $I \in \mathcal{I}$. Thus, as one can easily observe that the family

$$\tau^*(\mathcal{I}) = \{A \subseteq X : A \cap (X - A)^* = \phi\}$$

is a topology on X finer than τ. A subset A is closed in $(X, \tau^*(\mathcal{I}))$ if and only if $A^* - A = (X - A) \cap A^* = \phi$ if and only if $A^* \subseteq A$. Hence the closure and the interior operators in $(X, \tau^*(\mathcal{I}))$ satisfy easily

$$\tau^*\text{-cl}(A) = A \cup A^* \text{ and } \tau^*\text{-int}(A) = A - (X - A)^*.$$

Furthermore if $G_\alpha \in \tau$ and $I_\alpha \in \mathcal{I}$ for each index α of an arbitrary index set Λ we have $\cup\{G_\alpha - I_\alpha : \alpha \in \Lambda\} \in \tau^*(\mathcal{I})$, since

$$\bigcup_{\alpha\in\Lambda}(G_\alpha - I_\alpha) \cap (X - \bigcup_{\alpha\in\Lambda}(G_\alpha - I_\alpha))^*) \subseteq \bigcup_{\alpha\in\Lambda}((G_\alpha - I_\alpha) \cap (X - (G_\alpha - I_\alpha))^*) =$$

$$= \bigcup_{\alpha\in\Lambda}((G_\alpha - I_\alpha) \cap (X - G_\alpha)^*) = \phi.$$

Conversely, for any $A \in \tau^*(\mathcal{I})$ and for each $x \in A$ there exists $G_x \in \tau_x$ such that $I_x = G_x - A = G_x \cap (X - A) \in \mathcal{I}$ and thus $A = \cup\{A \cap G_x : x \in A\} = \cup\{G_x - I_x : x \in A\}$. Therefore, the topology $\tau^*(\mathcal{I})$ is nothing but the unique topology generated by *base family* $\beta(\tau,\mathcal{I}) = \{G - I : G \in \tau, I \in \mathcal{I}\}$. It is now easy to observe that $A^*(\tau,\mathcal{I}) = A^*(\tau^*(\mathcal{I}),\mathcal{I})$. Furthermore $\tau^*(\mathcal{I}_1) \subseteq \tau^*(\mathcal{I}_2)$ holds if $\mathcal{I}_1 \subseteq \mathcal{I}_2$. An ideal $\mathcal{I}$ is called *compatible with* τ or compatible in (X,τ) and $\mathcal{I} \sim \tau$ is written [N] if the condition $A \cap A^*(\mathcal{I}) = \phi$ gives $A \in \mathcal{I}$; that means if there exists a $G_x \in \tau_x$ for each $x \in A$ such that $G_x \cap A \in \mathcal{I}$ then $A \in \mathcal{I}$ necesssarily holds. Notice that if $G \cap A \in \mathcal{I}$ for each member G of an open family $\mathcal{G}$ and $\mathcal{I} \sim \tau$ then $\cup\{G \cap A : G \in \mathcal{G}\} \in \mathcal{I}$. The family $\beta(\tau,\mathcal{I})$ is in fact closed under the arbitrary unions and thus $\tau^*(\mathcal{I}) = \beta(\tau,\mathcal{I})$ if $\mathcal{I} \sim \tau$. For, if $G_\alpha \in \tau$ and $I_\alpha \in \mathcal{I}$ $(\alpha \in \Lambda)$ and $\mathcal{I} \sim \tau$ then $\cup\{G_\alpha - I_\alpha : \alpha \in \Lambda\} = G_0 - I_0$, where $G_0 = \cup\{G_\alpha : \alpha \in \Lambda\}$ and $I_0 = G_0 - \cup\{G_\alpha - I_\alpha : \alpha \in \Lambda\} \in \mathcal{I}$. In fact for each $x \in I_0$ we have an $\alpha_x \in \Lambda$ such that $x \in G_{\alpha_x}$ and $x \in G_{\alpha_x} \cap I_0 = G_{\alpha_x} - \cup\{G_\alpha - I_\alpha : \alpha \in \Lambda\} \subseteq G_{\alpha_x} - (G_{\alpha_x} - I_{\alpha_x}) \subseteq I_{\alpha_x} \in \mathcal{I}$. Since

$$A - A^* = A \cap (X - A^*) = \cup\{A \cap G : G \in \tau, A \cap G \in \mathcal{I}\}$$

always holds, one can easily derive that $A - A^* \in \mathcal{I}$ for any $A \subseteq X$ if $\mathcal{I} \sim \tau$. Conversely if $A - A^* \in \mathcal{I}$ then the condition $A \cap A^* \in \mathcal{I}$ evidently yields

$A = (A - A^*) \cup (A \cap A^*) \in \mathcal{I}$. Thus we get the important equivalency

$$\mathcal{I} \sim \tau \text{ if and only if } \forall A \subseteq X, A - A^* \in \mathcal{I}.$$

It is not dificult to prove that $A^*(\{\phi\}) = \mathrm{cl}(A)$, $A^*(\mathcal{P}(X)) = \phi$ and $A^*(\mathcal{I}_n) = \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))$. We have already observed that $A^* = A^*(\mathcal{I})$ is closed in (X, τ) for any ideal $\mathcal{I}$ and subset $A \subseteq X$. If $\mathcal{I}_n \subseteq \mathcal{I}$ holds particularly , then a subset A satisfying $A = A^*$ is actually regularly closed since $\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A))) \subseteq \mathrm{cl}(A) = A = A^*(\mathcal{I}) \subseteq A^*(\mathcal{I}_n) = \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))$. One additionally observes that if $\mathcal{I} \sim \tau$ then $A^* = A^{**}$ holds for each $A \subseteq X$, since $A^* - A^{**} \subseteq (A - A^{**})^* = (A - (A^* - (A^* - A^{**})))^* = (A - A^*)^* \cup (A \cap (A^* - A^{**}))^* = \phi$ by the compatibility of $\mathcal{I}$. Therefore if $\mathcal{I}_n \subseteq \mathcal{I}$ and $\mathcal{I} \sim \tau$ then $A^*(\mathcal{I})$ is a regularly closed subset for each $A \subseteq X$. The most typical compatible ideals in (X, τ) are $\{\phi\}$ and $\mathcal{I}_n$. The compatibility of $\mathcal{I}_n$ derives the well known equality

$$\mathrm{cl}(\mathrm{int}(\mathrm{cl}(\cup\{A \cap G : G \in \mathcal{G}\}))) = \mathrm{cl}(\cup\{\mathrm{int}(\mathrm{cl}(A \cap G)) : G \in \mathcal{G}\})$$

which holds for *any* $A \subseteq X$ and *any* open family $\mathcal{G}$ in (X, τ). The family of all meager sets $\mathcal{I}_m$ is also compatible and this interesting fact is known as Banach Category Theorem.

If $\mathrm{int}(I) = \phi$ for each $I \in \mathcal{I}$ or equivalently $\tau \cap \mathcal{I} = \{\phi\}$ then as one can easily observe that $\tau^*\text{-}\mathrm{cl}(U) = \mathrm{cl}(U)$ holds for each $U \in \tau^*(\mathcal{I})$ and hence $\tau^*\text{-}\mathrm{int}(K) = \mathrm{int}(K)$ for each closed K in $(X, \tau^*(\mathcal{I}))$. Thus $\tau^*\text{-}\mathrm{int}(\tau^*\text{-}\mathrm{cl}(A)) = \mathrm{int}(\tau^*\text{-}\mathrm{cl}(A)) \subseteq \mathrm{int}(\mathrm{cl}(A))$ and $\mathcal{I}_n \subseteq \mathcal{I}_n(\tau^*(\mathcal{I}))$ are obtained whenever $\tau \cap \mathcal{I} = \{\phi\}$. But in general as one can easily see we have the interesting inclusion

$$\mathcal{I} \cup \mathcal{I}_n(\tau^*(\mathcal{I})) \cup \mathcal{I}_n \subseteq \{A \subseteq X : \mathrm{int}(A^*(\mathcal{I})) = \phi\} = \widetilde{\mathcal{I}}.$$

Since $\mathrm{int}(A \cup B)^*(\mathcal{I}) = \mathrm{int}(A^*(\mathcal{I}) \cup \mathrm{int}(B^*(\mathcal{I})))$, one easily observes that $\widetilde{\mathcal{I}}$ is an ideal. Furthermore $\mathrm{int}(A^*(\mathcal{I})) = \mathrm{int}(A^*(\widetilde{\mathcal{I}}))$ holds for each $A \subseteq X$. $\mathcal{I} \subseteq \widetilde{\mathcal{I}}$ gives $\mathrm{int}(A^*(\widetilde{\mathcal{I}})) \subseteq \mathrm{int}(A^*(\mathcal{I}))$; if $x \in \mathrm{int}(A^*(\mathcal{I}))$ there exists $W_x \in \tau_x$ with $W_x \subseteq A^*(\mathcal{I})$ and thus $G_x \cap W_x \subseteq \mathrm{int}(G_x \cap A^*(\mathcal{I})) \subseteq \mathrm{int}(G_x \cap A)^*(\mathcal{I})$ holds for each $G_x \in \tau_x$ proving $G_x \cap A \notin \widetilde{\mathcal{I}}$ i.e. $x \in A^*(\widetilde{\mathcal{I}})$. Furthermore $\widetilde{\mathcal{I}}$ is compatible with τ since $A \cap A^*(\widetilde{\mathcal{I}}) = \phi$ gives $A \in \widetilde{\mathcal{I}}$ by $\mathrm{int}(A^*(\mathcal{I})) = \mathrm{int}(A^*(\mathcal{I})) \cap A^*(\mathcal{I}) \subseteq (\mathrm{int}(A^*(\widetilde{\mathcal{I}})) \cap A)^*(\mathcal{I}) \subseteq \phi^* = \phi$. We additionally have $A^*(\widetilde{\mathcal{I}}) = \mathrm{cl}(\mathrm{int}(A^*(\widetilde{\mathcal{I}}))) = \mathrm{cl}(\mathrm{int}(A^*(\mathcal{I})))$ since $\mathcal{I}_n \subseteq \widetilde{\mathcal{I}}$ and $\widetilde{\mathcal{I}} \sim \tau$. Therefore, each ideal $\mathcal{I}$ defined on (X, τ) is contained in a compatible ideal $\widetilde{\mathcal{I}}$. The ideal $\widetilde{\mathcal{I}}$ is called as *compatible extension* of $\mathcal{I}$ [JH]. We close this section with two simple but interesting equivalencies. It is easy to see first $\mathcal{I} \sim \tau$ if and only if $\mathcal{I} \sim \tau^*(\mathcal{I})$. Now if $\mathcal{I} = \widetilde{\mathcal{I}}$ then $\mathcal{I}_n \subseteq \mathcal{I}$ and $\mathcal{I} \sim \tau$ are evident. If conversely $\mathcal{I}_n \subseteq \mathcal{I}$, $\mathcal{I} \sim \tau$ and $A \in \widetilde{\mathcal{I}}$ then $A^*(\mathcal{I}_n) \in \mathcal{I}_n \subset \mathcal{I}$, $A - A^*(\mathcal{I}) \in \mathcal{I}$ and consequently $A \in \mathcal{I}$. Thus

$$\mathcal{I} = \widetilde{\mathcal{I}} \text{ if and only if } \mathcal{I}_n \subseteq \mathcal{I} \text{ and } \mathcal{I} \sim \tau.$$

Therefore $\mathcal{I}_n = \widetilde{\mathcal{I}}_n$ and $\mathcal{I}_m = \widetilde{\mathcal{I}}_m$ are obtained. The results of this introduction will be used in the sequel without explicit mentioning.

2. Results on compatible ideals

The following result improves slightly Corollary 3.6 of [JH].

PROPOSITION 1. *$\mathcal{I} = \mathcal{I}_n(\tau^*(\mathcal{I}))$ if and only if i)$\mathcal{I} \sim \tau$, ii) $\tau \cap \mathcal{I} = \{\phi\}$ and iii) $\mathcal{I}_n \subseteq \mathcal{I}$.*

Proof. The first and third conditions give easily $\mathcal{I}_n(\tau^*(\mathcal{I})) \subseteq \widetilde{\mathcal{I}} = \mathcal{I}$. Now let $A \in \mathcal{I} = \widetilde{\mathcal{I}}$. Then by the second condition we first have $\text{int}(A) = \phi$ and thus $\tau^*\text{-int}(\tau^*\text{-cl}(A)) = \text{int}(\tau^*\text{-cl}(A)) = \text{int}(\text{int}(A) \cup A^*(\mathcal{I})) = \phi$ and so $\mathcal{I} \subseteq \mathcal{I}_n(\tau^*(\mathcal{I}))$. Conversely let $\mathcal{I} = \mathcal{I}_n(\tau^*(\mathcal{I})) \sim \tau^*(\mathcal{I})$ and thus $\mathcal{I} \sim \tau$. Furthermore $\mathcal{I} \cap \tau \subseteq \mathcal{I}_n(\tau^*(\mathcal{I})) \cap \tau^*(\mathcal{I}) = \{\phi\}$ and by the last result we finally have $\mathcal{I}_n \subseteq \mathcal{I}_n(\tau^*(\mathcal{I})) = \mathcal{I}$.

COROLLARY 1 [JH]. *$\widetilde{\mathcal{I}} = \mathcal{I}_n(\tau^*(\mathcal{I}))$ if and only if $\tau \cap \mathcal{I} = \{\phi\}$. In particularly $\mathcal{I}_n = \mathcal{I}_n(\tau^*(\mathcal{I}_n))$. Furthermore in Baire spaces $\mathcal{I}_m = \mathcal{I}_n(\tau^*(\mathcal{I}_m))$.*

DEFINITION 1 [JH]. The $\mathcal{I}$-open sets is the family

$$\{A \subseteq X : A \subseteq \text{int}(A^*(\mathcal{I}))\} = \{A \subseteq X : A \subseteq \text{int}(A^*(\widetilde{\mathcal{I}}))\}.$$

The union of the all $\mathcal{I}$-open sets contained in A is written by $\mathcal{I}\text{-int}(A)$.

PROPOSITION 2. *For any $A \subseteq X$, we have $\text{int}(A^*(\mathcal{I})) = \text{int}(A^*(\widetilde{\mathcal{I}}))$, $A - \text{int}(A^*(\mathcal{I})) \in \widetilde{\mathcal{I}}$ and $\mathcal{I}\text{-int}(\mathcal{I}\text{-int}(A)) = \mathcal{I}\text{-int}(A) = A \cap \text{int}(A^*(\mathcal{I}))$.*

Proof. The first equality has already been proved in introduction. Note that $A - \text{int}(A^*(\mathcal{I})) = A - \text{int}(A^*(\widetilde{\mathcal{I}})) = [(A - A^*(\widetilde{\mathcal{I}})) \cup (A \cap A^*(\widetilde{\mathcal{I}})] - \text{int}(A^*(\widetilde{\mathcal{I}}))) \in \widetilde{\mathcal{I}}$ since $\widetilde{\mathcal{I}} \sim \tau$ and $A^*(\widetilde{\mathcal{I}}) - \text{int}(A^*(\widetilde{\mathcal{I}})) = \text{cl}(\text{int}(A^*(\mathcal{I}))) - \text{int}(A^*(\mathcal{I})) \in \mathcal{I}_n \subseteq \widetilde{\mathcal{I}}$. Thus $\text{int}(A - \text{int}(A^*(\mathcal{I}))^*(\mathcal{I}) = \phi$. The equality $\mathcal{I}-\text{int}(A) = A \cap \text{int}(A^*(\mathcal{I}))$ was proved in [JH]. Now since $A = (A \cap (\text{int}(A^*)) \cup (A - \text{int}(A^*))$ we finally have

$$\text{int}(A^*) = \text{int}((A \cap \text{int}(A^*))^* \cup \text{int}(A - \text{int}(A^*))^*) = \text{int}(A \cap \text{int}(A^*))^*,$$

$$\mathcal{I}\text{-int}(\mathcal{I}\text{-int}(A)) = A \cap \text{int}(A \cap \text{int}(A^*))^* = A \cap \text{int}(A^*) = \mathcal{I}\text{-int}(A).$$

COROLLARY 2. *$\mathcal{I}\text{-int}(A)$ is an $\mathcal{I}$-open set for any $A \subseteq X$.*

Proof. It is easy to observe that E is an $\mathcal{I}$-open set if and only if $E = \mathcal{I}\text{-int}(E)$.

COROLLARY 3. *The following are always equivalent*:
i) $A \in \widetilde{\mathcal{I}}$,
ii) $\mathcal{I}$-int$(A) \in \widetilde{\mathcal{I}}$,
iii) $\mathcal{I}$-int$(A) = \phi$.

Proof. i)⇒ii) is obvious; ii)⇒iii) is straightforward by $\text{int}(\mathcal{I}\text{-int}(A))^* = \text{int}(A^*) \supseteq \mathcal{I}\text{-int}(A)$ and iii) ⇒ i) is known in [JH].

PROPOSITION 3. *If $U \in \tau^*(\mathcal{I})$ and A is an $\mathcal{I}$-open set, then $U \cap A$ is an $\mathcal{I}$-open set.*

Proof. Let $U = \cup\{G_\alpha - I_\alpha : \alpha \in \Lambda\} \in \tau^*(\mathcal{I})$. Then

$$U \cap A \subseteq \cup\{((G_\alpha - I_\alpha) \cap \text{int}(A^*)) : \alpha \in \Lambda\} \subseteq$$
$$\subseteq \cup\{\text{int}(G_\alpha \cap A)^* : \alpha \in \Lambda\} \subseteq \text{int}(\cup\{((G_\alpha - I_\alpha) \cap A)^* : \alpha \in \Lambda\}) \subseteq$$
$$\subseteq \text{int}(\cup\{(G_\alpha - I_\alpha) \cap A : \alpha \in \Lambda\})^* = \text{int}(U \cap A)^*.$$

COROLLARY 4. *If $\tau \cap \mathcal{I} = \{\phi\}$, then each $U \in \tau^*(\mathcal{I})$ is an $\mathcal{I}$-open set.*

Proof. If $\tau \cap \mathcal{I} = \{\phi\}$ then we evidently have

$$X^* = X - \cup\{G \in \tau : G = G \cap X \in \mathcal{I}\} = X$$

and thus $X = X \cap \text{int}(X^*) = \mathcal{I}\text{-int}(X)$ is obtained. Then the above proposition is used.

Remark 1. Despite of the fact that the family of all $\mathcal{I}$-open sets is being closed under the arbitrary unions [JH], this family is not necessarily a topology on X. For instance the family of $\mathcal{I}_n$-open sets is nothing but $\{A \subseteq X : A \subseteq \text{int}(\text{cl}(A))\}$ which is not necessarily closed under the finite intersections.

The family of all countable subsets of A is written by $[A]^{\leq\omega_0}$ or by $[A]^{<\omega_1}$. An ideal is called σ-ideal if $\cup\mathcal{A} \in \mathcal{I}$ whenever $\mathcal{A} \in [\mathcal{I}]^{\leq\omega_0}$. $\mathcal{I}_c$, $\mathcal{I}_m$ and $\mathcal{P}(X)$ are typical σ-ideals. The σ-ideal generated by the ideal $\mathcal{I}$, i.e. the minimal σ-ideal containing $\mathcal{I}$ is $\mathcal{I}_\sigma = \{\cup\mathcal{A} : \mathcal{A} \in [\mathcal{I}]^{\leq\omega_0}\}$.

DEFINITION 2. Let κ be an infinite cardinal number. Then

$$[A]^{\leq\kappa} = \{B \subseteq A : \text{card}B \leq \kappa\} = \{B \subseteq A : \text{card}B < \kappa^+\}.$$

An ideal $\mathcal{I}$ is called *κ-ideal* or *k-dual filter* if $\mathcal{I} = \{\cup\mathcal{A} : \mathcal{A} \in [\mathcal{I}]^{\leq\kappa}\}$.

Remark 2. As one can easily observe that the minimal κ-ideal containing $\mathcal{I}$ is $\mathcal{I}_\kappa = \{\cup\mathcal{A} : \mathcal{A} \in [\mathcal{I}]^{\leq\kappa}\}$. Thus $\mathcal{I}$ is a κ-ideal (resp. σ-ideal) if and only if $\mathcal{I} = \mathcal{I}_\kappa$ (resp. $\mathcal{I} = \mathcal{I}_{\omega_0}$).

PROPOSITION 4. *Let $\mathcal{I}$ be any compatible ideal in (X,τ) and let κ be any infinite cardinal number. Then*

$$\cup(\tau^*(\mathcal{I}) \cap \mathcal{I}_\kappa) \in \mathcal{I}_\kappa.$$

Proof. By the well known Zorn Lemma, there exists a maximal cellular refinement $\mathcal{U}$ of the open family $\tau^*(\mathcal{I}) \cap I_\kappa$ in the space $(X,\tau^*(\mathcal{I}))$. Then

$$\cup\mathcal{U} \subseteq \cup(\tau^*(\mathcal{I}) \cap \mathcal{I}_\kappa) \subseteq \mathrm{cl}(\cup\mathcal{U}),$$
$$\mathrm{cl}(\cup(\tau^*(\mathcal{I}) \cap \mathcal{I}_\kappa)) - \cup\mathcal{U} = \tau^*\text{-}\mathrm{cl}(\cup\mathcal{U}) - \cup\mathcal{U} \in \mathcal{I}_n(\tau^*(\mathcal{I})) \subseteq \widetilde{\mathcal{I}} = \mathcal{I} \subseteq \mathcal{I}_\kappa.$$

It is easy to observe that each $U \in \mathcal{U}$ belongs to $\mathcal{I}_\kappa$ and thus $U = \cup\{I_\alpha(U) : \alpha < \kappa^+, I_\alpha(U) \in \mathcal{I}\}$. Now let us define

$$I_\alpha = \cup\{I_\alpha(U) : U \in \mathcal{U}\}, \quad \alpha < \kappa^+.$$

Then for each $\alpha < \kappa^+$ and $U \in \mathcal{U}$ we have $\mathcal{I}\text{-int}(I_\alpha) \cap U = \mathcal{I}\text{-int}(I_\alpha) \cap U \cap I_\alpha(U) \subseteq I_\alpha(U)$ since $U \cap I_\alpha(U^*) \subseteq U \cap U^* = \phi$ whenever U and U^* are different members of $\mathcal{U}$. Since $\mathcal{I}\text{-int}(I_\alpha) \cap U$ is an $\mathcal{I}$-open set by Corollary 2 and Proposition 3 and $I_\alpha(U) \in \mathcal{I} = \widetilde{\mathcal{I}}$, we have $\mathcal{I}\text{-int}(I_\alpha) \cap U \subseteq \mathcal{I}\text{-int}(I_\alpha(U)) \subseteq \mathrm{int}(I_\alpha(U))^* = \phi$. Thus $I_\alpha \in \widetilde{\mathcal{I}} = \mathcal{I}$ is obtained by Corollary 3 since

$$\mathcal{I}\text{-int}(I_\alpha) = \cup\{\mathcal{I}\text{-int}(I_\alpha) \cap U : U \in \mathcal{U}\} = \phi.$$

Therefore

$$I_0 = \cup\mathcal{U} = \cup\{I_\alpha(U) : U \in \mathcal{U}, \alpha < \kappa^+\} = \cup\{I_\alpha : \alpha < \kappa^+\} \in \mathcal{I}_\kappa,$$
$$\cup(\tau^*(\mathcal{I}) \cap \mathcal{I}_\kappa) \subseteq (\mathrm{cl}(\cup(\tau^*(\mathcal{I}) \cap \mathcal{I}_\kappa)) - \cup\mathcal{U}) \cup I_0 \in \mathcal{I}_\kappa.$$

COROLLARY 5. *For any compatible ideal $\mathcal{I}$ of (X,τ) and infinite cardinal number κ, we have $\cup(\tau \cap \mathcal{I}_\kappa) \in \mathcal{I}_\kappa$.*

COROLLARY 6. *For any ideal $\mathcal{I}$ and infinite cardianal number κ, we have*

$$\cup(\tau \cap (\widetilde{\mathcal{I}})_\kappa) \subseteq \cup(\tau^*(\widetilde{\mathcal{I}}) \cap (\widetilde{\mathcal{I}})_\kappa) \in (\widetilde{\mathcal{I}})_\kappa.$$

COROLLARY 7 (Janković and Hamlett [JH]). *For any ideal $\mathcal{I}$ we have $\cup(\tau \cap (\widetilde{\mathcal{I}})_\sigma) \in (\widetilde{\mathcal{I}})_\sigma$.*

Remark 3. Let $\mathcal{I}$ be any ideal and $A \subseteq X$. Then $\{I \in \mathcal{I} : I \subseteq A\} = \{I \cap A : I \in \mathcal{I}\}$ is evidently an ideal on A. This restricted ideal is written by $\mathcal{I}/A$. Observing $\mathcal{I}_\kappa/A = (\mathcal{I}/A)_\kappa$ is straightforward. It is also easy to see that $\mathcal{I}/A \sim \tau/A$ holds whenever $\mathcal{I} \sim \tau$. In fact if there exists $W_x \in \tau_x$ satisfying $W_x \cap A \in \mathcal{I}/A$ for each $x \in B \subseteq A$ then $W_x \cap B \in \mathcal{I}$ evidently holds.

PROPOSITION 5. *Let $\mathcal{I}$ be any compatible ideal in (X,τ) and let κ be any infinite cardinal number. Then $\mathcal{I}_\kappa$ is compatible.*

Proof. Let $A \cap A^*(\mathcal{I}_\kappa) = \phi$. Then for each $x \in A$ there exists $G_x \in \tau_x$ such that $A \cap G_x \in \mathcal{I}_\kappa$. Thus $A \cap G_x \in \tau/A \cap \mathcal{I}_\kappa/A = \tau/A \cap (\mathcal{I}/A)_\kappa$ and $A = \cup\{A \cap G_x : x \in A\} = \cup(\tau/A \cap (\mathcal{I}/A)_\kappa) \in (\mathcal{I}/A)_\kappa \subseteq \mathcal{I}_\kappa$, by Corollary 5.

COROLLARY 8. *$(\widetilde{\mathcal{I}})_\kappa$ is compatible.*

COROLLARY 9 (Generalized Banach Category Theorem; [JH]). *$(\widetilde{\mathcal{I}})_\sigma$ is compatible.*

3. Results on paracompactness modulo an ideal

In the last section we obtain some further results on paracompactness modulo an ideal which is recently studied in [HRJ] and [EN]. Let (X,τ) be given. If every open covering of X has a locally finite open refinement $\mathcal{U}$ satisfying $X = \mathrm{Cl}(\cup\mathcal{U})$ then X is called as *almost paracompact* [SA]. It is quite easy to observe that X is almost paracompact if and only if every open covering has a locally finite open refinement $\mathcal{U}$ such that $X - \cup\mathcal{U} \in \mathcal{I}_n$. X is called *paracompact modulo an ideal $\mathcal{I}$* or briefly X is *paracompact (mod $\mathcal{I}$)* if every open covering has a locally finite open refinement $\mathcal{U}$ such that $X - \cup\mathcal{U} \in \mathcal{I}$ [Z]. An ideal $\mathcal{I}$ is called τ-locally finite [HRJ] or locally finite additive in (X,τ) [EN] if the subfamily $\mathcal{I}_0 \subseteq \mathcal{I}$ is locally finite in (X,τ) then $\cup\mathcal{I}_0 \in \mathcal{I}$. It is well known that $\mathcal{I}_n$ is locally finite additive. It is proved in [EN] that $\mathcal{I}_m$ is also locally finite additive in (X,τ). The ideal $\mathcal{I}$ is called as *weakly compatible ideal* in (X,τ) if $A^*(\mathcal{I}) = \phi$ if and only if $A \in \mathcal{I}$.

PROPOSITION 6 (Hamlett, Rose and Janković [HRJ]).

i) *Every compatible ideal is weakly compatible and every weakly compatible ideal is locally finite additive.*

ii) *Let X be paracompact (mod $\mathcal{I}$). Then $\mathcal{I}$ is locally finite additive in (X,τ) if and only if $\mathcal{I}$ is weakly compatible in (X,τ).*

PROPOSITION 7. *Let (X,τ) be having a σ-locally finite base and $\mathcal{I}$ be a σ-ideal. Then the following are equivalent in (X,τ):*

i) *$\mathcal{I}$ is compatible;*

ii) *$\mathcal{I}$ is weakly compatible;*

iii) *$\mathcal{I}$ is locally finite additive.*

Proof. Let $\mathcal{I}$ be a locally finite additive ideal and $\mathcal{B} = \cup\{\mathcal{B}_n : n \in N\}$ be the σ-locally finite base in (X,τ). Let $A \cap A^*(\mathcal{I}) = \phi$. Then $A \in \mathcal{I}$ and thus $\mathcal{I}$ is compatible, since $\cup\{A \cap B : B \in \mathcal{B}_n^*\} \in \mathcal{I}$ whereas $\mathcal{B}_n^* = \{B \in \mathcal{B}_n : A \cap B \in \mathcal{I}\}$ for each $n \in N$. And thus

$$A \subseteq \cup\{A \cap B : B \in \mathcal{B}_n^*, n \in N\} \in \mathcal{I}.$$

PROPOSITION 8. *Let $\mathcal{I}_n \subseteq \mathcal{I}$ and $\mathcal{I}$ be a locally finite additive in (X,τ). Then X is paracompact (mod $\mathcal{I}$) if and only if every open covering has a locally finite refinement (which is not necessarily open) A such that $X - \cup\mathcal{A} \in \mathcal{I}$.*

Proof. Necessity is obvious. Now let $\mathcal{G}$ be any open covering of X, satisfying the sufficiency condition. Then there exists a locally finite refinement $\mathcal{A}$ with $X - \cup\mathcal{A} \in \mathcal{I}$. Thus for each $A \in \mathcal{A}$ there exists a uniquely determined $G_A \in \mathcal{G}$ such that $A \subseteq G_A$. $\mathcal{W} = \{\text{int}(\text{cl}(A)) \cap G_A : A \in \mathcal{A}\}$ is evidently open and locally finite refinement of $\mathcal{G}$. Then $\text{int}(\text{cl}(A)) \subseteq G_A \cup (\text{int}(\text{cl}(G_A)) - G_A)$ holds for each $A \in \mathcal{A}$ and we have $N_A = \text{int}(\text{cl}(A)) \cap (\text{int}(\text{cl}(G_A)) - G_A) \in \mathcal{I}_n$. Thus

$$I_0 = \cup\{N_A \cap \partial(\text{cl}(A)) : A \in \mathcal{A}\} \cup (X - \text{cl}(\cup\mathcal{A})) \in \mathcal{I}$$

since the family $\{N_A \cap \partial(\text{cl}(A)) : A \in \mathcal{A}\}$ is locally finite in (X,τ) and $X - \text{cl}(\cup\mathcal{A}) \subseteq X - \cup\mathcal{A} \in \mathcal{I}$. In here ∂ denotes the boundary operator. Consequently $X - \cup\mathcal{W} \in \mathcal{I}$ is obtained by

$$\begin{aligned} X &= \text{cl}(\cup\mathcal{A}) \cup (X - \text{cl}(\cup\mathcal{A})) \\ &= \cup\{\text{int}(\text{cl}(A)) : A \in \mathcal{A}\} \cup \{\partial(\text{cl}(A)) : A \in \mathcal{A}\} \cup (X - \text{cl}(\cup\mathcal{A})) \\ &= \cup\{\text{int}(\text{cl}(A)) \cap G_{A:} : A \in \mathcal{A}\} \cup I_0. \end{aligned}$$

PROPOSITION 9 (Hamlett, Rose and Janković [HRJ]). *Let $\mathcal{I}_n \subseteq \mathcal{I}$ and $\tau \cap \mathcal{I} = \{\phi\}$. Then X is paracompact (mod $\mathcal{I}$) if and only if every open covering has a σ-locally finite open refinement $\mathcal{V} = \cup\{\mathcal{V}_n : n \in N\}$ such that $X = \cup\{\text{int}(\text{cl}(\cup\mathcal{V}_n)) : n \in N\}$.*

PROPOSITION 10. *Let $\mathcal{I}$ be a compatible σ-additive ideal in (X,τ). Then X is paracompact (mod $\mathcal{I}$) if and only if every open covering has a σ-locally finite refinement (which is not necessarily open) $\mathcal{A} = \cup\{\mathcal{A}_n : n \in N\}$ such that $X = \cup\{\text{int}(\text{cl}(\cup\mathcal{A}_n)) : n \in N\}$.*

Proof. Since $\tau \cap \mathcal{I} = \{\phi\}$, the proof of the necessity is obvious. Now let us prove the sufficiency. Let $\mathcal{G}$ be an open covering of X. Let also $G_A \in \mathcal{G}$ be defined for each $A \in \mathcal{A}$ just as in the above proof. Then

$$\mathcal{W} = \{\text{int}(\text{cl}(A)) \cap G_A - \bigcup_{k<n} \text{cl}(\cup\mathcal{A}_k) : A \in \mathcal{A}_n, n \in N\}$$

is open and locally finite refinement of $\mathcal{G}$ since X is the union of open sets $\text{int}(\text{cl}(\cup\mathcal{A}_n))$. Our claim is to prove that $X - \cup\mathcal{W} \in \mathcal{I}$. Now let us define

$$W_0 = \phi, \quad W_{n-1} = \cup\{\text{int}(\text{cl}(\cup\mathcal{A}_k)) : k < n\} \quad (n \in N).$$

One should notice in here that

$$\mathrm{cl}(W_{n-1}) = \cup\{\mathrm{cl}(\cup\mathcal{A}_k) : k < n\} - \cup\{\mathrm{cl}(\cup\mathcal{A}_k) - \mathrm{cl}(W_{n-1}) : k < n\},$$
$$\cup\{\mathrm{cl}(\cup\mathcal{A}_k) - \mathrm{cl}(W_{n-1}) : k < n\} \subseteq \cup\{\mathrm{cl}(\cup\mathcal{A}_k) - \mathrm{cl}(\mathrm{int}(\mathrm{cl}(\cup\mathcal{A}_k))) : k < n\}$$
$$\in \mathcal{I}_n \subseteq \mathcal{I}$$

since as is well known $\mathrm{cl}(E)-\mathrm{cl}(\mathrm{int}(\mathrm{cl}(E)))$ is nowhere dense for each $E \subseteq X$. Now let $x \in X$ be given. Then there exists a uniquely determined positive integer $n(x)$ such that

$$x \in \mathrm{int}(\mathrm{cl}(\cup\mathcal{A}_{n(x)})) - \cup\{\mathrm{int}(\mathrm{cl}(\cup\mathcal{A}_n)) : n < n(x)\}.$$

Since the family $\mathcal{A}_{n(x)}$ is locally finite then there necessarily exists an $A \in \mathcal{A}_{n(x)}$ such that $x \in \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))$ and consequently we have

$$\begin{aligned} x &\in \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A))) \cap \mathrm{cl}(G_A) - W_{n(x)-1} \\ &= [(\mathrm{int}(\mathrm{cl}(A)) \cap G_A) \cup N_A^1)] - (\mathrm{cl}(W_{n(x)-1}) - \partial(W_{n(x)-1})) \\ &= ((\mathrm{int}(\mathrm{cl}(A)) \cap G_A) - \mathrm{cl}(W_{n(x)-1})) \cup N_A^2 \\ &= ((\mathrm{int}(\mathrm{cl}(A)) \cap G_A) - \cup\{\mathrm{cl}(\cup\mathcal{A}_k) : k < n(x)\}) \cup N_A^3. \end{aligned}$$

In above all the numbered N_A sets are nowhere dense and contained in $\mathrm{cl}(A)$. Thus by defining

$$I_0 = \cup\{N_A^3 : A \in \mathcal{A}_n, n \in N\} \in \mathcal{I}$$

we have $X = \cup\mathcal{W} \cup I_0$ which is nothing but the required result.

COROLLARY 10. *Let X be a Baire space. Then X is paracompact (mod $\mathcal{I}_m$) if and only if every open covering has a σ-locally finite refinement (which is not necessarily open) $\mathcal{A} = \cup\{\mathcal{A}_n : n \in N\}$ such that $X = \cup\{\mathrm{int}(\mathrm{cl}(\cup\mathcal{A}_n)) : n \in N\}$.*

Proof. As is well known a topological space (X, τ) is called as a Baire space if $\mathcal{I}_m \cap \tau = \{\phi\}$ or equivalently the σ-ideal $\mathcal{I}_m$ is compatible.

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N. Ergun
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCES
ISTANBUL UNIVERSITY
VEZNECILER 34459 - ISTANBUL, TURKEY
T. Noiri
DEPARTMENT OF MATHEMATICS
YATSUSHIRO COLLEGE OF TECHNOLOGY
YATSUSHIRO , KUMAMOTO, 866 JAPAN

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