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ON UPPER AND LOWER ALMOST α-CONTINUOUS MULTIFUNCTIONS

In this paper, the authors define a multifunction $F : X \to Y$ to be upper (lower) almost α-continuous if $F^+(V)$ ($F^-(V)$) is α-open in X for every regular open set V of Y. They obtain some characterizations and several properties concerning upper (lower) almost α-continuous multifunctions. The relationships between these multifunctions and α-closed graphs are investigated.

1. Introduction

In 1965, Njåstad [11] introduced a weak form of open sets called α-sets. In [18, 24] the authors investigated a class of functions called almost α-continuous or almost feebly continuous. In 1986, Neubrunn [10] introduced the notion of upper (lower) α-continuous multifunctions. The purpose of the present paper is to define upper (lower) almost α-continuous multifunctions and to obtain some characterizations of upper (lower) almost α-continuous multifunctions and several properties of such multifunctions.

2. Preliminaries

Let X be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A is said to be *α-open* [11] (resp. *semi-open* [6], *preopen* [9], *β-open* [1] or *semi-preopen* [2]) if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}(A))$, $A \subset \mathrm{Int}(\mathrm{Cl}(A))$, $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$). The family of all semi-open (resp. α-open) sets of X containing a point $x \in X$ is denoted by $\mathrm{SO}(X,x)$ (resp. $\alpha(X,x)$). The family of all α-open (resp. semi-open, preopen, semi-preopen) sets in X is denoted by $\alpha(X)$ (resp. $\mathrm{SO}(X), \mathrm{PO}(X), \mathrm{SPO}(X)$). For these four families, it is shown in [16, Lemma 3.1] that $\mathrm{SO}(X) \cap \mathrm{PO}(X) = \alpha(X)$ and it is obvious that $\mathrm{SO}(X) \cup \mathrm{PO}(X) \subset \mathrm{SPO}(X)$. Since $\alpha(X)$ is a topology for X [11, Proposition 2], by $\alpha\,\mathrm{Cl}(A)$ (resp. $\alpha\,\mathrm{Int}(A)$) we denote the closure (resp. interior) of A

with respect to $\alpha(X)$. The complement of a semi-open (resp. α-open) set is said to be *semi-closed* (resp. α-closed). The intersection of all semi-closed sets of X containing A is called the *semi-closure* [3] of A and is denoted by $\mathrm{sCl}(A)$. The union of all semi-open sets of X contained in A is called the *semi-interior* of A and is denoted by $\mathrm{sInt}(A)$. A subset A is said to be *feebly open* [5] if there exists an open set U such that $U \subset A \subset \mathrm{sCl}(U)$. It is shown in [16, Lemma 4.12] that the notion of feebly open sets is equivalent to that of α-open sets. A subset A of a space X is said to *regular open* (resp. *regular closed*) if $A = \mathrm{Int}(\mathrm{Cl}(A))$ (resp. $A = \mathrm{Cl}(\mathrm{Int}(A))$). The family of regular open (resp. regular closed) sets of X is denoted by $\mathrm{RO}(X)$ (resp. $\mathrm{RC}(X)$). Maheshwari et al. [7] defined a function to be *almost feebly continuous* if the inverse image of every regular open set is feebly open. Noiri [18] defined a function $f : X \to Y$ to be *almost α-continuous* if $f^{-1}(V) \in \alpha(X)$ for every $V \in \mathrm{RO}(Y)$ and pointed out that almost feeble continuity is equivalent to almost α-continuity.

Throughout this paper, spaces (X, τ) and (X, σ) (or simply X and Y) always mean topological spaces and $F : X \to Y$ (resp. $f : X \to Y$) presents a multivalued (resp. single valued) function. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a set G of Y by $F^+(G)$ and $F^-(G)$, respectively, that is

$$F^+(G) = \{x \in X : F(x) \subset G\} \quad \text{and} \quad F^-(G) = \{x \in X : F(x) \cap G \neq \emptyset\}.$$

3. Characterizations

DEFINITION 1. A multifunction $F : X \to Y$ is said to be

(a) *upper almost α-continuous* (briefly u.a.α.c.) at a point $x \in X$ if for each $U \in \mathrm{SO}(X, x)$ and each open set V containing $F(x)$, there exists a nonempty open set $G \subset U$ such that $F(G) \subset \mathrm{sCl}(V)$;

(b) *lower almost α-continuous* (briefly l.a.α.c.) at a point $x \in X$ if for each $U \in \mathrm{SO}(X, x)$ and each open set V such that $F(x) \cap V \neq \emptyset$, there exists a nonempty open set $G \subset U$ such that $F(g) \cap \mathrm{sCl}(V) \neq \emptyset$ for every $g \in G$;

(c) *upper (lower) almost α-continuous* if F has this property at every point of X.

THEOREM 1. *The following are equivalent for a multifunction $F : X \to Y$:*

(1) *F is u.a.α.c. at a point $x \in X$;*

(2) *for any open set V of Y containing $F(x)$, there exists $S \in \alpha(X, x)$ such that $F(S) \subset \mathrm{sCl}(V)$;*

(3) *$x \in \alpha\,\mathrm{Int}(F^+(\mathrm{sCl}(V)))$ for every open set V containing $F(x)$;*

(4) *$x \in \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^+(\mathrm{sCl}(V)))))$ for every open set V containing $F(x)$.*

Proof. (1)⇒(2): Let V be any open set of Y containing $F(x)$. For each $U \in \mathrm{SO}(X,x)$, there exists a nonempty open set G_U such that $G_U \subset U$ and $F(G_U) \subset \mathrm{sCl}(V)$. Let $W = \bigcup\{G_U : U \in \mathrm{SO}(X,x)\}$. Put $S = W \bigcup \{x\}$, then W is open in X, $x \in \mathrm{sCl}(W)$ and $F(W) \subset \mathrm{sCl}(V)$. Therefore, we have $S \in \alpha(X,x)$ [26, Lemma 2.1] and $F(S) \subset \mathrm{sCl}(V)$.

(2)⇒(3): Let V be any open set of Y containing $F(x)$. Then there exists $S \in \alpha(X,x)$ such that $F(S) \subset \mathrm{sCl}(V)$. Thus we obtain $x \in S \subset F^+(\mathrm{sCl}(V))$ and hence $x \in \alpha\,\mathrm{Int}(F^+(\mathrm{sCl}(V)))$.

(3)⇒(4): Let V be any open set of Y containing $F(x)$. Now put $U = \alpha\,\mathrm{Int}(F^+(\mathrm{sCl}(V)))$. Then $U \in \alpha(X)$ and $x \in U \subset F^+(\mathrm{sCl}(V))$. Thus we have $x \in U \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^+(\mathrm{sCl}(V)))))$.

(4)⇒(1): Let $U \in \mathrm{SO}(X,x)$ and V be any open set of Y containing $F(x)$. Then we have $x \in \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^+(\mathrm{sCl}(V))))) = \mathrm{sCl}(\mathrm{Int}(F^+(\mathrm{sCl}(V))))$. It follows from [13, Lemma 3] and [12, Lemma 1] that $\emptyset \neq U \cap \mathrm{Int}(F^+(\mathrm{sCl}(V))) \in \mathrm{SO}(X)$. Put $G = \mathrm{Int}(U \cap \mathrm{Int}(F^+(\mathrm{sCl}(V))))$. Then G is a nonempty open set of Y [12, Lemma 4], $G \subset U$ and $F(G) \subset \mathrm{sCl}(V)$.

THEOREM 2. *The following are equivalent for a multifunction* $F : X \to Y$:

(1) *F is l.a.α.c. at a point x of X;*

(2) *for any open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists $S \in \alpha(X,x)$ such that $F(s) \cap \mathrm{sCl}(V) \neq \emptyset$ for every $s \in S$;*

(3) *$x \in \alpha\,\mathrm{Int}(F^-(\mathrm{sCl}(V)))$ for every open set V of Y such that $F(x) \cap V \neq \emptyset$;*

(4) *$x \in \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^-(\mathrm{sCl}(V)))))$ for every open set V of Y such that $F(x) \cap V \neq \emptyset$.*

Proof. The proof is similar to that of Theorem 1.

THEOREM 3. *The following are equivalent for a multifunction* $F : X \to Y$:

(1) *F is u.a.α.c.;*

(2) *for each $x \in X$ and each open set V of Y containing $F(x)$, there exists $U \in \alpha(X,x)$ such that $F(U) \subset \mathrm{sCl}(V)$;*

(3) *for each $x \in X$ and each $V \in \mathrm{RO}(Y)$ containing $F(x)$, there exists $U \in \alpha(X,x)$ such that $F(U) \subset V$;*

(4) *$F^+(V) \in \alpha(X)$ for every $V \in \mathrm{RO}(Y)$;*

(5) *$F^-(K)$ is α-closed in X for every $K \in \mathrm{RC}(Y)$;*

(6) *$F^+(V) \subset \alpha\,\mathrm{Int}(F^+(\mathrm{sCl}(V)))$ for every open set V of Y;*

(7) *$\alpha\,\mathrm{Cl}(F^-(\mathrm{sInt}(K))) \subset F^-(K)$ for every closed set K of Y;*

(8) *$\alpha\,\mathrm{Cl}(F^-(\mathrm{Cl}(\mathrm{Int}(K)))) \subset F^-(K)$ for every closed set K of Y;*

(9) *$\alpha\,\mathrm{Cl}(F^-(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(B))))) \subset F^-(\mathrm{Cl}(B))$ for every subset B of Y;*

(10) $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^-(\mathrm{Cl}(\mathrm{Int}(K)))))) \subset F^-(K)$ *for every closed set* K *of* Y;
(11) $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^-(\mathrm{sInt}(K))))) \subset F^-(K)$ *for every closed set* K *of* Y;
(12) $F^+(V) \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^+(\mathrm{sCl}(V)))))$ *for every open set* V *of* Y.

Proof. (1)⇒(2): The proof follows immediately from Theorem 1.

(2)⇒(3): This is obvious.

(3)⇒(4): Let $V \in \mathrm{RO}(Y)$ and $x \in F^+(V)$. Then $F(x) \subset V$ and there exists $U_x \in \alpha(X, x)$ such that $F(U_x) \subset V$. Therefore, we have $x \in U_x \subset F^+(V)$ and hence $F^+(V) \in \alpha(X)$.

(4)⇒(5): This follows from the fact that $F^+(Y - B) = X - F^-(B)$ for every subset B of Y.

(5)⇒(6): Let V be any open set of Y and $x \in F^+(V)$. Then we have $F(x) \subset V \subset \mathrm{sCl}(V)$ and hence $x \in F^+(\mathrm{sCl}(V)) = X - F^-(Y - \mathrm{sCl}(V))$. Since $Y - \mathrm{sCl}(V) \in \mathrm{RC}(Y)$, $F^-(Y - \mathrm{sCl}(V))$ is α-closed in X. Therefore, $F^+(\mathrm{sCl}(V)) \in \alpha(X, x)$ and hence $x \in \alpha\,\mathrm{Int}(F^+(\mathrm{sCl}(V)))$. Consequently, we obtain $F^+(V) \subset \alpha\,\mathrm{Int}(F^+(\mathrm{sCl}(V)))$.

(6)⇒(7): Let K be any closed set of Y. Then, since $Y - K$ is open, we obtain

$$\begin{aligned} X - F^-(K) = F^+(Y - K) &\subset \alpha\,\mathrm{Int}(F^+(\mathrm{sCl}(Y - K))) \\ &= \alpha\,\mathrm{Int}(F^+(Y - \mathrm{sInt}(K))) \\ &= \alpha\,\mathrm{Int}(X - F^-(\mathrm{sInt}(K))) = X - \alpha\,\mathrm{Cl}(F^-(\mathrm{sInt}(K))). \end{aligned}$$

Therefore, we obtain $\alpha\,\mathrm{Cl}(F^-(\mathrm{sInt}(K))) \subset F^-(K)$.

(7)⇒(8): The proof is obvious since $\mathrm{sInt}(K) = \mathrm{Cl}(\mathrm{Int}(K))$ for every closed set K.

(8)⇒(9): The proof is obvious.

(9)⇒(10): It follows from [26, Lemma 2.2] that $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A))) \subset \alpha\,\mathrm{Cl}(A)$ for every subset A. Thus for every closed set $K \subset Y$, we have

$$\begin{aligned} &\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^-(\mathrm{Cl}(\mathrm{Int}(K)))))) \subset \alpha\,\mathrm{Cl}(F^-(\mathrm{Cl}(\mathrm{Int}(K)))) \\ &= \alpha\,\mathrm{Cl}(F^-(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(K))))) \subset F^-(\mathrm{Cl}(K)) = F^-(K). \end{aligned}$$

(10)⇒(11): The proof is obvious since $\mathrm{sInt}(K) = \mathrm{Cl}(\mathrm{Int}(K))$ for every closed set K.

(11)⇒(12): Let V be any open set of Y. Then $Y - V$ is closed in Y and we have

$$\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^-(\mathrm{sInt}(Y - V))))) \subset F^-(Y - V) = X - F^+(V).$$

Moreover, we have

$$\begin{aligned} &\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^-(\mathrm{sInt}(Y - V)))))) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^-(Y - \mathrm{sCl}(V))))) = \\ &\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(X - F^+(\mathrm{sCl}(V))))) = X - \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^+(\mathrm{sCl}(V))))). \end{aligned}$$

Therefore, we obtain $F^+(V) \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^+(\mathrm{sCl}(V)))))$.

(12)⇒(1): Let x be any point of X and V be any open set of Y containing $F(x)$. Then $x \in F^+(V) \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^+(\mathrm{sCl}(V)))))$ and hence F is u.a.α.c. at x by Theorem 1.

THEOREM 4. *The following are equivalent for a multifunction $F : X \to Y$:*

(1) *F is u.a.α.c.;*
(2) *$\alpha\,\mathrm{Cl}(F^-(V)) \subset F^-(\mathrm{Cl}(V))$ for every $V \in \mathrm{SPO}(Y)$;*
(3) *$\alpha\,\mathrm{Cl}(F^-(V)) \subset F^-(\mathrm{Cl}(V))$ for every $V \in \mathrm{SO}(Y)$;*
(4) *$F^+(V) \subset \alpha\,\mathrm{Int}(F^+(\mathrm{Int}(\mathrm{Cl}(V))))$ for every $V \in \mathrm{PO}(Y)$.*

Proof. (1)⇒(2): Let V be any semi-preopen set of Y. Since $\mathrm{Cl}(V) \in \mathrm{RC}(Y)$, by Theorem 3 $F^-(\mathrm{Cl}(V))$ is α-closed in X and $F^-(V) \subset F^-(\mathrm{Cl}(V))$. Therefore, we obtain $\alpha\,\mathrm{Cl}(F^-(V)) \subset F^-(\mathrm{Cl}(V))$.

(2)⇒(3): This is obvious since $\mathrm{SO}(Y) \subset \mathrm{SPO}(Y)$.

(3)⇒(1): Let $K \in \mathrm{RC}(Y)$. Then $K \in \mathrm{SO}(Y)$ and hence $\alpha\,\mathrm{Cl}(F^-(K)) \subset F^-(K)$. Therefore, $F^-(K)$ is α-closed in X and hence F is u.a.α.c. by Theorem 3.

(1)⇒(4): Let V be arbitrary preopen set of Y. Since $\mathrm{Int}(\mathrm{Cl}(V)) \in \mathrm{RO}(Y)$, by Theorem 3 we have $F^+(\mathrm{Int}(\mathrm{Cl}(V))) \in \alpha(X)$ and hence

$$F^+(V) \subset F^+(\mathrm{Int}(\mathrm{Cl}(V))) = \alpha\,\mathrm{Int}(F^+(\mathrm{Int}(\mathrm{Cl}(V)))).$$

(4)⇒(1): Let V be any regular open set of Y. Since $V \in \mathrm{PO}(Y)$, we have

$$F^+(V) \subset \alpha\,\mathrm{Int}(F^+(\mathrm{Int}(\mathrm{Cl}(V)))) = \alpha\,\mathrm{Int}(F^+(V))$$

and hence $F^+(V) \in \alpha(X)$. It follows from Theorem 3 that F is u.a.α.c.

THEOREM 5. *The following are equivalent for a multifunction $F : X \to Y$:*

(1) *F is l.a.α.c.;*
(2) *for each $x \in X$ and each open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists $U \in \alpha(X, x)$ such that $U \subset F^-(\mathrm{sCl}(V))$;*
(3) *for each $x \in X$ and each $V \in \mathrm{RO}(Y)$ such that $F(x) \cap V \neq \emptyset$, there exists $U \in \alpha(X, x)$ such that $U \subset F^-(V)$;*
(4) *$F^-(V) \in \alpha(X)$ for every $V \in \mathrm{RO}(Y)$;*
(5) *$F^+(K)$ is α-closed in X for every $K \in \mathrm{RC}(Y)$;*
(6) *$F^-(V) \subset \alpha\,\mathrm{Int}(F^-(\mathrm{sCl}(V)))$ for every open set V of Y;*
(7) *$\alpha\,\mathrm{Cl}(F^+(\mathrm{sInt}(K))) \subset F^+(K)$ for every closed set K of Y;*
(8) *$\alpha\,\mathrm{Cl}(F^+(\mathrm{Cl}(\mathrm{Int}(K)))) \subset F^+(K)$ for every closed set K of Y;*
(9) *$\alpha\,\mathrm{Cl}(F^+(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(B))))) \subset F^+(\mathrm{Cl}(B))$ for every subset B of Y;*
(10) *$\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(\mathrm{Cl}(\mathrm{Int}(K)))))) \subset F^+(K)$ for every closed set K of Y;*
(11) *$\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(\mathrm{sInt}(K))))) \subset F^+(K)$ for every closed set K of Y;*
(12) *$F^-(V) \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^-(\mathrm{sCl}(V)))))$ for every open set V of Y.*

Proof. The proof is similar to that of Theorem 3.

THEOREM 6. *The following are equivalent for a multifunction* $F : X \to Y$:

(1) F *is l.a.α.c.*;
(2) $\alpha\,\mathrm{Cl}(F^+(V)) \subset F^+(\mathrm{Cl}(V))$ *for every* $V \in \mathrm{SPO}(Y)$;
(3) $\alpha\,\mathrm{Cl}(F^+(V)) \subset F^+(\mathrm{Cl}(V))$ *for every* $V \in \mathrm{SO}(Y)$;
(4) $F^-(V) \subset \alpha\,\mathrm{Int}(F^-(\mathrm{Int}(\mathrm{Cl}(V))))$ *for every* $V \in \mathrm{PO}(Y)$.

Proof. The proof is similar to that of Theorem 4.

A function $f : X \to Y$ is said to be *almost α-continuous* [18] if $f^{-1}(V) \in \alpha(X)$ for every $V \in \mathrm{RO}(Y)$.

COROLLARY 1 (Maheshwari et al. [7], Noiri [18] and Popa [24]). *The following are equivalent for a function* $f : X \to Y$:

(1) f *is almost α-continuous*;
(2) *for each* $x \in X$ *and each open set* V *of* Y *containing* $f(x)$, *there exists* $U \in \alpha(X,x)$ *such that* $f(U) \subset \mathrm{sCl}(V)$;
(3) *for each* $x \in X$ *and each* $V \in \mathrm{RO}(Y)$ *containing* $f(x)$, *there exists* $U \in \alpha(X,x)$ *such that* $f(U) \subset V$;
(4) *for each* $x \in X$ *and each open set* V *of* Y *containing* $f(x)$, *there exists* $U \in \alpha(X,x)$ *such that* $f(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$;
(5) $f^{-1}(K)$ *is α-closed in* X *for every* $K \in \mathrm{RC}(Y)$;
(6) $f^{-1}(V) \subset \alpha\,\mathrm{Int}(f^{-1}1(\mathrm{sCl}(V)))$ *for every open set* V *of* Y;
(7) $\alpha\,\mathrm{Cl}(f^{-1}(\mathrm{sInt}(K))) \subset f^{-1}(K)$ *for every closed set* K *of* Y;
(8) $\alpha\,\mathrm{Cl}(f^{-1}(\mathrm{Cl}(\mathrm{Int}(K)))) \subset f^{-1}(K)$ *for every closed set* K *of* Y;
(9) $\alpha\,\mathrm{Cl}(f^{-1}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(B))))) \subset f^{-1}(\mathrm{Cl}(B))$ *for every subset* B *of* Y;
(10) $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(f^{-1}(\mathrm{Cl}(\mathrm{Int}(K)))))) \subset f^{-1}(K)$ *for every closed set* K *of* Y;
(11) $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(f^{-1}(\mathrm{sInt}(K))))) \subset f^{-1}(K)$ *for every closed set* K *of* Y;
(12) $f^{-1}(V) \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(f^{-1}(\mathrm{sCl}(V)))))$ *for every open set* V *of* Y.

COROLLARY 2. *The following are equivalent for a function* $f : X \to Y$:

(1) f *is almost α-continuous*;
(2) $\alpha\,\mathrm{Cl}(f^{-1}(V)) \subset f^{-1}(\mathrm{Cl}(V))$ *for every* $V \in \mathrm{SPO}(Y)$;
(3) $\alpha\,\mathrm{Cl}(f^{-1}(V)) \subset f^{-1}(\mathrm{Cl}(V))$ *for every* $V \in \mathrm{SO}(Y)$;
(4) $f^{-1}(V) \subset \alpha\,\mathrm{Int}(f^{-1}(\mathrm{Int}(\mathrm{Cl}(V))))$ *for every* $V \in \mathrm{PO}(Y)$.

4. Almost α-continuity and α-continuity

DEFINITION 2. A multifunction $F : X \to Y$ is said to be

(a) *upper α-continuous* [26] (resp. *upper weakly α-continuous* [28]) at a point x of X if for each open set V of Y containing $F(x)$, there exists $U \in \alpha(X,x)$ such that $F(U) \subset V$ (resp. $F(U) \subset \mathrm{Cl}(V)$);

(b) *lower α-continuous* [26] (resp. *lower weakly α-continuous* [28]) at $x \in X$ if for each open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists $U \in \alpha(X, x)$ such that $F(u) \cap V \neq \emptyset$ (resp. $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$) for every $u \in U$;

(c) *upper (or lower) α-continuous* [10] (resp. *weakly α-continuous* [28]) if it is upper (or lower) α-continuous (resp. weakly α-continuous) at every point of X.

DEFINITION 3. A multifunction $F : X \to Y$ is said to be

(a) *upper almost continuous* [22] (resp. *upper weakly continuous* [21, 32]) at $x \in X$ if for each open set V of Y containing $F(x)$, there exists an open set U of X containing x such that $F(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$ (resp. $F(U) \subset \mathrm{Cl}(V)$);

(b) *lower almost continuous* [22] (resp.*lower weakly continuous* [21, 32]) if for each open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists an open set U of X containing x such that $F(u) \cap \mathrm{Int}(\mathrm{Cl}(V)) \neq \emptyset$ (resp. $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$) for every $u \in U$;

(c) *upper* (or *lower*) *almost continuous* [22] (resp. *weakly continuous* [21, 32]) if it is upper (or lower) almost continuous (resp. weakly continuous) at every point of X.

THEOREM 7. (1) *A multifunction $F : (X, \tau) \to (X, \sigma)$ is upper α-continuous (resp. u.a.α.c., upper weakly α-continuous) if and only if $F : (X, \tau^\alpha) \to (Y, \sigma)$ is upper continuous (resp. upper almost continuous, upper weakly continuous).*

(2) *A multifunction $F : (X, \tau) \to (Y, \sigma)$ is lower α-continuous (resp. l.a.α.c., lower weakly α-continuous) if and only if $F : (X, \tau^\alpha) \to (Y, \sigma)$ is lower continuous (resp. lower almost continuous, lower weakly continuous).*

Proof. The proof is obvious from the definitions.

DEFINITION 4. A subset A of a space X is said to be

(a) *α-paracompact* [35] if every cover of A by open sets of X is refined by a cover of A which consists of open sets of X and is locally finite in X;

(b) *α-regular* [5] if for each point $x \in A$ and each open set U of X containing x, there exists an open set G of X such that $x \in G \subset \mathrm{Cl}(G) \subset U$.

THEOREM 8. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$ such that $F(x)$ is an α-regular α-paracompact set for each $x \in X$, the following are equivalent:*

(1) *F is upper weakly α-continuous;*

(2) *F is u.a.α.c.;*

(3) *F is upper α-continuous.*

Proof. (1)⇒(3): By Theorem 7, $F : (X, \tau^{\alpha}) \to (Y, \sigma)$ is upper weakly continuous and hence upper continuous [25, Theorem 1]. Thus, it follows from Theorem 7 that $F : (X, \tau) \to (Y, \sigma)$ is upper α-continuous.

THEOREM 9. *For a multifunction $F : X \to Y$ such that $F(x)$ is an α-regular set for every $x \in X$, the following are equivalent:*

(1) *F is lower weakly α-continuous;*
(2) *F is l.a.α.c.;*
(3) *F is lower α-continuous.*

Proof. (1)⇒(3): By Theorem 7, $F : (X, \tau^{\alpha}) \to (Y, \sigma)$ is lower weakly continuous and hence lower continuous [25, Theorem 2]. Thus, it follows from Theorem 7 that $F : (X, \tau) \to (Y, \sigma)$ is lower α-continuous.

DEFINITION 5. A subset A of a space X is said to be *α-semi-regular* [25] if for each point $a \in A$ and each open set U containing a, there exists $V \in \mathrm{RO}(X)$ such that $a \in V \subset U$.

THEOREM 10. *Let $F : (X, \tau) \to (Y, \alpha)$ be a multifunction such that $F(x)$ is an α-semi-regular set for each $x \in X$. Then F is l.a.α.c. if and only if F is lower α-continuous.*

Proof. Suppose that $F : (X, \tau) \to (Y, \sigma)$ is l.a.α.c. By Theorem 7, $F : (X, \tau^{\alpha}) \to (Y, \sigma)$ is lower almost continuous. It follows from [25, Theorem 5] that $F : (X, \tau^{\alpha}) \to (Y, \alpha)$ is lower continuous. Therefore, by Theorem 7 $F : (X, \tau) \to (Y, \sigma)$ is lower α-continuous.

DEFINITION 6. A space X is said to be

(a) *semi-regular* if for each point x of X and each open set U containing x, there exists $V \in \mathrm{RO}(X)$ such that $x \in V \subset U$.
(b) *rim-compact* if each point of X has a base of neighbourhoods with compact frontiers.

COROLLARY 3. *Every l.a.α.c. multifunction $F : X \to Y$ is lower α-continuous if Y is semi-regular.*

COROLLARY 4 (Maheshwari et al. [7] and Thakur and Paik [33]). *Every almost α-continuous function $f : X \to Y$ is α-continuous if Y is semi-regular.*

THEOREM 11. *If Y is a rim-compact space and $F : X \to Y$ is a compact valued multifunction with the closed graph, then the following are equivalent:*

(1) *F is upper weakly α-continuous;*
(2) *F is u.a.α.c.;*
(3) *F is upper α-continuous.*

Proof. Suppose that F is upper weakly α-continuous. Let $x \in X$ and V be any open set of Y containing $F(x)$. Since Y is rim-compact, for each $z \in F(x)$ there exists an open set $W(z)$ such that $z \in W(z) \subset V$ and the frontier $\mathrm{Fr}(W(z))$ is compact. The family $\{W(z) : z \in F(x)\}$ is a cover of $F(x)$ by open sets of Y. Since $F(x)$ is compact, there exists a finite number of points, say, $z_1, z_2, \ldots, z_n$ in $F(x)$ such that $F(x) \subset \cup\{W(z_j) : 1 \leq j \leq n\}$. Let $W = \cup\{W(z_j); 1 \leq j \leq n\}$, then we have $\mathrm{Fr}(W)$ is compact, $F(x) \subset W \subset V$, and

$$F(x) \cap \mathrm{Fr}(W) = F(x) \cap \mathrm{Cl}(W) \cap \mathrm{Cl}(Y - W) \subset F(x) \cap (Y - W) = \emptyset.$$

For each $y \in \mathrm{Fr}(W)$, $(x, y) \in X \times Y - G(F)$. Since $G(F)$ is closed, there exist open sets $U(y) \subset X$ and $V(y) \subset Y$ containing x and y, respectively, such that $F(U(y)) \cap V(y) = \emptyset$. The family $\{V(y) : y \in \mathrm{Fr}(W)\}$ is a cover of $\mathrm{Fr}(W)$ by open sets of Y. Since $\mathrm{Fr}(W)$ is compact, there exists a finite subset K of $\mathrm{Fr}(W)$ such that $\mathrm{Fr}(W) \subset \cup\{V(y) : y \in K\}$. Since F is upper weakly α-continuous, there exists $U_0 \in \alpha(X, x)$ such that $F(U_0) \subset \mathrm{Cl}(W)$. Put $U = U_0 \cap [\cap\{U(y) : y \in K\}]$. Then we obtain $U \in \alpha(X, x)$ [15, Lemma 3.3], $F(U) \subset \mathrm{Cl}(W)$ and $F(U) \cap \mathrm{Fr}(W) = \emptyset$. Therefore, we obtain $F(U) \subset W \subset V$. This shows that F is upper α-continuous.

COROLLARY 5 (Popa [24]). *If Y is a rim-compact space and $f : X \to Y$ is an almost α-continuous function with the closed graph, then f is α-continuous.*

THEOREM 12. *If (Y, α) is rim-compact Hausdorff, then for a multifunction $F : (X, \tau) \to (Y, \alpha)$ the following are equivalent:*

(1) *F is lower weakly α-continuous;*
(2) *F is l.a.α.c.;*
(3) *F is lower α-continuous.*

Proof. Suppose that F is lower weakly α-continuous. It follows from Theorem 7 that $F : (X, \tau^\alpha) \to (X, \sigma)$ lower weakly continuous. Since (Y, α) is rim-compact Hausdorff, it is regular [14, Theorem 4] and hence $F : (X, \tau^\alpha) \to (Y, \sigma)$ is lower continuous [21, Theorem 2]. Therefore, $F : (X, \tau) \to (Y, \sigma)$ is lower α-continuous by Theorem 7.

DEFINITION 7. The *semi-frontier* [4], $\mathrm{sFr}(A)$, of a subset A of a space X is defined as follows: $\mathrm{sFr}(A) = \mathrm{sCl}(A) \cap \mathrm{sCl}(X - A) = \mathrm{sCl}(A) - \mathrm{sInt}(A)$.

DEFINITION 8. A multifunction $F : X \to Y$ is said to be *complementary almost quasi continuous* [27] if for each open set V of Y, $F^-(\mathrm{sFr}(V))$ is a closed set of X.

THEOREM 13. *If $F : X \to Y$ is u.a.α.c. and complementary almost quasi continuous, then it is upper α-continuous.*

Proof. Let $x \in X$ and V be any open set of Y such that $F(x) \subset V$. By Theorem 3, there exists $G \in \alpha(X, x)$ such that $F(G) \subset \mathrm{sCl}(V)$. Now, put $U = G \cap [X - F^{-}(\mathrm{sFr}(V))]$. Since $F^{-}(\mathrm{sFr}(V))$ is closed in $X, U \in \alpha(X)$ [15, Lemma 3.3]. Moreover, we have $F(x) \cap \mathrm{sFr}(V) = \emptyset$ and hence $x \in X - F^{-}(\mathrm{sFr}(V))$. Therefore, we obtain $x \in U \in \alpha(X)$ and $F(U) \subset V$ since $F(U) \subset F(G) \subset \mathrm{sCl}(V)$ and $F(U) \subset Y - \mathrm{sFr}(V)$. So, F is upper α-continuous.

COROLLARY 6 (Popa [24]). *If $f : X \to Y$ is an almost α-continuous function and $f^{-1}(\mathrm{sFr}(V))$ is closed in X for each open set V of Y, then f is α-continuous.*

5. Properties

DEFINITION 9. A multifunction $F : X \to Y$ is said to be

(a) *upper precontinuous* [23] if $F^{+}(V) \in \mathrm{PO}(X)$ for each open set V of Y;

(b) *lower precontinuous* [23] if $F^{-}(V) \in \mathrm{PO}(X)$ for each open set V of Y.

DEFINITION 10. A multifunction $F : X \to Y$ is said to be

(a) *upper almost quasi continuous* [27] at a point $x \in X$ if for each open set U containing x and each open set V containing $F(x)$, there exists a nonempty open set G of X such that $G \subset U$ and $F(G) \subset \mathrm{sCl}(V)$;

(b) *lower almost quasi continuous* [27] at a point $x \in X$ if for each open set U containing x and each open set V such that $F(x) \cap V \neq \emptyset$, there exists a nonempty open set G of X such that $G \subset U$ and $F(g) \cap \mathrm{sCl}(V) \neq \emptyset$ for every $g \in G$;

(c) *upper almost quasi continuous* (*lower almost quasi continuous*) if F has the property at every point of X.

THEOREM 14. *If a multifunction $F : X \to Y$ is upper precontinuous and upper almost quasi continuous, then it is u.a.α.c.*

Proof. Let V be a regular open set of Y. Since F is upper precontinuous, $F^{+}(V) \in \mathrm{PO}(X)$. Since F is upper almost quasi continuous, $F^{+}(V) \in \mathrm{SO}(X)$ [27, Theorem 3.3] and hence $F^{+}(V) \in \alpha(X)$ [16, Lemma 3.1]. Therefore, F is u.a.α.c.

THEOREM 15. *If a multifunction is lower precontinuous and lower almost quasi continuous, then it is l.a.α.c.*

Proof. The proof is similar to that of Theorem 14.

COROLLARY 7 (Popa [24]). *If a function $f : X \to Y$ is almost continuous (in the sense of Husain) and almost quasi continuous [20], then f is almost α-continuous.*

DEFINITION 11. A subset S of a space X is called an A-set [34] if $S = U - V$, where U is an open set and $V \in \mathrm{RO}(X)$.

LEMMA 1 (Tong [34]). *A subset of a space X is open in X if and only if it is both α-open and A-set.*

DEFINITION 12. A multifunction $F : X \to Y$ is said to be *upper* (resp. *lower*) *almost A-continuous* if $F^+(V)$ (resp. $F^-(V)$) is an A-set of X for each $V \in \mathrm{RO}(Y)$.

It follows from [24, Remark 1] that every upper almost continuous (resp. lower almost continuous) multifunction is upper almost A-continuous (resp. lower almost A-continuous) but the converse need not be true.

THEOREM 16. *A multifunction $F : X \to Y$ is upper almost continuous (resp. lower almost continuous) if and only if it is both u.a.α.c. (resp. l.a.α.c.) and upper almost A-continuous (resp. lower almost A-continuous).*

Proof. This follows from Lemma 1 and [22, Theorem 2,4] (resp. [22, Theorem 2.2]).

COROLLARY 8 (Popa [24]). *A function $f : X \to Y$ is almost continuous (in the sense of Singal [31]) if and only if it is both almost feebly continuous and almost A-continuous.*

DEFINITION 13. A multifunction $F : X \to Y$ is said to be

(a) *upper β-continuous* [26] if for each $x \in X$ and each open set V of Y such that $F(x) \subset V$, there exists a β-open set U containing x such that $F(U) \subset V$;

(b) *lower β-continuous* [26] if for each $x \in X$ and each open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a β-open set U containing x such that $F(u) \cap V \neq \emptyset$ for every $u \in U$.

LEMMA 2 (Popa and Noiri [26]. *A multifunction $F : X \to Y$ is upper β-continuous (resp. lower β-continuous) if and only if* $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^-(B)))) \subset F^-(\mathrm{Cl}(B))$ (*resp.* $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^+(B)))) \subset F^+(\mathrm{Cl}(B))$) *for every subset B of Y.*

THEOREM 17. *If a multifunction $F : X \to Y$ is l.a.α.c. and upper β-continuous, then it is lower weakly continuous.*

Proof. Let V be any open set of Y such that $F(x) \cap V \neq \emptyset$. Since F is l.a.α.c., by Theorem 2 $x \in \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^-(\mathrm{sCl}(V)))))$. Let

$$U = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^-(\mathrm{sCl}(V))))),$$

then U is an open set containing x. Since F is upper β-continuous, by Lemma 2 we have $U \subset F^-(\mathrm{Cl}(\mathrm{sCl}(V))) \subset F^-(\mathrm{Cl}(V))$. This shows that F is lower weakly continuous.

COROLLARY 9 (Popa and Noiri [26]). *If a multifunction is lower α-continuous and upper β-continuous, then it is lower weakly continuous.*

COROLLARY 10 (Neubrunn [10]). *If a multifunction is lower α-continuous and upper quasi continuous, then it is lower weakly continuous.*

COROLLARY 11. *If a multifunction is l.a.α.c. and upper precontinuous, then it is lower weakly continuous.*

COROLLARY 12 (Neubrunn [10]). *If a multifunction $F : X \to Y$ is lower almost continuous and upper precontinuous and Y is regular, then F is lower continuous.*

Proof. This follows from Corollary 11 and [21, Theorem 2].

THEOREM 18. *If a multifunction is u.a.α.c. and lower β-continuous, then it is upper weakly continuous.*

Proof. The proof is similar to that of Theorem 17.

COROLLARY 13 (Popa and Noiri [26]). *If a multifunction is upper α-continuous and lower β-continuous, then it is upper weakly continuous.*

COROLLARY 14 (Neubrunn [10]). *If a multifunction is upper α-continuous and lower quasi continuous, then it is upper weakly continuous.*

COROLLARY 15. *If a multifunction is u.a.α.c. and lower precontinuous, then it is upper weakly continuous.*

DEFINITION 14. A subset A of a space X is said to be *quasi H-closed relative to X* [29] if for every cover $\{V_\alpha : \alpha \in \nabla\}$ of A by open sets of X, there exists a finite subset ∇_0 of ∇ such that $A \subset \bigcup\{\mathrm{Cl}(V_\alpha) : \alpha \in \nabla_0\}$. If X is quasi H-closed relative to X, then the space X is called *quasi H-closed*. A subset A is said to be *quasi H-closed* if the subspace A is quasi H-closed. A quasi H-closed Hausdorff space is called *H-closed*.

DEFINITION 15. A space X is said to be *α-compact* [8] if every cover of X by α-open sets of X has a finite subcover.

THEOREM 19. *Let $F : X \to Y$ be an upper weakly α-continuous surjective multifunction such that $F(x)$ is compact for each $x \in X$. If X is α-compact, then Y is quasi H-closed.*

Proof. Let $\{V_\lambda : \lambda \in \Lambda\}$ be any open cover of Y. For each $x \in X, F(x)$ is compact and hence there exists a finite subset $\Lambda(x)$ of Λ such that $F(x) \subset \bigcup\{V_\lambda : \lambda \in \Lambda(x)\}$. Since F is upper weakly α-continuous, there exists

$U(x) \in \alpha(X)$ such that $F(U(x)) \subset \bigcup\{\mathrm{Cl}(V_\lambda) : \lambda \in \Lambda(x)\}$. Since X is α-compact, there exist a finite number of points, say, $x_1, x_2, \ldots, x_n$ in X such that $X = \bigcup\{U(x_i) : 1 \le i \le n\}$. Therefore, we obtain

$$Y = F(X)$$
$$= F\Big(\bigcup\{U(x_i) : 1 \le i \le n\}\Big) \subset \bigcup\{\mathrm{Cl}(V_\lambda) : \lambda \in \Lambda(x_i), \quad 1 \le i \le n\}.$$

This shows that Y is quasi H-closed.

THEOREM 20. *Let $F : (X,\tau) \to (Y,\sigma)$ be a surjective connected valued multifunction. If F is upper weakly α-continuous (or lower weakly α-continuous) and (X,τ) is connected, then (Y,σ) is connected.*

Proof. Since (X,τ) is connected (X,τ^α) is connected [30, Theorem 2]. By Theorem 7, $F : (X,\tau^\alpha) \to (Y,\sigma)$ is upper weakly continuous (or lower weakly continuous) and hence (Y,σ) is connected [32, Theorem 11].

COROLLARY 16 (Noiri [17]). *If $f : X \to Y$ is a weakly α-continuous surjection and X is connected, then Y is connected.*

DEFINITION 16. A multifunction $F : X \to Y$ has an *α-closed graph* if for each $(x,y) \in X \times Y - G(F)$, there exist $U \in \alpha(X,x)$ and an open set V containing y such that $[U \times \mathrm{Cl}(V)] \cap G(F) = \emptyset$.

LEMMA 3. *A multifunction $F : X \to Y$ has an α-closed graph if and only if for each $(x,y) \in X \times Y - G(F)$, there exist $U \in \alpha(X,x)$ and an open set V of Y containing y such that $F(U) \cap \mathrm{Cl}(V) = \emptyset$.*

THEOREM 21. *If $F : X \to Y$ is an u.a.α.c. compact valued multifunction and Y is Hausdorff, then F has an α-closed graph.*

Proof. Let $(x,y) \in X \times Y - G(F)$, then $y \in Y - F(x)$. For each $a \in F(x)$, there exist open sets $V(a)$ and $W(a)$ containing a and y, respectively, such that $V(a) \cap W(a) = \emptyset$. The family $\{V(a) : a \in F(x)\}$ is an open cover of $F(x)$ and there exist a finite number of points in $F(x)$, say, $a_1, a_2, \ldots, a_n$ such that $F(x) \subset \cup\{V(a_i) : 1 \le i \le n\}$. Set $V = \cup\{V(a_i) : 1 \le i \le n\}$ and $W = \cap\{W(a_i) : 1 \le i \le n\}$. Then $F(x) \subset V, V \cap W = \emptyset$ and $V \cap \mathrm{Cl}(W) = \emptyset$. Thus $F(x) \subset Y - \mathrm{Cl}(W)$. Since W is open, $\mathrm{Cl}(W)$ is regular closed and $Y - \mathrm{Cl}(W) \in \mathrm{RO}(Y)$. Theorem 3, there exists $U \in \alpha(X,x)$ such that $F(U) \subset Y - \mathrm{Cl}(W)$, thus $F(U) \cap \mathrm{Cl}(W) = \emptyset$ and by Lemma 3 F has an α-closed graph.

COROLLARY 17. *If $F : X \to Y$ is an upper α-continuous multifunction into a Hausdorff space Y and $F(x)$ is compact for each $x \in X$, then F has an α-closed graph.*

THEOREM 22. *If a multifunction $F : X \to Y$ has an α-closed graph, then F has the following property:*

(P) *For each set K quasi H-closed relative to Y, $F^-(K)$ is an α-closed set of X.*

Proof. Let $G(F)$ be α-closed. Suppose that there exists a set K quasi H-closed relative to Y such that $F^-(K)$ is not α-closed in X. Then there exists $x \in \alpha\,\mathrm{Cl}(F^-(K)) - F^-(K)$. Since $x \in X - F^-(K)$, we have $F(x) \cap K = \emptyset$ and hence $(x, y) \in X \times Y - G(F)$ for each $y \in K$. Since $G(F)$ is α-closed, there exist $U(y) \in \alpha(X, x)$ and an open set $V(y)$ of Y containing y such that $F(U(y)) \cap \mathrm{Cl}(V(y)) = \emptyset$. The family $\{V(y) : y \in K\}$ is an open cover of K. Since K is quasi H-closed relative to Y, there exist a finite number of points in K, say, $y_1, y_2, \dots, y_n$ such that $K \subset \cup\{\mathrm{Cl}(V(y_i)) : 1 \leq i \leq n\}$. Let $U = \cap\{U(y_i) : 1 \leq i \leq n\}$. Then $U \in \alpha(X, x)$ and $F(U) \cap K = \emptyset$. Therefore, we have $U \cap F^-(K) = \emptyset$. This contradicts the fact that $x \in \alpha\,\mathrm{Cl}(F^-(K))$.

COROLLARY 18. *If a multifunction $F : X \to Y$ has an α-closed graph and Y is quasi H-closed, then F is u.a.α.c.*

Proof. Let K be a regular closed set of Y. Since Y is quasi H-closed, K is quasi H-closed relative to Y and by Theorem 22 $F^-(K)$ is α-closed in X. Therefore, F is u.a.α.c. by Theorem 3.

DEFINITION 17. A Hausdorff space X is said to be *locally H-closed* [19] if every point of X has an H-closed neighborhood.

THEOREM 23. *Let Y be a locally H-closed space. If a multifunction $F : X \to Y$ is compact valued and has the following property:*

(P*) *For each quasi H-closed set of Y, $F^-(K)$ is α-closed in X,*

then F has an α-closed graph.

Proof. Let Y be locally H-closed and $(x, y) \in X \times Y - G(F)$, then $y \in Y - F(x)$. Since Y is Hausdorff and $F(x)$ is compact for every $x \in X$, as in Theorem 21 there exist open sets U and V such that $y \in U, F(x) \subset V$ and $U \cap V = \emptyset$. Since Y is locally H-closed, there exists an H-closed neighborhood W of y. So there exists an open set W_0 such that $y \in W_0 \subset W$. Let $G = U \cap W_0$, then G is open, $y \in G$ and $G \cap V = \emptyset$ which implies $\mathrm{Cl}(G) \cap V = \emptyset$. Since Y is Hausdorff and W is H-closed, W is closed and hence $\mathrm{Cl}(G) \subset W$. Since G is open in Y, G is open in W and $\mathrm{Cl}(G)$ is regular closed in W. Since W is H-closed, $\mathrm{Cl}(G)$ is H-closed. Since F has the property $(P^*), F^-(\mathrm{Cl}(G))$ is α-closed in X. Let $H = X - F^-(\mathrm{Cl}(G))$. Then we obtain $H \in \alpha(X, x)$ and $F(H) \cap \mathrm{Cl}(G) = \emptyset$ because $V \cap \mathrm{Cl}(G) = \emptyset$. Thus F has an α-closed graph.

COROLLARY 19. *Let Y be an H-closed space. Then for a compact valued multifunction $F : X \to Y$, the following are equivalent:*

(1) *F is u.a.α.c.;*
(2) *F has an α-closed graph;*
(3) *F has the property (P);*
(4) *F has the property P^*).*

Proof. This is an immediate consequence of Theorems 21, 22 and 23 and Corollary 18.

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