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FK SPACES THAT INCLUDE STRONGLY *A*-SUMMABLE SEQUENCES

1. Introduction

Let c, ℓ^∞ and ϕ denote the spaces of convergent sequences, bounded sequences and finite sequences, respectively. By c_A we denote the convergence domain of the infinite matrix $A = (a_{nk})$, i.e., $c_A = \{x = (x_k) : \; Ax \in c\}$. It is known that c_A can be made into a locally convex FK space ([11], Chapter 4). For $0 < p < \infty$ let $w(p)$ denote the space of strongly C_1 summable sequences with index p, i.e.,

$$w(p) = \left\{x = (x_k) : \lim_n \frac{1}{n} \sum_{k=1}^{n} |x_k - L|^p = 0 \text{ for some } L\right\}.$$

If $L = 0$, we then write $w_0(p)$ instead of $w(p)$.

In 1946, Kuttner [4] proved that, if $0 < p < 1$ and A is a regular matrix, then there is always a sequence which is strongly Cesàro summable with index p but which is not A-summable. Maddox [5] proved that, if $c_A \supset w(p), (0 < p < 1)$, then $c_A \supset \ell^\infty$ which includes Kuttner's result. Recently Thorpe [8] has shown that Maddox's result remains true, if c_A is replaced by any locally convex FK space (see also [7]).

Recall that the space of strongly A-summable sequences with index p, $0 < p < \infty$, is denoted by $w(A,p)$. Thus

$$w(A,p) = \left\{x = (x_k) : \lim_n \sum_k a_{nk} |x_k - L|^p = 0 \text{ for some } L\right\},$$

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(with the usually summation convention k changing from 1 to n), where A is a non-negative regular matrix [3], [6]. If $L = 0$, we write $w_0(A,p)$ for $w(A,p)$. The space $w_0(A,p)$ is a topological vector space paranormed by

$$g(x) = \sup_n \sum_k a_{nk}|x_k|^p, \quad 0 < p < 1,$$

(see [6], p. 190). Actually it is an FK space.

The aim of this paper is to give necessary and sufficient conditions for a locally convex FK space X to include ℓ^∞, whenever it includes $w_0(A,p)$, $0 < p < 1$.

Throughout the paper A is a non-negative regular matrix, $e = (1,1,\ldots)$, $\delta^k = (0,\ldots,0,1,0,\ldots)$, (1 in the k-th place), X is locally convex FK space with continuous dual X', $X \supset \phi$ and $X^f := \{(f(\delta^k)) : f \in X'\}$.

2. The main results

Before proving the main result we introduce a sequence space related to $w_0(A,p)$. Let

$$N(p) := \left\{x = (x_k) : \lim_n \sum_k (a_{nk})^{\frac{1}{p}}|x_k| = 0\right\}, \quad 0 < p < \infty.$$

The following theorem gives some properties of $N(p)$ which we shall need in the sequel.

THEOREM 1. *Let* $0 < p < 1$; *then*

(i) $N(p)$ *is a locally convex FK space under the paranorm*

$$q(x) = \sup_n \sum_k (a_{nk})^{\frac{1}{p}}|x_k|;$$

(ii) $\{\delta^k\}$ *is a Schauder basis for* $N(p)$;
(iii) $w_0(A,p)$ *is a closed subspace of* $N(p)$;
(iv) $w_0(A,p)^f = N(p)^f$;
(v) $N(p) \supset \ell^\infty$ *if and only if* $e \in N(p)$.

Proof. It is routine to establish that (i) and (ii) hold, and (v) is trivial.
(iii) If $0 < p < 1$, then

$$\sum_k (a_{nk})^{\frac{1}{p}}|x_k| \le \left(\sum_k a_{nk}|x_k|^p\right)^{\frac{1}{p}}$$

which yields that $w_0(A,p) \subset N(p)$. Now let $x = (x_i) \in \overline{w_0(A,p)}$. Then there exists $x^m = (x_k^m) = (x_1^m, x_2^m, \ldots, x_k^m, \ldots) \in w_0(A,p)$ such that $q(x^m - x) \to 0$, as $m \to \infty$. Thus given $\varepsilon > 0$, there is $N_1(\varepsilon)$ so that for all

$m \geq N_1(\varepsilon)$ and for all n, we have

$$(1) \qquad \sum_k (a_{nk})^{\frac{1}{p}} |x_k^m - x_k| < \varepsilon.$$

Using the well-known inequality $(a+b)^{\frac{1}{p}} \leq H(a^{\frac{1}{p}} + b^{\frac{1}{p}})$, where $a \geq 0, b \geq 0$, $\frac{1}{p} > 1$ and H is a positive constant, we get, by (1), that

$$H^{-p} \sum_k a_{nk} |x_k^m - x_k|^p \leq \Big\{ \sum_k (a_{nk})^{\frac{1}{p}} |x_k^m - x_k| \Big\}^p < \varepsilon^p.$$

Hence $g(x^m - x) \longrightarrow 0$, as $m \to \infty$, which yields that $(x^m - x) \in w_0(A,p)$. Since $x = (x - x^m) + x^m \in w_0(A,p)$, (iii) is proved. Combining (iii) and Theorem 7.2.6 of [11], we get (iv).

We now prove the main theorem.

THEOREM 2. *Let* $0 < p < 1$ *and* X *be a locally convex FK space. Then the following assertions are equivalent.*

(i) $e \in N(p)$, *i.e.*, $\lim_n \sum_k (a_{nk})^{\frac{1}{p}} = 0$,
(ii) $X \supset \ell^\infty$ *whenever* $X \supset w_0(A,p)$.

Proof. Suppose that (i) holds and $X \supset w_0(A,p)$. It follows from Theorem 7.2.6 of [11] that $X^f \subset w_0(A,p)^f$ whence $X^f \subset N(p)^f$, by Theorem 1(iv). Also, by Theorem 1(ii), ϕ is dense in $N(p)$. Now Theorem 4 of [9] implies that $X \supset N(p)$. Thus, by Theorem 1(v), we have $X \supset \ell^\infty$, and so, (i) implies (ii).

Suppose now that (ii) holds but $\lim_n \sum_k (a_{nk})^{\frac{1}{p}} \neq 0$. Since $\{\sum_k (a_{nk})^{\frac{1}{p}}\}_{n=1}^\infty$ is bounded, there exists a subsequence $\{n(j)\}$ of positive integers such that

$$\lim_j \sum_k (a_{n(j),k})^{\frac{1}{p}} = L \neq 0.$$

Now define $b_{jk} := (a_{n(j),k})^{\frac{1}{p}}$, $0 < p < 1$. It is obvious that the matrix $B = (b_{jk})$ is conservative, i.e., maps c into c. Since

$$\chi(B) = \lim_j \sum_k b_{jk} = \sum_k \lim_j b_{jk} = L \neq 0,$$

B is a co-regular matrix. Moreover, we have

$$|(Bx)_j| = \Big| \sum_k b_{jk} x_k \Big| \leq \sum_k (a_{n(j),k})^{\frac{1}{p}} |x_k| \leq \Big(\sum_k a_{n(j),k} |x_k|^p \Big)^{\frac{1}{p}},$$

hence B maps $w_0(A,p)$ into c, so $c_B \supset w_0(A,p)$. But c_B is a locally convex FK space (see, e.g. [11], Chapter 4), so (ii) implies that $c_B \supset \ell^\infty$ which is a contradiction to the fact that a coregular matrix cannot sum all bounded sequences [10]; so (ii) implies (i).

Note that if we take $A = C_1$ in Theorem 2, then (i) is automatically satisfied. Thus we immediately get the following main result of Thorpe [8].

COROLLARY 3. *If $0 < p < 1$ and X is locally convex FK space with $X \supset w_0(p)$, then $X \supset \ell^\infty$.*

We remark that Theorem 2 is very useful in order to show that some sequence spaces cannot be given FK topology. To make such an application we pause to collect some further notations.

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r := k_r - k_{r-1} \to \infty$, as $r \to \infty$. The intervals determined by θ will be denoted by $I_r := (k_{r-1}, k_r]$.

The number sequence $\{x_k\}$ is S_θ-convergent to L provided that for every $\varepsilon > 0$

$$\lim_r \frac{1}{h_r} \left| \{k \in I_r : |x_k - L| \geq \varepsilon\} \right| = 0,$$

where the bold vertical bars indicate the number of elements in the enclosed set. By S_θ we denote the set of all S_θ-convergent sequences. It is known that $c \subset S_\theta$ and S_θ contains some bounded divergent sequences as well as it contains some unbounded ones (for details, see [1], [2]). Theorem.3.1 of [3] implies that $S_\theta \supset w_0(C^\theta, p)$, where $p > 0$ and $C^\theta = (c^\theta_{rk})$ is a non-negative regular matrix given by

$$c^\theta_{rk} = \begin{cases} \frac{1}{h_r}, & \text{if } k \in I_r, \\ 0, & \text{if } k \notin I_r. \end{cases}$$

Furthermore

$$\lim_r \sum_k (c^\theta_{rk})^{\frac{1}{p}} = 0, \qquad 0 < p < 1.$$

If S_θ is given locally convex FK topology, then the present Theorem 2 implies $S_\theta \supset \ell^\infty$ which is impossible [1], [2]. So we have already proved the following theorem.

THEOREM 4. *The set of all S_θ-convergent sequences cannot be given locally convex FK topology.*

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