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s-STABILITY AND RANDOM INTEGRAL REPRESENTATIONS OF LIMIT LAWS

Introduction

In the first part, the notion of s-stability is presented using a semigroup of non-linear shrinking transformations. This may have some potential applications in technical sciences. Then, two open problems associated with these transformations are discussed. In the second part, the advantages of random integral representations are illustrated on classes $\mathcal{U}_\beta$, which are limit distributions of some averages of independent Lévy processes. It is shown that the random integral mappings induced by these representations are homeomorphisms between appropriate convolution semigroups. Stable measures are characterized as invariant elements of these random integral mappings. Using the classes $\mathcal{U}_\beta$ a subclassification of the class ID of all infinitely divisible measures is given.

1. s-stable distributions

Contemplating the notion of stability, in the probability theory, one can notice that it is featured by three properties. First, one deals with samples of observations (i.e., independent identically distributed random variables). Second, the observed variables are modified (normalized) by some transformations (mappings on the space where the random variables take on their values). Third, on the modified sample of observations the operation in underlying space is performed and then the limit distributions of such constructed sequences are investigated. To be more specific let us illustrate the above steps by well-known examples.

EXAMPLE 1 (*stable distributions*). The underlying space is $\mathbf{R}$, $T_a x := ax$, $a > 0$, $x \in \mathbf{R}$, are the transformations on $\mathbf{R}$ and "+" is the operation in $\mathbf{R}$. So, $(X_1, X_2, \ldots, X_k)$ is an observed sample, $(T_{a_k}X_1, T_{a_k}X_2, \ldots, T_{a_k}X_k)$ is a modified sample and therefore we get the sequence

(1) $$T_{a_k}X_1 + T_{a_k}X_2 + \ldots + T_{a_k}X_k + b_k,$$

where b_k are some real shifts. The limits of (1) are called *stable distributions* (*measures*), and are studied for about 70 years; cf. Loéve (1955), Zolotarev (1986), Linde (1986).

EXAMPLE 2 (*max-stable distrubutions*). Replacing in Example 1 the operation "+" by "$\vee$", which is the maximum in $\mathbf{R}$, we obtain sequences of the form

(2) $$T_{a_k}X_1 \vee T_{a_k}X_2 \vee \ldots \vee T_{a_k}X_k + b_k.$$

The limits of (2) are called *max-stable* (or *extremal*) *distributions* and were introduced by Gnedenko, cf. Leadbetter-Lindgren-Rootzen (1983).

EXAMPLE 3 (*operator-stable distributions*). The underlying space is $\mathbf{R}^d$ (or any linear vector space E) with "+" as the operation. Modifications of observed samples are done by invertible matrices A_k on $\mathbf{R}^d$ (or bounded linear operators on E). So, this leads to sequences

(3) $$A_kX_1 + A_kX_2 + \ldots + A_kX_k + b_k,$$

with $b_k \in \mathbf{R}^d$, whose limits are called *operator-stable distributions* cf. Sharpe (1969), Krakowiak (1979), Sakovič (1965); Jurek–Mason (1993).

The formulas (1) and (3) are usually written as a modification of partial sums and the formula (2) as a modification of the maximum of a sample. It is so, because the transformation in question are distributive with respect to the operation in the underlying space. Moreover, it is important that $T_a, a > 0$, are invertible operators and form a group of transformations.

The example below of s-operations lacks both these properties.

Let E be a Banach space and U_r be *shrinking operation* (for short: *s-operation*) from E to E given by the formula

(4) $$U_rx := (0 \vee (\|x\| - r))x/\|x\| \quad \text{for } x \neq 0,\ U_r0 = 0,$$

where $x \in E$ and $r > 0$. Of course, U_r are non-linear and form an one-parameter semigroup, because

$$U_s(U_r(x)) = (0 \vee (\|U_rx\| - s))x/\|x\| = (0 \vee (\|x\| - r - s))x/\|x\| = U_{r+s}(x).$$

For $E = \mathbf{R}$, the formula (4) gives the following

$$U_rx = \begin{cases} 0, & \text{for } |x| \leq r, \\ x - r, & \text{for } x > r, \\ x + r, & \text{for } x < -r. \end{cases}$$

So, if C_r denotes the censoring (truncation) at the level r, then $x = C_rx + U_rx$, i.e., s-operation is a complementary to the censoring. Furthermore, if x represents a true signal and a measuring gauge is not very sensitive we read off zero when the signal is "small", i.e., $|x| \leq r$, and we receive only $x - r$, i.e., the excess above the level r, when $x > r$. Thus, it models many real problems in technical sciences. Also, the shrinking operation may be viewed in terms of erosion function when applied to sets; cf. Matheron (1975). But the main objective was the theoretical question how far one can go with limits of sequences of the form (1), (2) or (3) when dealing with the semigroup of non-linear transformations U_r. This problem was raised by K. Urbanik in 1972 and was completely solved by the end of 1976; cf. Jurek (1981).

Let us take Banach space E with addition "+" as the underlying space and s-operations $U_r, r > 0$, as the transformations on E. Thus, a sample $(X_1, X_2, \ldots, X_k)$ of E-valued variables produces a sequence

$$U_{r_k}X_1 + U_{r_k}X_2 + \ldots U_{r_k}X_k + b_k, \quad b_k \in E, \tag{5}$$

which weak limits are called *s-stable distributions* (not to be confused with a stable law with an exponent s !). Additionally we assume that rv's X_j are not uniformly bounded and the triangular array $U_{r_k}X_j, 1 \leq j \leq k$, is uniformly infinitesimal, i.e., for each $\varepsilon > 0$

$$\lim_{k\to\infty} \max_{1\leq j\leq k} P\{\|U_{r_k}X_j\| \geq \varepsilon\} = 0.$$

Hence, all s-stable distributions are infinitely divisible. They are charaterized in [3] as follows.

THEOREM 1. *A measure μ on a Hilbert space H is s-stable if and only if either μ is purely Gaussian or μ is infinitely divisible without Gaussian component, its Lévy spectral measure M is finite and of the following form*

$$M(A) = \int_S \int_0^\infty 1_A(tx)e^{-\alpha t}\, dt\, m(dx),$$

where m is a finite Borel measure on the unit sphere S in H and α is a positive constant.

The proofs from [3] can be extended to Banach space setting using the General Central Limit Theorem, Theorem 5.9, from Araujo-Giné (1980), Chapter 3. Some of them may require additional assumptions on a geometry of the Banach space.

PROBLEM 1. Constructing distributions of X_j from Gaussian covariance operator and the standard normal distributions on $\mathbf{R}^+$ and then using the solutions of the equation

$$x^3 \exp(x^2/2) = Kn, \qquad K \text{ is a constant},$$

as r_n's we obtain Gaussian measure γ as the s-stable distribution, i.e., γ is a weak limit of (5); cf. Lemma 5.2 in [3]. What is a condition on the distribution of X_j (probably moment condition) and what are canonical r_n's such that Gaussian measure is the only limit in (5)? In other words, find an analogue of CLT for s-operations U_r. (Of course, in the classical CLT second moment is the condition, and $a_n = n^{-1/2}$ are the canonical constants in (1).)

PROBLEM 2. Looking at the extremal distributions (Example 2) as limit of maximum of modified sample, one might ask for limits of sequences

$$U_{r_k}X_1 \vee U_{r_k}X_2 \vee \ldots \vee U_{r_k}X_k + b_k, \tag{7}$$

where $b_k \in \mathbf{R}$, $X_1, \ldots, X_k$ are i.i.d $\mathbf{R}$-valued and U_r's are the shrinking operations. So, the modification of the sample is done by U_r instead of T_a, as it is in (2).

2. Random integral representations

The Fourier transform (characteristic function) is the main analytic tool, in probability theory, for description of laws or limit laws. In some cases the Choquet-Krein-Millman Theorem was used to prove such characterizations; cf. Urbanik (1975), Jurek (1981), Section 6. However, in recent years many classes of limit distributions were identified as classes of probability distributions of certain random integrals. Usually one integrates a deterministic function with respect to some *Lévy processes* (processes with independent and stationary increments). Random integral representation immediately gives the Fourier transform and provides the connection between the theory of stochastic processes and the theory of limit distributions.

We are going to illustrate the above ideas on classes $\mathcal{U}_\beta$ of limit distributions.

For a $\beta \in \mathbf{R}$, we say that a measure $\mu \in \mathcal{U}_\beta$ provided it is a weak limit of a sequence

$$(\xi_1(n^{-\beta}) + \xi_2(n^{-\beta}) + \ldots + \xi_n(n^{-\beta}))/n, \tag{8}$$

where ξ_k's are independent Lévy processes with $\xi_k(0) = 0$ a.s. So, if $\nu_j = \mathcal{L}(\xi_j(1))$ denotes the probability distribution of $\xi_j(1)$, then the probability distribution of (8) is equal to

$$T_{n^{-1}}(\nu_1 \star \nu_2 \star \ldots \star \nu_n)^{\star n^{-\beta}}. \tag{9}$$

From [5], Theorem 1.1, we obtain the following

THEOREM 2. (*Convolution equation characterization*) *A measure* $\mu \in \mathcal{U}_\beta$ *if and only if for each* $0 < c < 1$ *there exists a measure* μ_c *such that*

$$\mu = T_c \mu^{\star c^\beta} \star \mu_c. \tag{10}$$

The equation (10) can be expressed in terms of stochastic processes. Namely, if ξ is a Lévy process with $\mu = \mathcal{L}(\xi(1))$ and process η is independent of ξ, then

$$\xi(1) \stackrel{d}{=} c\xi(c^\beta) + \eta(c), \quad \text{for } 0 < c < 1,$$

above $\stackrel{d}{=}$ means equality in distribution.

From Theorem 2 we see that $\mathcal{U}_0 = L_0$ is the Lévy class of *selfdecomposable distributions*, i.e., these are the limits of (1) when instead of dealing with samples one has sequences of independent rv's such that the triangular array $T_{a_k}X_j, 1 \le j \le k$, is uniformly infinitesimal. Of course, in such a setting one loses the immediate applications in mathematical statistics which almost exclusively deals with samples. Yet, DeConinck (1984) proved that measures from class L_0 are limit distributions in the Ising model for ferromagnetism. Moreover, he showed that except Cauchy distribution (stable measure with exponent $p = 1$) none of stable distributions can be obtained that way.

Class $\mathcal{U}_1$ coincides with so called *s-selfdecomposable measures*. These are limits of (5) when once again samples are subsituted by sequences of independent rv's such that triangular array $U_{r_k}X_j$, $1 \le j \le k$, is uniformly infinitesimal; cf. Jurek (1985), Corollary 2.3. Of course, s-stable measures are elements of the class $\mathcal{U}_1$.

Another conclusion from Theorem 2 is that $\mathcal{U}_\beta$'s form an increasing sequence of closed convolution subsemigroups of the semigroup, ID, of all infinitely divisible measures. Furthermore, if $\mu \ne \delta(x) \in \mathcal{U}_\beta$, then $\beta \ge -2$ and $\mathcal{U}_{-2}$ consists of all Gaussian measures; cf. Jurek (1985), Corollary 1.1 and (1989), Proposition 1.1.

The main objective of this section is the following characterization.

THEOREM 3 (*Random integral representations*). (a) *For* $\beta > 0$, $\mu \in \mathcal{U}_\beta$ *if and only if there exists a unique Lévy process* Y *such that*

$$\mu = \mathcal{L}\Big(\int_{(0,1)} t\, dY(t^\beta)\Big).$$

(b) *A measure* $\mu \in \mathcal{U}_0 \equiv L_0$ *if and only if there exists a unique Lévy process* Y *such that* $\mathbf{E}[\log(1 + \|Y(1)\|)] < \infty$ *and*

$$\mu = \mathcal{L}\Big(\int_{(0,\infty)} e^{-t}\, dY(t)\Big) = \mathcal{L}\Big(-\int_{(0,1)} t\, dY(-\ell n t)\Big).$$

(c) *For* $-1 < \beta < 0$ *and measure* μ *on a Hilbert space* H *we have* $\mu \in \mathcal{U}_\beta$ *iff there exists a unique Lévy process in* H *such that* $\mathbf{E}[\|Y(1)\|^{-\beta}] < \infty$ *and*

$$\mu = \gamma_\beta \star \mathcal{L}\Big(\int_{(0,1)} t\, dY(t^\beta)\Big),$$

where γ_β *is a strictly stable measure with exponent* $(-\beta)$.

(d) *For* $-2 < \beta \leq -1$ *and symmetric measures* μ *on a Hilbert space the characterization from* (c) *is true.*

These random integral representations were proved in a sequence of papers: part (a) in Jurek (1988), part (b) in Jurek-Vervaat (1983) and parts (c) and (d) in Jurek (1989). Also, let us note that these representations can be viewed as probability distribution of integral functionals of Lévy processes with changed time scale. Moreover, the random integrals above, are defined by the formal formula of integration by parts, i.e.,

$$\int_{(a,b]} f(t)\, dY(t) := f(t)Y(t)|_{t=a}^{t=b} - \int_{(a,b]} Y(t)\, df(t)),$$

and the integral on the right-hand side exists for a function f with bounded variation because Y has its paths in Skorohod space $D_E[0,\infty)$.

From the random integral representation we get immediately the characterizations in terms of the Fourier transform. Simply one needs to calculate the Fourier transforms of appropriate random integrals, cf. Jurek (1988). Furthermore, we obtain a subclassification of the class ID of all infinitely divisible measures. Namely, we have

COROLLARY 1. $ID = \overline{\bigcup_{\beta>0} \mathcal{U}_\beta}$ (*closure in weak topology*).

Proof. From Theorem 3 (a),

$$\lim_{\beta\to+\infty} \int_{(0,1)} t\, dY(t^\beta) = \lim_{t\to\infty} \int_{(0,1)} t^{1/\beta}\, dY(t) = \int_{(0,1)} dY(t) = Y(1-),$$

and we may take any infinitely divible measure as the probability distribution of $Y(1)$. (In fact, in Corollary 1 it is enough to sum up over any sequence $\beta_n \to +\infty$).

Each random integral representation in Theorem 3 indicates an appropriate *random integral mapping*. Namely, for $\beta > 0$ we have

$$(11) \qquad \mathcal{I}^\beta : ID \to \mathcal{U}_\beta, \quad \text{where } \mathcal{I}^\beta(\nu) := \mathcal{L}\Big(\int_{(0,1)} t\, dY(t^\beta)\Big),$$

where Y is a Lévy process such that $\mathcal{L}(Y(1)) = \nu$. For $\beta = 0$, the random integral mapping $\mathcal{I}^0$ is given by the integral in part (b) of Theorem 3 and its domain is the subset of infinitely divisible measures with finite logarithmic moments. Let us quote from Jurek (1988) the following

THEOREM 4. (a) *For $\beta > 0$, the random integral mapping $\mathcal{I}^\beta$ is a homeomorphism between topolgical convolution semigroups ID and $\mathcal{U}_\beta$.*

(b) *If ν is a stable measure with exponent, p then $\mathcal{I}^\beta(\nu) = \nu^{\star\beta(p+\beta)} \star \delta(x)$ for some $x \in E$. Conversely, if μ has the property that $\mathcal{I}^\beta(\mu) = \mu^{\star c} \star \delta(z)$ for some $c > 0$ and $z \in E$, then μ is stable with the exponent $\beta(1-c)c^{-1}$.*

Note that the above property (b), of $\mathcal{I}^\beta$-invariance, may be used as the characterization of stability using the random integral representation technique and the appropriate random integral mapping.

In case of the class $\mathcal{U}_0 = L_0$ of selfdecomparable measures, cf. Theorem 3(b), we have the processes

$$Z(t) := \int_{(0,t]} e^{-s}\, dY(s) \stackrel{\mathrm{d}}{=} \int_{(0,t]} e^{-(t-s)}\, dY(s), \qquad t \geq 0, \tag{12}$$

which are of the *Ornstein-Uhlenbeck type*, (take Brownian motion as Y). Their infinitesimal generators were described in Sato and Yamazato (1984). Thus the Lévy class L_0 concides with limits distribution of (12), when $t \to \infty$.

The survey of results, in Section 2 of this note, clearly indicates how useful are random integral representations and how they connect the theory of limit distributions and the theory of stochastic processes (stochastic integration). On the other hand, the random integral mappings provide homeomorphism between classes of limit distributions and convolution subsemigroups of the semigroup ID of all infinitely divisible measures. The examples of classes $\mathcal{U}_\beta$ discussed above, as well as some others, lead to the formulation of the following conjucture:

Each class of limit distributions, derived from sequences of independent random variables, is the image of some subset of ID by some mapping defined as a random integral.

Cf. Jurek (1985), p. 607 and Jurek (1988), p. 474.

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