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SOME CHARACTERIZATION OF DUAL SPACE OF ORLICZ-SLOBODECKII SPACE

1. Let Ω be a nonempty open set in the n-dimensional real Euclidean space $\Re^n$. By a function $M : [0,\infty) \to [0,\infty)$ we mean a map which is convex, vanishing and continuous at zero. If M satisfies additionally the conditions

$$\frac{M(u)}{u} \to 0 \quad \text{as } u \to 0 \quad \text{and} \quad \frac{M(u)}{u} \to \infty \quad \text{as } u \to \infty$$

then M is called an N-function.

We say that M satisfies the condition Δ_2 if there exists a constant $K > 0$ such that $M(2u) \le K \cdot M(u)$ for every u. If M is an N-function, then we define the complementary (conjugate) N-function N by

$$N(u) = \sup\{uv - M(v);\ v > 0\} \quad \text{for } u \ge 0.$$

Let k be an arbitrary positive, noninteger number and $k = [k] + \lambda$, where $[k]$ denotes the entire part of k and $0 < \lambda < 1$. Denote by X the vector space of all real-valued, measurable functions defined on Ω with equality almost everywhere on Ω. Define a functional I on X in the following manner:

$$I(f) = \sum_{|\alpha|\le[k]} \Big(\int_\Omega M(|D^\alpha f(x)|)\,dx) + \int_\Omega\int_\Omega M\Big(\frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^\lambda}\Big)\frac{dxdy}{|x-y|^n}\Big)$$

where $D^\alpha f$ is the distributional derivative of f. Note that this functional I is a convex modular on X.

By the Orlicz-Slobodeckii space $B^{k,M}$ we mean the set of all functions $f \in X$, possessing distributional derivatives $D^\alpha f$ up to order $[k]$, for which there exists a constant $a > 0$, depending of f such that $I(af) < \infty$, [3]. The space $B^{k,M}$ equipped with the Luxemburg norm generated by the modular I, is a Banach space, (see [2]).

Define $B = \{(x,y) \in \Omega \times \Omega : x = y\}$. For any set A in the σ-algebra of Lebesgue measurable subsets of $\Omega \times \Omega$, $\Omega \subset \Re^n$, consider the nonnegative measure ν given by

$$\nu(A) = \iint_A |x-y|^{-n} dx dy \quad \text{and} \quad \nu(B) = 0.$$

The measure ν is separable and σ-finite, [3]. Let $L^M(\Omega \times \Omega, \nu)$ denote the Orlicz space of all real and measurable functions F defined on $\Omega \times \Omega$, generated by the modular

$$J(F) = \int_\Omega \int_\Omega M\left(|F(x,y)|\right) d\nu(x,y), \qquad \text{(see [3])}.$$

Put

$$l = \sum_{|\alpha| \leq [k]} 1$$

and

$$\mathcal{L}^M = \prod_{i=1}^{l} \left(L^M(\Omega) \times L^M(\Omega \times \Omega, \nu)\right).$$

For any $u \in \mathcal{L}^M$, $u = (u_i, U_i)_{i=1}^l$, we define

$$\rho(u) = \sum_{i=1}^{l} \Big(\int_\Omega M(|u_i(x)|)\, dx + \int_\Omega \int_\Omega M(|U_i(x,y)|)\, d\nu(x,y) \Big).$$

Then ρ is a convex modular in $\mathcal{L}^M$. The space $\mathcal{L}^M$ with the Luxemburg norm $\|\cdot;\mathcal{L}^M\|$ generated by ρ is complete. We also define on $\mathcal{L}^M$ the Orlicz norm by

$$^o\|u;\mathcal{L}^M\| =$$

$$\sup\left\{ \Big| \sum_{i=1}^{l} \Big(\int_\Omega u_i(x) v_i(x)\, dx + \int_\Omega \int_\Omega U_i(x,y) V_i(x,y)\, d\nu(x,y) \Big) \Big| : \|v;\mathcal{L}^N\| \leq 1 \right\}$$

where $v \in \mathcal{L}^M$ and the function N is complementary to M.

Let N be complementary to M. Then the following inequality

$$(1) \quad \Big| \sum_{i=1}^{l} \Big(\int_\Omega u_i(x) v_i(x)\, dx + \int_\Omega \int_\Omega U_i(x,y) V_i(x,y)\, d\nu(x,y) \Big) \Big| \leq$$
$$\leq {}^o\|v;\mathcal{L}^N\| \cdot \|u;\mathcal{L}^M\|$$

holds for any $u \in \mathcal{L}^M$ and $v \in \mathcal{L}^N$.

If an N-function M satisfies the condition Δ_2 then every continuous linear functional over $\mathcal{L}^M$ is of the form

$$f^*(u) = \sum_{i=1}^{l} \Big(\int_\Omega v_i(x)u_i(x)\,dx + \int_\Omega\int_\Omega V_i(x,y)U_i(x,y)\,\frac{dxdy}{|x-y|^n}\Big)$$

for $u = (u_i, U_i)_{i=1}^l \in \mathcal{L}^M$, where $v = (v_i, V_i)_{i=1}^l \in \mathcal{L}^N$ and N is complementary to M, [3]. Moreover $\|f^*\| = {}^o\|v; \mathcal{L}^N\|$.

Suppose that the l multiindices α satisfying $|\alpha| \le [k]$ are linearly ordered so that with each $u \in B^{k,M}(\Omega)$ we may associate a well-defined vector Pu in $\mathcal{L}^M$ given by

$$Pu = \Big(D^\alpha u, \frac{D^\alpha u(x) - D^\alpha u(y)}{|x-y|^\lambda} \Big)_{|\alpha|\le[k]}.$$

Then $\|u; B^{k,M}\| = \|Pu; \mathcal{L}^M\|$ for any $u \in B^{k,M}(\Omega)$. So P is an isometric isomorphism of $B^{k,M}(\Omega)$ onto a subspace $P\left(B^{k,M}(\Omega)\right)$ of $\mathcal{L}^M$. Hence for $u \in B^{k,M}(\Omega)$ we have

$$(2)\quad \Big| \sum_{|\alpha|\le[k]} \Big(\int_\Omega D^\alpha u(x) v_\alpha(x)\,dx + \int_\Omega\int_\Omega \frac{D^\alpha u(x) - D^\alpha u(y)}{|x-y|^\lambda} V_\alpha(x,y)\,d\nu(x,y) \Big) \Big| \le {}^o\|v; \mathcal{L}^N\| \cdot \|u; B^{k,M}\|.$$

Thus each $v \in \mathcal{L}^N$ defines a continuous linear functional on $B^{k,M}$ by

$$(3)\quad L(u) = \sum_{|\alpha|\le[k]} \Big(\int_\Omega D^\alpha u(x) v_\alpha(x)\,dx + \int_\Omega\int_\Omega \frac{D^\alpha u(x) - D^\alpha u(y)}{|x-y|^\lambda} V_\alpha(x,y)\,\frac{dxdy}{|x-y|^n} \Big),$$

where $v = (v_\alpha, V_\alpha)_{|\alpha|\le[k]}$.

Let M be an N-function satisfying the condition Δ_2 and L be arbitrary continuous linear functional over $B^{k,M}(\Omega)$. Then there exists $v \in \mathcal{L}^N$, $v = (v_\alpha, V_\alpha)_{|\alpha|\le[k]}$, such that L is of the form (3) and $\|L\| = \inf {}^o\|v; \mathcal{L}^N\|$, where the infimum is taken over, the set of all $v \in \mathcal{L}^N$ such that L can be expressed by formula (3).

Every element L of the space $(B^{k,M}(\Omega))^*$ is an extension to $B^{k,M}(\Omega)$ of a distribution $T \in \mathcal{D}'(\Omega)$. To see this suppose that L is given by (3) for some $v \in \mathcal{L}^N$, $v = (v_\alpha, V_\alpha)_{|\alpha|\le[k]}$ and consider

$$T_{(v_\alpha, V_\alpha)}(\varphi) = \int_\Omega v_\alpha(x)\varphi(x)\,dx + \int_\Omega\int_\Omega V_\alpha(x,y)\frac{\varphi(x) - \varphi(y)}{|x-y|^\lambda}\,d\nu(x,y)$$

for $\varphi \in \mathcal{D}(\Omega)$ and every α, $|\alpha| \leq [k]$, and

$$T = \sum_{|\alpha|\leq[k]} (-1)^{|\alpha|} D^\alpha T_{(v_\alpha, V_\alpha)}. \tag{4}$$

For every $\varphi \in \mathcal{D}(\Omega) \subset B^{k,M}(\Omega)$ we have

$$T(\varphi) = \sum_{|\alpha|\leq[k]} T_{(v_\alpha, V_\alpha)}(D^\alpha \varphi) = L(\varphi),$$

so that L is clearly an exitension of T. But if T is any element of $\mathcal{D}'(\Omega)$ having the form (4) for some $v \in \mathcal{L}^N$, then the continuous extension of T to $B^{k,M}(\Omega)$ may be not unique. However, T possesses a unique extension to $B_0^{k,M}(\Omega)$, where $B_0^{k,M}(\Omega)$ denotes the closure in $B^{k,M}(\Omega)$ of the set $C_0^\infty(\Omega)$ with respect to the norm generated by the modular I.

Now suppose T is any element of $\mathcal{D}'(\Omega)$ having the form (4) for some $v \in \mathcal{L}^N$. Let $u \in B_0^{k,M}(\Omega)$. Then there exists a sequence (φ_n) in $C_0^\infty(\Omega)$ such that $\|u - \varphi_n\, ;\, B^{k,M}\| \to 0$, as $n \to \infty$. We have

$$|T\varphi_n - T\varphi_m| = \Big| \sum_{|\alpha|\leq[k]} T_{(v_\alpha, V_\alpha)}(D^\alpha \varphi_n - D^\alpha \varphi_m)\Big| \leq$$
$$\leq {}^{o}\|v; \mathcal{L}^N\| \cdot \|\varphi_n - \varphi_m; B^{k,M}\| \to 0$$

as $m, n \to \infty$. Therefore $(T\varphi_n)$ is a Cauchy sequence and so it is convergent. Observe that it's limit equals $L(u)$,

$$L(u) = \lim_{n\to\infty} T(\varphi_n),$$

since it is clear that if also $(\psi_n) \subset C_0^\infty(\Omega)$ and $\|u - \psi_n; B^{k,M}\| \to 0$, then $T(\varphi_n) - T(\psi_n) \to 0$ as $n \to \infty$, where

$$u = \lim_{n\to\infty} \varphi_n.$$

So L is a linear functional on $B_0^{k,M}(\Omega)$. Furthermore

$$|L(u)| \leq \lim_{n\to\infty} {}^{o}\|v; \mathcal{L}^N\| \cdot \|\varphi_n; B^{k,M}\| = q\|v; \mathcal{L}^N\| \cdot \|u; B^{k,M}\|.$$

Hence L is bounded and $\|L\| \leq q\|v; \mathcal{L}^N\|$.

Denote by $B^{-k,N}(\Omega)$ the space consisting of those distributions $T \in \mathcal{D}'(\Omega)$ satisfying (4) for some $v \in \mathcal{L}^N$, $v = (v_\alpha, V_\alpha)_{|\alpha|\leq[k]}$, normed by

$$\|T\| = \inf\{{}^{o}\|v; \mathcal{L}^N\| : v \text{ satisfies } (4)\}.$$

Then the following theorem holds:

THEOREM 1. *The dual space $(B_0^{k,M}(\Omega))^*$ is isometrically isomorphic to the space $B^{-k,N}(\Omega)$.*

The space $B^{-k,N}(\Omega)$ is complete. If N statisfies Δ_2 then $B^{-k,N}(\Omega)$ is separable and reflexive provided that M and its complementary function N satisfy Δ_2.

Let M and N be complementary N-functions and both the functions satisfy the condition Δ_2. Each element $v = (v_0, V_0) \in L^N(\Omega) \times L^N(\Omega \times \Omega, \nu)$ determines a functional $L_v \in (B_{k,M}(\Omega))^*$ of the from

$$(5) \qquad L_v(u) = \int_\Omega u(x)v_0(x)\,dx + \int_\Omega\int_\Omega \frac{u(x) - u(y)}{|x - y|^\lambda} V_0(x,y)\,d\nu(x,y).$$

Let us define the norm of $v \in L^N(\Omega) \times L^N(\Omega \times \Omega, \nu)$ as the norm of L_v given by (5), that is

$$\|v\|_{-k,N} = \sup\{|L_v(u)| : u \in B^{k,M}(\Omega) \quad \text{and} \quad \|u; B^{k,M}\| \le 1\}.$$

Clearly, for any $u \in B^{k,M}(\Omega)$ and $v \in L^N(\Omega) \times L^N(\Omega \times \Omega, \nu)$ we have

$$\|v\|_{-k,N} \le \mathfrak{q}\|v; L^N(\Omega) \times L^N(\Omega \times \Omega, \nu)\|.$$

Let

$$W = \left\{L_v : v \in L^N(\Omega) \times L^N(\Omega \times \Omega, \nu)\right\}.$$

Thus W is a vector subspace of $\left(B^{k,M}(\Omega)\right)^*$. We shall show that W is dense in $\left(B^{k,M}(\Omega)\right)^*$.

LEMMA ([1] p.51). *Let X be a subspace of vector space Y^*. The density of X in Y^* is equivalent to that if $F \in Y^{**}$ and $F(x) = 0$ for every $x \in X$, then $F = 0$ in Y^{**}.*

Let $U \in (B^{k,M}(\Omega))^{**}$. Since $B^{k,M}(\Omega)$ is reflexive, [3], there exists $u \in B^{k,M}(\Omega)$ such that

$$\int_\Omega u(x)v_0(x)\,dx + \int_\Omega\int_\Omega \frac{u(x) - u(y)}{|x - y|^\lambda} V_0(x,y)\,d\nu(x,y) = L_v(u) = U(L_v) = 0$$

for every $v = (v_0, V_0) \in L^N(\Omega) \times L^N(\Omega \times \Omega, \nu)$. Hence $u(x) = 0$ almost everywhere in Ω. Then $u = 0$ in $B^{k,M}(\Omega)$ and $U = 0$ in $\left(B^{k,M}(\Omega)\right)^{**}$.

Let $H^{-k,N}(\Omega)$ denotes the completion of $L^N(\Omega) \times L^N(\Omega \times \Omega, \nu)$ with respect to the norm $\|\cdot\|_{-k,N}$. Then we have

THEOREM 2. *The dual space $(B^{k,M}(\Omega))^*$ is isomorphic to the space $H^{-k,N}(\Omega)$.*

Proof. Let H denote the closure of $L^N(\Omega) \times L^N(\Omega \times \Omega, \nu)$ with respect to the norm $\|\cdot\|_{-k,N}$.Then the spaces $W_1 = \{L_v : v \in H\}$ and $H^{-k,M}(\Omega)$ are isomorphic. From the density of W in $(B^{k,M}(\Omega))^*$ it follows that $W_1 = (B^{k,M}(\Omega))^*$. Hence $H^{-k,M}(\Omega)$ and $(B^{k,M}(\Omega))^*$ are isomorphic.

2. Remarks

Let Ω be an open, bounded and convex subset in $\Re^n$. Let us consider a function $M : \Omega \times [0,\infty) \to [0,\infty)$ which is convex, vanishing and continuous at zero, not identically equal zero for a.e. $t \in \Omega$ and measurable for every $u \geq 0$. If the function M satisfies the conditions

$$\frac{M(t,u)}{u} \to 0 \quad \text{as } u \to 0 \quad \text{and} \quad \frac{M(t,u)}{u} \to \infty \quad \text{as } u \to \infty$$

for a.e. $t \in \Omega$ then it is called an N-function with parametr t.

Furthemore, the follownig conditions for the function M will be used:

$$\int_\Omega M(t,u)\,dt < \infty \quad \text{for every } u \geq 0, \quad \text{and} \tag{6}$$

Δ_2: there exists a constant $K > 0$ such that $M(t,2u) \leq K \cdot M(t,u)$ for a.e. $t \in \Omega$ and every $u \geq 0$.

Then, for any function M, let us define on X a modular I generating the space $B^{k,M}(\Omega)$ by

$$I(f) = \sum_{|\alpha|\leq[k]} \Bigg(\int_\Omega M(x, |D^\alpha f(x)|)\,dx + \\ + \int_\Omega\int_\Omega M\Big(\frac{x+y}{2}, \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^\lambda}\Big) \frac{dx\,dy}{|x-y|^n}\Bigg).$$

In the sequel by the space $L^M(\Omega \times \Omega, \nu)$ we mean the Orlicz space of all real and measurable functions F on $\Omega \times \Omega$, generated by the modular

$$J(F) = \int_\Omega\int_\Omega M\Big(\frac{x+y}{2}, |F(x,y)|\Big) d\nu(x,y), \qquad \text{(see [3])}.$$

If a function M satisfies the condition (6), then the inclusion $C_0^\infty(\Omega) \subset B^{k,M}(\Omega)$ holds for every $k > 0$ and $k \neq 1, 2, \ldots$ The inclusion follows from the fact that for every $u \in C_0^\infty(\Omega)$ and α, $|\alpha| \leq [k]$,

$$I_\alpha = \int_\Omega \Bigg(\int_Z M\Big(\frac{x+y}{2}, \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\lambda}\Big) \frac{dx}{|x-y|^n}\Bigg) dy$$

is finite, where Z denotes the support of u. Applying the mean-value theorem and Jensen's inequality, we obtain

$$I_\alpha \leq c_1 \int_{\Re^n} \Bigg(\int_\Omega M\Big(t, \frac{|D^\alpha u(2t-y) - D^\alpha u(y)|}{|2t-2y|^\lambda}\Big) \frac{dt}{|2t-2y|^n}\Bigg) dy \leq$$

$$\leq c_1 \int\limits_{|h|\leq 1} \Big(\int\limits_{\Omega} M\Big(t, \frac{|D^\alpha u(t+\frac{h}{2}) - D^\alpha u((t-\frac{h}{2})|}{|h|^\lambda}\Big) \frac{dt}{|h|^n}\Big) dh +$$

$$+ c_1 \int\limits_{|h|>1} \Big(\int\limits_{\Omega} M\Big(t, \frac{|D^\alpha u(t+\frac{h}{2}) - D^\alpha u((t-\frac{h}{2})|}{|h|^\lambda}\Big) \frac{dt}{|h|^n}\Big) dh \leq$$

$$\leq c_1 \Big(\int\limits_{|h|\leq 1} \frac{dh}{|h|^{n-1+\lambda}} + \int\limits_{|h|>1} \frac{dh}{|h|^{n+\lambda}} \Big) \int\limits_{\Omega} M(t,c)\, dt < \infty.$$

Obviously $\int_\Omega M(x, |D^\alpha u(x)|)\, dx = \int_Z M(x, |D^\alpha u(x)|) dx < \infty$ for every $|\alpha| \leq [k]$ and $u \in C_0^\infty(\Omega)$. This shows that $C_0^\infty(\Omega) \subset B^{k,M}(\Omega)$. The same inclusion holds also if the set Ω has finite measure.

The facts relating to the space $\mathcal{L}^M$ and the dual space $(B^{k,M}(\Omega))^*$ used in Section 1 hold also in the case when M depends on the parameter. By an argument similar to that in Section 1 the dual space $(B_0^{k,M}(\Omega))^*$ can be characterized as the space consisting of those distributions $T \in \mathcal{D}'(\Omega)$ satisfying (4) for some $v \in \mathcal{L}^N$ and the dual space $(B^{k,M}(\Omega))^*$ as the completion of $L^N(\Omega) \times L^N(\Omega \times \Omega, \nu)$ with respect to the norm $\|\cdot\|_{-k,M}$.

References

[1] R. A. Adams, *Sobolev spaces*. Academic Press, 1975.

[2] M. Liskowski, *On generalized Besov spaces* $B^{k,M}(\Omega)$, Funct. Approx., 15 (1986), 175–184.

[3] M. Liskowski, *Separability, duality and reflexivity of Orlicz-Besov spaces*, Comm. Math., 30 (1991), 377–382.

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