DEMONSTRATIO MATHEMATICA
Vol. XXIX No 2 1996

Jan Ambrosiewicz

THE INEQUALITY cn(PSL(n, K) ≤ 4 HOLDS FOR $n = 2, 3$ AND INFINITE FIELD K

Let G be a group. The smallest integer m satisfying $C^m = G$ for each noncentral conjugacy class of G is called the covering number of G and is denoted by cn(G). If there is no such m, denote cn(G) $= \infty$. In [4] it was proved that cn(PSL($2, K$) ≥ 3 for $K = \mathbb{Q}, \mathbb{R}$ (rational and real numbers). In [3] it was proved that if $n = 2, 3$ and $K = \mathbb{R}, \mathbb{C}$ then cn(PSL(n, K) ≤ 4. In this paper we will prove that if $n = 2$ or 3 and K is any infinite field, then cn(PSL(n, K)) ≤ 4 (Corollary 5.1 and 6.1).

The proof of the inequality cn(PSL(n, K)) ≤ 4 ($n = 2, 3$, $K = \mathbb{R}, \mathbb{C}$) is based on the following:

THEOREM 1. *If $A \in GL(n, K)$ ($n = 2, 3$; $K = \mathbb{R}, \mathbb{C}$), $A \notin Z$, then there exists $X \in SL(n, K)$ such that the eigenvalues of $AX^{-1}AX$ are different* (see Lemmas 2, 3 of [3]).

Theorem 1 raises the following two questions:

(i) Is Theorem 1 true for arbitrary n and arbitrary K?
(ii) Is Theorem 1 true for $n = 2, 3; K = \mathbb{Q}$?

In this paper it is proved that the answer to the first question is negative (Theorem 2) and that to the second one is positive (Theorems 3, 4).

The following notations will be used, C_V denotes the conjugacy class of the matrix V, Z denotes the center of a group G and $A^X = X^{-1}AX$. The remaining notations are standard.

THEOREM 2. *If $n \geq 4$, then there exists $A \in GL(n, K)$ such that there is no matrix $X \in SL(n, K)$ such that the eigenvalues of AA^X are different.*

Proof. Let

(1) $$A = \mathrm{diag}(a, \dots, a, b) \in GL(n, K), \qquad a \neq b$$

and $X = [t_{ij}]$ be any matrix of $SL(n, K)$. Then $X^{-1} = [T_{ji}]$ and

$$w(t) = \det(AA^X - tE) = \det C, \tag{2}$$

where

$$c_{ii} = \begin{cases} a^2 - t + (ab - a^2)T_{ni}t_{ni}, & i = 1, \dots, n-1 \\ ab - t + (b^2 - ab)T_{ni}t_{ni}, & i = n \end{cases}$$

and for $i \neq j$

$$c_{ij} = (ab - a^2)T_{ni}t_{nj}, \ i = 1, \dots, n-1; \ j = 1, \dots, n$$

$$c_{nj} = (b^2 - ab)T_{nn}t_{nj}, \ j = 1, \dots, n-1.$$

One can prove the following

LEMMA 1. *If*

$$d(x) = \begin{vmatrix} a_{11}(x), \dots, a_{1n}(x) \\ \dots \\ a_{n1}(x), \dots, a_{nn}(x) \end{vmatrix},$$

then

$$d'(x) = \begin{vmatrix} a'_{11}(x), \dots, a'_{1n}(x) \\ a_{21}(x), \dots, a_{2n}(x) \\ \dots\dots\dots \\ a_{n1}(x), \dots, a_{nn}(x) \end{vmatrix} + \dots + \begin{vmatrix} a_{11}(x), \dots, a_{1n}(x) \\ \dots\dots\dots \\ a_{n-11}(x), \dots, a_{n-1n}(x) \\ a'_{n1}(x), \dots, a'_{nn}(x) \end{vmatrix}.$$

Using (2) and Lemma 1 it can easily be shown that $w(a^2) = w'(a^2) = \dots = w^{(n-3)}(a^2) = 0$, $w^{(n-2)}(a^2) = \pm a^2(b-a)^2(1 - T_{nn}t_{nn})$. Thus the element a^2 is a root of $w(t)$ of at least multiplicity $n-2$.

It can be observed that Lemma 4 in [3] is not true for the matrix (1), by Theorem 2. Then the proof of Theorem 3 is incorect in [3].

THEOREM 3. *If K is a field, $|K| \neq 2, 3, 5$ and A is a non-central matrix of $GL(2, K)$, then there exist matrices $S, T \in SL(2, K)$ such that eigenvalues of $A^S A^T$ are different and belong to K.*

Proof. Let $N = A^P$ denote the rational canonical form of A in the group $GL(n, K)$. Note that if all eigenvalues of NN^X are distinct, where $X \in SL(n, K)$, then all eigenvalues of $A^{PU}A^{PXU}$ will also be distinct and we can choose the matrix U (for example $U = P^{-1}$) such that $det(PU) = det(PXU) = 1$. Therefore to prove Theorem 3 it is sufficient to show that there exists a matrix $X \in SL(2, K)$ such that the eigenvalues of NN^X are distinct, where $N = \begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix}$.

It is easy to see that there exists $m \in K^*$ such that for

$$X = \begin{bmatrix} 0 & -m^{-1} \\ m & 0 \end{bmatrix}$$

the eigenvalues $-m^2$, $-a^2m^{-2}$ of NN^X are distinct and belong to K.

LEMMA 2. *If K is an infinite field and $A \in GL(n, K)$ has rational canonical form*

$$N = \begin{bmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ & 0 & & 1 \\ a_1 & & \dots & a_n \end{bmatrix} \quad \text{or} \quad N_1 = \begin{bmatrix} N & 0 \\ 0 & c \end{bmatrix}$$

then there exists a matrix $X \in SL(n,K)$ such that eigenvalues of AA^X are distinct.

Proof. As in the proof of Theorem 3, it is sufficient to show that eigenvalues of NN^X and $N_1N_1^Y$ are distinct for some $X, Y \in SL(n,K)$.

If we take

$$X = \begin{bmatrix} 0 & & sx_1 \\ & \cdot^{\cdot^{\cdot}} & \\ x_n & & 0 \end{bmatrix}, x_1 \dots x_n = 1,$$

where $s = (-1)^{\frac{n(n-1)}{2}}$, then the elements

(3) $$sx_1x_2^{-1}, x_2x_3^{-1}, \dots, x_{n-1}x_n^{-1}, a_1^2x_nx_1^{-1}s$$

are eigenvalues of NN^X.

If we take

$$Y = \begin{bmatrix} X & 0 \\ 0 & y \end{bmatrix}, \quad x_1 \dots x_n y = 1 \text{ with } s = (-1)^{\frac{n(n-1)}{2}},$$

then the elements

(4) $$sx_1x_2^{-1},\ x_2x_3^{-1},\ \dots,\ x_{n-1}x_n^{-1},\ a_1^2x_nx_1^{-1}s,\ c^2$$

are eigenvalues of $N_1N_1^Y$.

Since K is an infinite field then it is obvious that there exist $x_1, \dots, x_n \in K$ such that the numbers in sequences (3) and (4) are distinct.

Non-central matrices of $GL(3,K)$ are similar to matrices of the form N or N_1. Hence from Lemma 2 it follows that:

THEOREM 4. *If K is an infinite field and A is a non-central matrix of $GL(3,K)$, then there exist matrices $S, T \in SL(3,K)$ such that eigenvalues of A^SA^T are different and belong to K.*

THEOREM 5. *If C is any non-central conjugacy class of $SL(2,K)$ and there exists $m \in K^*$ such that $m^4 \neq 1$, then $SL(2,K) = C^4 \cup Z$.*

Proof. Let $A \in SL(2,K)$ and $A \notin Z$. From Theorem 3 it follows that there exist matrices $S, T \in SL(2,K)$ such that the eigenvalues v, v^{-1} of $A^SA^T = B$ are all distinct. The matrix B is similar to the matrix $V = \mathrm{diag}(v, v^{-1})$ in the group $SL(2,K)$. Hence $C_V = C_B \subseteq C_A^2$ and $C_V^2 \subseteq C_A^4$. Now from following lemma proved in [1] and [2]:

LEMMA 4. *If $V = \mathrm{diag}(v_1, \dots, v_n)$, $W = \mathrm{diag}(w_1, \dots, w_n)$, $v_1 \neq v_j$, $w_1 \neq w_j$, for $i \neq j$ and $V, W \in SL(n,K)$, then $SL(n,K) = C_VC_W \cup Z$ it* *results that*

$$SL(2,K) - Z \subseteq C_V^2 \subseteq C_A^4 \quad \text{for any } A \notin Z.$$

COROLLARY 5.1. *If C is any non-identity conjugacy class of* PSL$(2,K)$ *and if there exists $m \subset K^*$ such that $m^4 \neq 1$, then*

$$\mathrm{PSL}(2,K) = C^4.$$

Proof. The matrices $V = \mathrm{diag}(v, v^{-1})$ and V^{-1} are similar in the group PSL$(2,K)$, so $Z \subseteq C_V^2 \subseteq C_V^4$. Thus Corollary 5.1 results from Theorem 5.

THEOREM 6. *If K is an infinite field and C is any non-central conjugacy class of $SL(3,K)$, then $SL(3,K) = C^4 \cup Z$.*

Proof. Let $A \in SL(3,K)$, $A \notin Z$. By Theorem 4 it follows that there exist matrices $S,T \in SL(3,K)$ such that the eigenvalues v_1, v_2, v_3 of $A^S A^T = B$ are all distinct. The matrix B is similar to the matrix $V = diag(v_1, v_2, v_3)$ in the group $SL(3,K)$. The remaining part of the proof is the same as in the proof of Theorem 5.

COROLLARY 6.1. *If K is an infinite field and C is any non-central conjugacy class of* PSL$(3,K)$, *then* PSL$(3,K) = C^4$.

Proof. Note that if $N, N_1 \in SL(3,K)$, then it is possible to choose X and Y such that the eigenvalues (3) of NN^X and (4) of $N_1 N_1^Y$ are different and are of the form $1, v, v^{-1}$. By Lemma 2 this means that for any non-central conjugacy class C, C^2 contains the matrix $V = \mathrm{diag}(1, v, v^{-1})$ and V^{-1}.

The matrices V and V^{-1} are similar in $SL(3,K)$. Therefore C^4 contains E, so if $C \in$ PSL$(3,K)$, then PSL$(3,K) = C^4$, by Theorem 6.

References

[1] J. Ambrosiewicz, *On the square of sets of the group $SL(3,F)$*, PSL$(3,F)$, Demonstratio Math. 18 (1985), 963–968.

[2] J. Ambrosiewicz, *Products of sets in linear groups*, Demonstratio Math. 22 (1989), 677–698.

[3] J. Ambrosiewicz, *The equation for $SL(n,K) = C^4 \cup Z$, $n = 2,3,4$; $K = \mathbb{R}, \mathbb{C}$*, Publ. Math. Debrecen, 45/1-2 (1994), 153–158.

[4] J. L. Brenner, *Covering theorems for nonabelian simple groups IV*, Inanabha, section A, 3(1973), 77–84.

INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY OF BIAŁYSTOK
ul. Wiejska 45A
15-351 BIAŁYSTOK, POLAND

Received July 28, 1994.