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CERTAIN CLASSES OF p-VALENT ANALYTIC FUNCTIONS

1. Introduction

Let A_p, with fixed integer $p > 0$, be the class of functions

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \tag{1.1}$$

analytic and p-valent in the unit disc $E = \{z : |z| < 1\}$.

Let Ω denote the class of bounded analytic functions $w(z)$ in E satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ for all z in E.

If the functions (1.1) and $g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k}$ belong to A_p, the convolution or Hadamard product of $f(z)$ and $g(z)$ is defined by the power series

$$(f * g)(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k}.$$

With the convolution above, we define

$$D^{n+p-1} f(z) = \frac{z^p}{(1-z)^{n+p}} * f(z), \qquad f(z) \in A_p, \tag{1.2}$$

where n is any integer greater than $-p$. The symbol D^n (i.e., (1.2) for $p = 1$) was introduced by Ruscheweyh [13] and D^{n+p-1} by Goel and Sohi [5]. Therefore, we call $D^{n+p-1} f(z)$ the Ruscheweyh derivative of $(n+p-1)$-th order. It follows from (1.2) that

$$z(D^{n+p-1} f(z))' = (n+p) D^{n+p} f(z) - n D^{n+p-1} f(z). \tag{1.3}$$

A function $f(z) \in A_p$ is said to be in the class $V^{\lambda}_{n,p}(A, B, \alpha)$, if it satisfies

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the condition

$$\frac{(D^{n+p-1}f(z))'}{z^{p-1}} \prec \frac{p+\{pB+(A-B)(p-\alpha)e^{-i\lambda}\cos\lambda\}z}{1+Bz}, \quad z \in E,$$

where $-1 \leq B < A \leq 1$, $0 \leq \alpha \leq p$, $|\lambda| < \pi/2$ and the symbol $\prec$ denotes subordination. From the definition of subordination, it follows that $f(z) \in V_{n,p}^{\lambda}(A,B,\alpha)$ has a representation of the form

$$(1.4) \quad \frac{(D^{n+p-1}f(z))'}{z^{p-1}} = \frac{p+\{pB+(A-B)(p-\alpha)e^{-i\lambda}\cos\lambda\}w(z)}{1+Bw(z)}, \quad w(z) \in \Omega.$$

From this, we note that $f(z) \in V_{n,p}^{\lambda}(A,B,\alpha)$ if and only if

$$\left|\frac{(D^{n+p-1}f(z))' - pz^{p-1}}{(A-B)(p-\alpha)e^{-i\lambda}z^{p-1}\cos\lambda - B((D^{n+p-1}f(z))' - pz^{p-1})}\right| < 1, \quad z \in E.$$

Clearly, for $n = 0$, $p = 1$, $A = \beta$ and $B = -\beta$, $0 < \beta \leq 1$, we get the class introduced and studied by Ahuja [1] which in turn reduces to the class studied in [9] for $\lambda = 0$. Further, $n = \alpha = \lambda = 0$ and replacement of A by 1 and B by $\frac{1-\delta}{\delta}$, $\delta > \frac{1}{2}$, gives the class studied by Sohi [14].

We further observe that, for special choice of the parameters A, B and λ, our class gives rise to the following new subclasses of p-valent analytic functions:

$$V_{n,p}^{\lambda}(1,-1,\alpha) \equiv V_{n,p}^{\lambda}(\alpha) =$$
$$= \left\{f(z) \in A_p : \ \operatorname{Re} e^{i\lambda}\frac{(D^{n+p-1}f(z))'}{z^{p-1}} > \alpha\cos\lambda,\ 0 \leq \alpha < p,\ |\lambda| < \frac{\pi}{2},\ z \in E\right\},$$

$$V_{n,p}^{\lambda}(1,\frac{1-\delta}{\delta},0) \equiv V_{n,p,\delta}^{\lambda} = \left\{f(z) \in A_p : \right.$$
$$\left.\left|\left\{(1+i\operatorname{tg}\lambda)\frac{(D^{n+p-1}f(z))'}{z^{p-1}} - i\operatorname{tg}\lambda\right\} - \delta\right| < \delta,\ \delta > \frac{1}{2},\ |\lambda| < \frac{\pi}{2},\ z \in E\right\},$$

$$V_{n,p}^{\lambda}(\sigma,0,0) \equiv (V_{n,p}^{\lambda})^{\sigma} = \left\{f(z) \in A_p : \right.$$
$$\left.\left|\left\{(1+i\operatorname{tg}\lambda)\frac{(D^{n+p-1}f(z))'}{z^{p-1}} - i\operatorname{tg}\lambda\right\} - 1\right| < \sigma,\ 0 < \sigma \leq 1,\ |\lambda| < \frac{\pi}{2},\ z \in E\right\},$$

$$V_{n,p}^{\lambda}(\gamma,-\gamma,0) \equiv (V_{n,p}^{\lambda})_{\gamma} = \Bigg\{ f(z) \in A_p :$$

$$\left| \frac{(1+i\,\mathrm{tg}\,\lambda)\frac{(D^{n+p-1}f(z))'}{z^{p-1}} - i\,\mathrm{tg}\,\lambda - 1}{(1+i tg\lambda)\frac{(D^{n+p-1}f(z))'}{z^{p-1}} - i\,\mathrm{tg}\,\lambda + 1} \right| < \gamma,\ 0<\gamma\le 1,\ |\lambda| < \frac{\pi}{2},\ z\in E \Bigg\}.$$

As noticed above, the class $V_{n,p}^{\lambda}(A,B,\alpha)$ includes various subclasses of p-valent analytic functions; a study of its properties will lead to a unified study of these classes. In the present paper, we first obtain the basic inclusion relation $V_{n+1,p}^{\lambda}(A,B,\alpha) \subset V_{n,p}^{\lambda}(A,B,\alpha)$. Then we obtain class preserving integral operators and a sufficient condition in terms of coefficients for a function to be in $V_{n,p}^{\lambda}(A,B,\alpha)$. We also obtain sharp coefficient estimates and closure theorems for these classes. Papers [1], [3], [4], [9], [14] follow as special cases of our results.

Unless otherwise mentioned, in the sequel we assume that $-1 \le B < A \le 1$, $0 \le \alpha < p$ and $|\lambda| < \pi/2$.

2. Preliminary lemmas

LEMMA 1. *A function $f(z) \in A_p$ belongs to the class $V_{n,p}^{\lambda}(A,B,\alpha)$, if and only if*

$$\left| \frac{(D^{n+p-1}f(z))'}{z^{p-1}} - m \right| < M, \qquad z \in E, \tag{2.1}$$

where

$$\begin{cases} m = p - \dfrac{B(A-B)(p-\alpha)e^{-i\lambda}\cos\lambda}{1-B^2}, \\ M = \dfrac{(A-B)(p-\alpha)\cos\lambda}{1-B^2}. \end{cases} \tag{2.2}$$

Proof. First, suppose that $f(z) \in V_{n,p}^{\lambda}(A,B,\alpha)$. Then, by using (2.2) in (1.4), we have

$$\frac{(D^{n+p-1}f(z))'}{z^{p-1}} - m = \tag{2.3}$$
$$= \frac{(p-m) + \{B(p-m) + (A-B)(p-\alpha)e^{-i\lambda}\cos\lambda\}w(z)}{1+Bw(z)} = Mh(z),$$

where

$$h(z) = e^{-i\lambda}\frac{B+w(z)}{1+Bw(z)}.$$

It is clear that $|h(z)| < 1$ for $z \in E$. Hence (2.1) follows from (2.3).

Conversely, suppose that the condition (2.1) holds. Then, we have

$$\frac{1}{M}\left|\frac{(D^{n+p-1}f(z))'}{z^{p-1}} - m\right| < 1.$$

Let

$$g(z) = \frac{e^{i\lambda}}{M}\left\{\frac{(D^{n+p-1}f(z))'}{z^{p-1}} - m\right\}.$$

Then, by (2.3),

$$w(z) = \frac{g(z) - g(0)}{1 - g(0)g(z)} = \tag{2.4}$$

$$= \frac{(D^{n+p-1}f(z))' - pz^{p-1}}{(A-B)(p-\alpha)e^{-i\lambda}\cos\lambda \cdot z^{p-1} - ((D^{n+p-1}f)z))' - pz^{p-1})}.$$

Clearly, $w(0) = 0$ and $|w(z)| < 1$ for $z \in E$. Rearranging (2.4), we arrive at (1.4). Hence $f(z) \in V_{n,p}^{\lambda}(A, B, \alpha)$. This completes the proof of Lemma 1.

LEMMA 2 (cf. [6]). *If the function $w(z)$ is analytic for $|z| \le r < 1, w(0) = 0$ and $|w(z_0)| = \max_{|z|=r}|w(z)|$, then $z_0 w'(z_0) = kw(z_0)$, where k is a real number such that $k \ge 1$.*

3. Main results

THEOREM 1. *Let n be any integer greater than $-p$. Then*

$$V_{n+1,p}^{\lambda}(A, B, \alpha) \subset V_{n,p}^{\lambda}(A, B, \alpha).$$

Proof. Let $f(z) \in V_{n+1,p}^{\lambda}(A, B, \alpha)$. Suppose that

$$\frac{(D^{n+p-1}f(z))'}{z^{p-1}} = \frac{p + \{pB + (A-B)(p-\alpha)e^{-i\lambda}\cos\lambda\}w(z)}{1 + Bw(z)}, \tag{3.1}$$

where $w(0) = 0$ and $w(z)$ is either analytic or meromorphic in E. Differentiating (3.1) and using (1.3), (2.2), we obtain

$$\frac{(D^{n+p}f(z))'}{z^{p-1}} - m = \frac{p - m + \{B(p-m) + (A-B)(p-\alpha)e^{-i\lambda}\cos\lambda\}w(z)}{1 + Bw(z)}$$
$$+ \frac{(A-B)(p-\alpha)e^{-i\lambda}\cos\lambda}{n+p} \cdot \frac{zw'(z)}{(1 + Bw(z))^2}.$$

Let r^* be the distance from the origin to the nearest pole of $w(z)$ in E. Then $w(z)$ is analytic in the disc $\{z : |z| < r_0 = \min(r^*, 1)\}$.

By Lemma 2, for $|z| < r$, $r < r_0$, there exists a point z_0 in E such that

$$z_0 w'(z_0) = kw(z_0), \quad k \ge 1. \tag{3.3}$$

From (3.2) and (3.3) we have

$$\frac{(D^{n+p}f(z))'}{z^{p-1}} - m = \frac{N(z_0)}{R(z_0)}, \tag{3.4}$$

where

$$N(z_0) = Me^{-i\lambda}[(n+p)B + \{(n+p) + (n+p)B^2 + k(1-B^2)\}w(z_0) + (n+p)Bw(w(z_0))^2],$$
$$R(z_0) = (n+p)\{1 + 2Bw(z_0) + B^2(w(z_0))^2\}.$$

Now suppose that $\max_{|z|=r}|w(z)| = 1$ for some r, $r \leq r_0 < 1$. At the point z_0, where this occurs, we would have $|w(z_0)| = 1$. Then, by using the identities

$$p - m = BMe^{-i\lambda}, \quad (p-m)B + (A-B)(p-\alpha)e^{-i\lambda}\cos\lambda = Me^{-i\lambda},$$

we deduce that

$$|N(z_0)|^2 - M^2|R(z_0)|^2 = x + 2y\,\mathrm{Re}\,w(z_0), \tag{3.5}$$

where

$$x = k(1-B^2)M^2\{k(1-B^2) + 2(n+p)(1+B^2)\},$$
$$y = 2k(n+p)BM^2(1-B^2).$$

From (3.5), it follows that

$$|N(z_0)|^2 - M^2|R(z_0)|^2 > 0, \tag{3.6}$$

provided $x \pm 2y > 0$. Now, in view of the fact that $M > 0$ and $-1 \leq B < 1$, we deduce that

$$x + 2y = k(1-B^2)M^2\{k(1-B^2) + 2(n+p)(1+B)^2\} > 0,$$
$$x - 2y = k(1-B^2)M^2\{k(1-B^2) + 2(n+p)(1-B)^2\} > 0.$$

Thus, from (3.4) and (3.6) we get

$$\left|\frac{(D^{n+p}f(z_0))'}{z_0^{p-1}} - m\right| > M.$$

But, in view of Lemma 1, this is a contradiction to the fact that $f(z) \in V^{\lambda}_{n+1,p}(A, B, \alpha)$. So, we cannot have $|w(z_0)| = 1$. Thus, $|w(z)| \neq 1$ in $|z| < r_0$. Since $w(0) = 0$, $|w(z)|$ is continuous and $|w(z)| \neq 1$ in $|z| < r_0$, we cannot have a pole at $|z| = r_0$

Therefore, $w(z)$ is analytic in E and $|w(z)| < 1$ for $z \in E$. Hence, from (3.1), $f(z) \in V^{\lambda}_{n,p}(A, B, \alpha)$.

In the next theorem, we study the class preserving integral operators for the class $V^{\lambda}_{n,p}(A, B, \alpha)$.

THEOREM 2. *Let p be a positive integer and n be any integer greater than $-p$. If $f(z) \in V_{n,p}^{\lambda}(A,B,\alpha)$ and $\operatorname{Re}(c+p) > 0$, then*

$$F(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t)\, dt \tag{3.7}$$

also belongs to $V_{n,p}^{\lambda}(A,B,\alpha)$.

Proof. From the definition (3.7) of $F(z)$, by (1.3) we have

$$z(D^{n+p-1}F(z))' = (c+p)D^{n+p-1}f(z) - cD^{n+p-1}F(z),$$

so that

$$z(D^{n+p-1}F(z))'' = (c+p)(D^{n+p-1}f(z))' - c(D^{n+p-1}F(z))'. \tag{3.8}$$

Let us suppose that, according to (1.4), we have

$$\frac{(D^{n+p-1}F(z))'}{z^{p-1}} = \frac{p + \{pB + (A-B)(p-\alpha)e^{-i\lambda}\cos\lambda\}w(z)}{1+Bw(z)}, \tag{3.9}$$

where the function $w(z)$ is either analytic or meromorphic in E and such that $w(0) = 0$. Differentiating (3.9) and then using (3.8), we deduce that

$$\begin{aligned} \frac{(D^{n+p-1}F(z))'}{z^{p-1}} - m = \\ = \frac{(A-B)(p-\alpha)e^{-i\lambda}\cos\lambda}{(c+p)} \frac{zw'(z)}{(1+Bw(z))^2} \\ + \frac{(p-m) + \{(p-m)B + (A-B)(p-\alpha)e^{-i\lambda}\cos\lambda\}w(z)}{1+Bw(z)}. \end{aligned} \tag{3.10}$$

The required result can be obtained now from (3.10) by using the same technique as applied in (3.2) in the proof of Theorem 1.

Remark. For $n = 0$, $p = 1$, $\alpha = 0$, $A = 1$ and $B = -1$. Theorem 2 improves a result of Bernardi [2], who proved it when c is a positive integer.

THEOREM 3. *Let p be a positive integer and n be any integer greater than $-p$. If $F(z) = \frac{n+p}{z^n}\int_0^z t^{n-1}f(t)\,dt$, then $F(z) \in V_{n+1,p}^{\lambda}(A,B,\alpha)$ if and only if $f(z) \in V_{n,p}^{\lambda}(A,B,\alpha)$.*

Proof. We have $(D^{n+p}F(z))' = (D^{n+p-1}f(z))'$ and the result follows.

Now we obtain coefficient inequalities for the class $V^{\lambda}_{n,p}(A,B,\alpha)$.

THEOREM 4. *If $f(z)$, given by (1.1), belongs to $V^{\lambda}_{n,p}(A,B,\alpha)$, then*

$$|a_{p+k}| \leq \frac{(A-B)(p-\alpha)\cos\lambda}{(p+k)\delta(n,k)}, \quad k \geq 1, \tag{3.11}$$

where $\delta(n,k)=(n+p-1)!/k!(n+p-1)!$. The results is sharp.

Proof. Since $f(z)\in V^{\lambda}_{n,p}(A,B,\alpha)$, the relation (1.4) hold with $w(z)=\sum_{j=1}^{\infty} w_j z^j$ analytic in E and such that $|w(z)|<1$ for $z\in E$. Substituting the power series expansion of $f(z)$ and $w(z)$ in (1.4), wehave

$$\sum_{j=1}^{\infty}(p+j)\delta(n,j)a_{p+j}z^j = \{(A-B)(p-\alpha)e^{-i\lambda}\cos\lambda - B\sum_{j=1}^{\infty}(p+j)\delta(n,j)a_{p+j}z^j\}\sum_{j=1}^{\infty} w_j z^j. \tag{3.12}$$

Comparing coefficients of the same powers of z on both sides of (3.12), we see that the coefficient a_{p+j} depends only on $a_{p+1}, a_{p+2}, \ldots, a_{p+j-1}$ on the right-hand side of (3.12). Hence, for $k=1,2,\ldots$, it follows from (3.12) that

$$\sum_{j=1}^{k}(p+j)\delta(n,j)a_{p+j}z^j + \sum_{j=k+1}^{\infty} c_j z^j = \\ = \left\{(A-B)(p-\alpha)e^{-i\lambda}\cos\lambda - B\sum_{j=1}^{k-1}(p+j)\delta(n,j)a_{p+j}z^j\right\}w(z),$$

where c_j are some complex numbers. Since $|w(z)|<1$, by using Parseval's identity [11], we get

$$\sum_{j=1}^{k}(p+j)^2(\delta(n,j))^2|a_{p+j}|^2 r^{2j} + \sum_{j=k+1}^{\infty}|c_j|^2 r^{2j} \\ \leq \{(A-B)(p-\alpha)\cos\lambda\}^2 + B^2\sum_{k=1}^{k-1}(p+j)^2(\delta(n,j))^2|a_{p+j}|^2 r^{2j} \\ \leq \{(A-B)(p-\alpha)\cos\lambda\}^2 + B^2\sum_{k=1}^{k-1}(p+j)^2((\delta(n,j))^2|a_{p+j}|^2.$$

Letting $r \to 1$ on the left-hand side of the above inequality, we obtain

$$\sum_{j=1}^{k}(p+j)^2(\delta(n,j))^2|a_{p+j}|^2$$
$$\leq \{(A-B)(p-\alpha)\cos\lambda\}^2 + B^2\sum_{j=1}^{k-1}(p+j)^2(\delta(n,j))^2|a_{p+j}|^2.$$

Thus, we get the inequality

$$(p+k)^2(\delta(n,k))^2|a_{p+k}|^2$$
$$\leq \{(A-B)(p-\alpha)\cos\lambda\}^2 - (1-B^2)\sum_{j=1}^{k-1}(p+j)^2(\delta(n,j))^2|a_{p+j}|^2$$
$$\leq \{(A-B)(p-\alpha)\cos\lambda\}^2$$

implying (3.11).

In order to establish the sharpness of (3.11), consider the functions $f_k(z)$ defined by

$$\frac{(D^{n+p-1}f_k(z))'}{z^{p-1}} = \frac{p + \{pB + (A-B)(p-\alpha)e^{-i\lambda}\cos\lambda\}z^k}{1+Bz^k}, \qquad k \geq 1.$$

Clearly, $f_k(z) \in V_{n,p}^{\lambda}(A,B,\alpha)$ for each $k \geq 1$. Also, it is easy to compute that the functions $f_k(z)$ has the expansions

$$f_k(z) = z^p + \frac{(A-B)(p-\alpha)e^{-i\lambda}\cos\lambda}{(p+k)\delta(n,k)}z^{p+k} + \dots$$

showing that the estimates (3.11) are sharp.

THEOREM 5. *Let $f(z)$, given by* (1.1), *belong to $V_{n,p}^{\lambda}(A,B,\alpha)$ and μ be any complex number. Then*

(3.13)
$$|a_{p+2} - \mu a_{p+1}^2|$$
$$\leq \frac{(A-B)(p-\alpha)\cos\lambda}{(p+2)\delta(n,2)}\times$$
$$\times \max\left\{1, |B + \mu\cdot\frac{(A-B)(p-\alpha)(p+2)\delta(n,2)e^{-i\lambda}\cos\lambda}{(p+2)^2(\sigma(n,1))^2}|\right\}.$$

The estimate is sharp.

Proof. Upon equating the coefficients of z and of z^2 in (3.12), we have

$$w_1 = \frac{(p+1)\delta(n,1)a_{p+1}}{(A-B)(p-\alpha)e^{-i\lambda}\cos\lambda} \qquad (3.14)$$

and

$$(3.15) \qquad w_2 = \frac{(p+2)\delta(n,2)a_{p+2}}{(A-B)(p-\alpha)e^{-i\lambda}\cos\lambda} + \left\{\frac{(p+1)\delta(n,1)}{(A-B)(p-\alpha)e^{-i\lambda}\cos\lambda}\right\}^2 a_{p+1}^2,$$

respectively. It is known [7] that for every complex number γ

$$(3.16) \qquad |w_2 - \gamma w_1^2| \le \max\{1, |\gamma|\},$$

so the estimate (3.13) is sharp. Now, using (3.14) and (3.15), we obtain

$$(3.17) \qquad |a_{p+2} - \mu a_{p+1}^2| \le \frac{(A-B)(p-\alpha)\cos\lambda}{(p+2)\delta(n,2)}|w_2 - \gamma w_1^2|,$$

where

$$\gamma = B + \mu\frac{(A-B)(p-\alpha)(p+2)\delta(n,2)e^{-i\lambda}\cos\lambda}{(p+1)^2(\delta(n,1))^2}.$$

The estimate (3.13) follows by using (3.16) in (3.17). The result is sharp as the estimate (3.16) is sharp.

In the following theorem, we obtain a sufficient condition, in terms of coefficients, for a function to be in $V_{n,p}^{\lambda}(A,B,\alpha)$.

THEOREM 6. *Let $f(z)$, given by* (1.1), *belong to A_p. If*

$$(3.18) \qquad \sum_{k=1}^{\infty}(p+k)\delta(n,k)|a_{p+k}| \le \frac{(A-B)(p-\alpha)\cos\lambda}{1+B}, \qquad B \ge 0,$$

$$(3.19) \qquad \sum_{k=1}^{\infty}(p+k)\delta(n,k)|a_{p+k}| \le \frac{(A-B)(p-\alpha)\cos\lambda}{1-B}, \qquad B \le 0,$$

then $f(z) \in V_{n,p}^{\lambda}(A,B,\alpha)$. The result is sharp.

Proof. Suppose that (3.18) holds. Then, for $|z| = r < 1$, we have

$$|(D^{n+p-1}f(z))' - pz^{p-1}| - |(A-B)(p-\alpha)e^{-i\lambda}z^{p-1}\cos\lambda - B\{(D^{n+p-1}f(z))' - pz^{p-1}\}| =$$

$$= \left| \sum_{k=1}^{\infty} (p+k)\delta(n,k) a_{p+k} z^{p+k-1} \right|$$

$$- \left| (A-B)(p-\alpha)e^{-i\lambda} z^{p-1} \cos\lambda - B \cdot \sum_{k=1}^{\infty} (p+k)\delta(n,k) a_{p+k} z^{p+k-1} \right|$$

$$\leq \sum_{k=1}^{\infty} (p+k)\delta(n,k) |a_{p+k}| r^{p+k-1}$$

$$- \left\{ (A-B)(p-\alpha) r^{p-1} \cos\lambda - B \sum_{k=1}^{\infty} (p+k)\delta(n,k) |a_{p+k}| r^{p+k-1} \right\}$$

$$\leq \left[\sum_{k=1}^{\infty} (1+B)(p+k)\delta(n,k) |a_{p+k}| - (A-B)(p-\alpha)\cos\lambda \right] r^{p-1} \leq 0,$$

by (3.18). Hence $f(z) \in V_{n,p}^{\lambda}(A, B, \alpha)$.

Next, we assume that (3.19) holds. Then

$$|(D^{n+p-1} f(z))' - p z^{p-1}|$$

$$- |(A-B)(p-\alpha)e^{-i\lambda} z^{p-1} \cos\lambda - B\{(D^{n+p-1} f(z))' - p z^{p-1}\}|$$

$$\leq \sum_{k=1}^{\infty} (p+k)\delta(n,k) |a_{p+k}| r^{p+k-1}$$

$$- \left\{ (A-B)(p-\alpha) r^{p-1} \cos\lambda + B \sum_{k=1}^{\infty} (p+k)\delta(n,k) |a_{p+k}| r^{p+k-1} \right\}$$

$$\leq \left[\sum_{k=1}^{\infty} (1-B)(p+k)\delta(n,k) |a_{p+k}| - (A-B)(p-\alpha)\cos\lambda \right] r^{p-1} \leq 0,$$

by (3.19). This proves that $f(z) \in V_{n,p}^{\lambda}(A, B, \alpha)$.

Remark. We observe that for $B \neq 0$ the converse of Theorem 6 may not be true. For instance, consider the function $f(z)$ defined in E by

$$\frac{(D^{n+p-1} f(z))'}{z^{p-1}} = \frac{p - \{pB + (A-B)(p-\alpha)e^{-i\lambda} \cos\lambda\} z}{1 - Bz}, \quad B > 0.$$

It is easily seen that $f(z) \in V_{n,p}^{\lambda}(A, B, \alpha)$ and

$$a_{p+k} = -\frac{B^{k-1}(A-B)(p-\alpha)e^{-i\lambda} \cos\lambda}{(p+k)\delta(n,k)}, \quad k \geq 1,$$

so that

$$\sum_{k=1}^{\infty}\frac{(1+B)(p+k)\delta(n,k)|a_{p+k}|}{(A-B)(p-\alpha)\cos\lambda}=(1+B)\sum_{k=1}^{\infty}B^{k-1}=\frac{1+B}{1-B}>1,$$

for A, B, λ satisfying $0 < B < A \leq 1$ and $|\lambda| < \pi/2$.

Further, consider the function $f(z)$ defined in E by

$$\frac{(D^{n+p-1}f(z))'}{z^{p-1}}=\frac{p+\{pB+(A-B)(p-\alpha)e^{-i\lambda}\cos\lambda\}z}{1+Bz},\qquad B<0.$$

It is easily seen that $f(z)\in V^{\lambda}_{n,p}(A,B,\alpha)$ and

$$a_{p+k}=\frac{(-B)^{k-1}(A-B)(p-\alpha)e^{-i\lambda}\cos\lambda}{(p+k)\delta(n,k)},\qquad k\geq 1.$$

But

$$\sum_{k=1}^{\infty}\frac{(1-B)(p+k)\delta(n,k)|a_{p+k}|}{(A-B)(p-\alpha)\cos\lambda}=(1-B)\sum_{k=1}^{\infty}(-B)^{k-1}=\frac{1-B}{1+B}>1,$$

for A, B, λ satisfying $-1 < B < 0 \leq A \leq 1$ and $|\lambda| < \pi/2$.

Motivated by Theorem 6, we introduce a new subclass of p-valent analytic functions in the unit disc E. We say that a function $f(z) \in A_p$ is in the class $\widetilde{V}^{\lambda}_{n,p}(A,B,\alpha)$ if and only if the conditions (3.18), (3.19) hold. Clearly, $\widetilde{V}^{\lambda}_{n,p}(A,B,\alpha)\subset V^{\lambda}_{n,p}(A,B,\alpha)$. Then the following theorem is true.

THEOREM 7. *If* $f(z)=z^p+\sum_{k=1}^{\infty}a_{p+k}z^{p+k}$ *and* $g(z)=z^p+\sum_{k=1}^{\infty}b_{p+k}z^{p+k}$ *belong to* $\widetilde{V}^{\lambda}_{n,p}(A,B,\alpha)$, *then so does* $F(z)$ *defined by*

$$F(z)=z^p+\sum_{k=1}^{\infty}a_{p+k}b_{p+k}z^{p+k}.$$

Proof. Since $f(z)\in \widetilde{V}^{\lambda}_{n,p}(A,B,\alpha)$, we have

$$\sum_{k=1}^{\infty}(p+k)\delta(n,k)|a_{p+k}|\leq\begin{cases}\dfrac{(A-B)(p-\alpha)\cos\lambda}{1+B}, & B\geq 0,\\[2mm] \dfrac{(A-B)(p-\alpha)\cos\lambda}{1-B}, & B\leq 0.\end{cases}$$

This yields

$$|a_{p+k}|\leq\begin{cases}\dfrac{(A-B)(p-\alpha)\cos\lambda}{(1+B)(p+k)\delta(n,k)}, & B\geq 0,\\[2mm] \dfrac{(A-B)(p-\alpha)\cos\lambda}{(1-B)(p+k)\delta(n,k)}, & B\leq 0,\end{cases}$$

for all $k \geq 1$. Therefore, it follows that

$$|a_{p+k}| < 1, \quad k \geq 1. \tag{3.21}$$

Using (3.21), we have

$$\sum_{k=1}^{\infty}(p+k)\delta(n,k)|a_{p+k}|^2 \leq \sum_{k=1}^{\infty}(p+k)\delta(n,k)|a_{p+k}. \tag{3.22}$$

Similarly, since $g(z) \in \widetilde{V}_{n,p}^{\lambda}(A, B, \alpha)$, we have

$$\sum_{k=1}^{\infty}(p+k)\delta(n,k)|b_{p+k}| \leq \begin{cases} \dfrac{(A-B)(p-\alpha)\cos\lambda}{1+B}, & B \geq 0, \\ \dfrac{(A-B)(p-\alpha)\cos\lambda}{1-B}, & B \leq 0, \end{cases} \tag{3.23}$$

and

$$\sum_{k=1}^{\infty}(p+k)\delta(n,k)|b_{p+k}|^2 \leq \sum_{k=1}^{\infty}(p+k)\delta(n,k)|b_{p+k}|. \tag{3.24}$$

Now, we have

$$\begin{aligned} &\sum_{k=1}^{\infty}(p+k)\delta(n,k)|a_{p+k}b_{p+k}| \\ &\leq \Big(\sum_{k=1}^{\infty}(p+k)\delta(n,k)|a_{p+k}|^2\Big)^{1/2}\Big(\sum_{k=1}^{\infty}(p+k)\delta(n,k)|b_{p+k}|^2\Big)^{1/2} \\ &\leq \Big(\sum_{k=1}^{\infty}(p+k)\delta(n,k)|a_{p+k}|\Big)^{1/2}\Big(\sum_{k=1}^{\infty}(p+k)\delta(n,k)|b_{p+k}|\Big)^{1/2}, \end{aligned} \tag{3.25}$$

where we have applied Cauchy-Schwarz inequality and the relations (3.22) and (3.24). By (3.20) and (3.23), the relation (3.25) becomes

$$\sum_{k=1}^{\infty}(p+k)\delta(n,k)|a_{p+k}b_{p+k}| \leq \begin{cases} \dfrac{(A-B)(p-\alpha)\cos\lambda}{1+B}, & B \geq 0, \\ \dfrac{(A-B)(p-\alpha)\cos\lambda}{1-B}, & B \leq 0. \end{cases}$$

This proves that $F(z) \in \widetilde{V}_{n,p}^{\lambda}(A, B, \alpha)$.

THEOREM 8. *If* $f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k}z^{p+k}$ *and* $g(z) = z^p + + \sum_{k+1}^{\infty} b_{p+k}z^{p+k}$ *belong to* $V_{n,p}^{\lambda}(A, B, \alpha)$, *then so does the function* $F(z)$

defined by

$$F(z) = z^p + \sum_{k=1}^{\infty} \left(\frac{p+k}{p}\right) \delta(n,k) a_{p+k} \cdot b_{p+k} z^{p+k}. \tag{3.26}$$

Proof. Since $f(z) \in V_{n,p}^{\lambda}(A, B, \alpha)$, we have for $f(z)$ the inequality (2.1) with (2.2).

It is known [11] that, if $h(z) = \sum_{n=0}^{\infty} c_n z^n$ is analytic in the unit disc E and $|h(z)| \leq l$, then

$$\sum_{n=0}^{\infty} |c_n|^2 \leq l^2. \tag{3.27}$$

Using (3.27) in (2.1), we get

$$|p-m|^2 + \sum_{k=1}^{\infty} (p+k)^2 (\delta(n,k))^2 |a_{p+k}|^2 \leq \left\{ \frac{(A-B)(p-\alpha)\cos\lambda}{1-B^2} \right\}^2$$

implying after simplification,

$$\sum_{k=1}^{\infty} (p+k)^2 (\delta(n,k))^2 |a_{p+k}|^2 \leq \frac{\{(A-B)(p-\alpha)\cos\lambda\}^2}{1-B^2}. \tag{3.28}$$

Similarly,

$$\sum_{k=1}^{\infty} (p+k)^2 (\delta(n,k))^2 |b_{p+k}|)^2 \leq \frac{\{(A-B)(p-\alpha)\cos\lambda\}^2}{1-B^2}. \tag{3.29}$$

Now, using the Cauchy-Schwarz inequality and then (3.28), (3.29) and (2.2), for $|z| < r$ we get

$$\begin{aligned}
&\left| \frac{(D^{n+p-1}F(z))'}{z^{p-1}} - m \right|^2 \\
&= |(p-m) + \frac{1}{p} \sum_{k=1}^{\infty} (p+k)^2 (\delta(n,k))^2 a_{p+k} b_{p+k} z^k|^2 \\
&\leq |p-m|^2 + \frac{1}{p^2} \Big[\sum_{k=1}^{\infty} (p+k)^2 (\delta(n,k))^2 |a_{p+k} b_{p+k}| r^k \Big]^2 \\
&\quad + \frac{2|p-m|}{p} \sum_{k=1}^{\infty} (p+k)^2 (\delta(n,k))^2 |a_{p+k} b_{p+k}| r^k \\
&\leq |p-m|^2 + \frac{1}{p^2} \Big(\sum_{k=1}^{\infty} (p+k)^2 (\delta(n,k))^2 |a_{p+k}|^2 r^k \Big) \times
\end{aligned}$$

$$\times \Big(\sum_{k=1}^{\infty}(p+k)^2(\delta(n,k))^2 \cdot |b_{p+k}|^2 r^k\Big)$$
$$+\frac{2|p-m|}{p}\Big(\sum_{k=1}^{\infty}(p+k)^2(\delta(n,k))^2|a_{p+k}|^2 r^k\Big)^{1/2}\times$$
$$\times \Big(\sum_{k=1}^{\infty}(p+k)^2(\delta(n,k))^2|b_{p+k}|^2 r^k\Big)^{1/2}$$
$$\leq |p-m|^2+\frac{1}{p^2}\Big(\sum_{k=1}^{\infty}(p+k)^2(\delta(n,k))^2|a_{p+k}|^2\Big)\times$$
$$\times \Big(\sum_{k=1}^{\infty}(p+k)^2(\delta(n,k))^2|b_{p+k}|^2\Big)$$
$$+\frac{2|p-m|}{p}\Big(\sum_{k=1}^{\infty}(p+k)^2(\delta(n,k))^2|a_{p+k}|^2\Big)^{1/2}\times$$
$$\times \Big(\sum k=1^{\infty}(p+k)^2(\delta(n,k))^2|b_{p+k}|^2\Big)^{1/2}$$
$$\leq \left\{\frac{B(A-B)(p-\alpha)\cos\lambda}{1-B^2}\right\}^2+\frac{\{(A-B)(p-\alpha)\cos\lambda\}^4}{(1-B^2)^2p^2}$$
$$+\frac{2B\{(A-B)(p-\alpha)\cos\lambda\}^3}{(1-B^2)^2p}$$
$$=\left\{\frac{(A-B)(p-\alpha)\cos\lambda}{1-B^2}\right\}^2\times$$
$$\times\left\{B^2+\frac{((A-B)(p-\alpha)\cos\lambda)^2}{p^2}+\frac{2B(A-B)(p-\alpha)\cos\lambda}{p}\right\}$$
$$=M^2\left\{B+\frac{(A-B)(p-\alpha)\cos\lambda}{p}\right\}^2.$$

Thus, $F(z)\in V_{n,p}^{\lambda}(A,B,\alpha)$, in view of Lemma 1, if

$$B+\frac{(A-B)(p-\alpha)\cos\lambda}{p}<1$$

which is certainly true. Hence $F(z)\in V_{n,p}^{\lambda}(A,B,\alpha)$.

Lastly, we establish a closure property for the class $V_{n,p}^{\lambda}(A,B,\alpha)$, its proof being obvious.

THEOREM 9. *If the functions $f(z)$ and $g(z)$ belong to the class $V_{n,p}^{\lambda}(A,B,\alpha)$ and $0\leq s\leq 1$, then the function $F(z)=sf(z)+(1-s)g(z)$ also belongs to $V_{n,p}^{\lambda}(A,B,\alpha)$.*

Remark. Taking appropriate values of the parameters A, B, α in the results of section 3, we may get the corresponding results for functions in the classes $V^{\lambda}_{n,p}(\alpha), V^{\lambda}_{n,p,\delta}, (V^{\lambda}_{n,p})^{\delta}$ and $(V^{\lambda}_{n,p})_{\gamma}$.

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