DEMONSTRATIO MATHEMATICA
Vol. XXIX No 2 1996

Marek Balcerzak

ON BOREL SETS MODULO A σ-IDEAL

Introduction

Let I denote a proper σ-ideal of subsets of an uncountable Polish space X. Through the paper we assume that I contains all countable subsets of X. By $\mathcal{B}$ we denote the family of all Borel subsets of X and, by $\mathcal{B}\triangle I$ — the smallest σ-algebra including $\mathcal{B} \cup I$; then $\mathcal{B}\triangle I$ consists of all symmetric differences $(B \setminus A) \cup (A \setminus B)$ (denoted by $B\triangle A$) where $B \in \mathcal{B}$ and $A \in I$. We say that I is *Borel supported* if, for each $A \in I$ there exists $B \in \mathcal{B} \cap I$ such that $A \subseteq B$.

In the paper we are interested in the following problem. For which σ-ideals I and J with $I \neq J$, can we infer that $\mathcal{B}\triangle I \neq \mathcal{B}\triangle J$? In certain cases connected with perfect sets, Bernstein sets make a good tool to get the positive answer. We recall some known results, extend them and give new applications. In a general case, we follow the method of Pelc [P] to show that an answer to the above question can depend on special axioms of set theory. Some of those results are applied to the studies of the alternate iteration of the operations $\mathcal{B}\triangle(\cdot)$ and $H(\cdot)$ where H sends a σ-algebra $\mathcal{S}$ to the σ-ideal of hereditary $\mathcal{S}$-measurable sets.

1. Application of Bernstein sets

We say that $B \subseteq X$ is a *Bernstein set* if it meets each perfect subset of X, and $X \setminus B$ has same property. The standard construction of a Bernstein set, based on a well-ordering of the family of perfect sets, can be repeated when X is replaced by a fixed set A that contains a perfect set (hence it contans $c = 2^{\aleph_0}$ perfect sets). Then B (included in A) will be called a *Bernstein set relatively to A*. The classical theorem states that each Bernstein set on the

---

AMS classification: 04A15, 28A05.

Key words and phrases: Borel sets, σ-ideal, σ-algebra, Bernstein sets, hereditary measurable sets.

real line is nonmeasurable in the Lebesgue sense and it does not possess the Baire property (see [O, Th. 5.4]). The following proposition is more general.

PROPOSITION 1.1 (Cf.[I]). *If a σ-ideal I is Borel supported then no Bernstein set $B \subseteq X$ is in $\mathcal{B}\triangle I$.* ∎

We are going to extend that result. Let TI denote the family of all totally imperfect sets of a given space X (a set is called *totally imperfect*, if it has no perfect subset). Note that a Bernstein set belongs to TI. Several interesting σ-ideals containing uncountable sets included in TI for $X = \mathbb{R}$ are described in [Mi].

PROPOSITION 1.2. *Let I be a σ-ideal of subsets of X and assume that $A \notin I$ contains a perfect set. The following conditions are equivalent:*

(1) *there is $E \in I$ such that $A \setminus E \in TI$,*

(2) *there is a Bernstein set relatively to A that belongs to I,*

(3) *there is a Bernstein set relatively to A that belongs to $\mathcal{B}\triangle I$.*

Proof. (1)⇒(2) We can extend $A\setminus E$ to a Bernstein set $B \subseteq A$ relatively to A. Then $A \setminus B$ is that Bernstein set which is required in (2).

(2)⇒(3) Obvious.

(3)⇒(1) Let a Bernstein set B satisfy (3). Thus $B = D\triangle E$ for some $D \in \mathcal{B}$ and $E \in I$. Suppose that (1) is false. Hence $A \setminus E$ contains a perfect set P. Then at least one of the sets $P \cap D$ and $P \setminus D$ contains a perfect set. The first case is impossible since $P\cap D \subseteq D\setminus E \subseteq B$ and B does not contain perfect sets. The second case is also impossible since $P \setminus D \subseteq (A \setminus E) \setminus D \subseteq A \setminus B$ and $A \setminus B$ does not contain perfect sets. ∎

COROLLARY 1.1. *If I is a Borel supported σ-ideal of subsets of X and $A \subseteq X$ is an analytic set such that $A \notin I$ then no Bernstein set B relatively to A is in $\mathcal{B}\triangle I$.*

Proof. Consider an arbitrary $E \in I$. Let $D \in I$ be a Borel set such that $E \subseteq D$. Thus $A \setminus D$ is analytic and $A \setminus D \notin I$. So, $A \setminus D$ contains a perfect set [K, p. 479]. Hence (1) is false and, consequently (3) is false. ∎

That corollary is known, however maybe never written down explicitly. Its scheme was applied in [C] several times. Let us give one more application.

Recall (cf. [Z]) that $E \in \mathbb{R}$ is said to be *porous at a point* $x \in \mathbb{R}$ if

$$\limsup_{r\to 0^+} \gamma(E,x,r)/r > 0$$

where $\gamma(E,x,r)$ is the length of the longest interval $(a,b) \subseteq (x-r,x+r)\setminus E$. A set E is called *porous* if it is porous at each of its points, and is called *σ-porous* if E is a countable union of porous sets. Porous sets are nowhere dense and of measure zero. The family of σ-porous subsets of $\mathbb{R}$ forms a σ-ideal denoted further by $\mathcal{M}$.

Consider additionally the following σ-ideals of subsets of $\mathbb{R}$:

$\mathcal{K}$ — the σ-deal of sets of the first category,

$\mathcal{L}$ — the σ-ideal of sets of Lebesgue measure zero,

$\mathcal{L}^*$ — the σ-ideal of sets that are contained in F_σ sets from $\mathcal{L}$.

Recall that:

- $\mathcal{M}$ is Borel supported, $\mathcal{M} \subsetneq \mathcal{K} \cap \mathcal{L}$, and there exists a perfect set of measure zero which is not σ-porous (see [Z]);
- $\mathcal{L}^* \subsetneq \mathcal{K} \cap \mathcal{L}$, and there exists a G_δ nowhere dense set of measure zero which is not in $\mathcal{L}^*$ (see e.g. [BBH]);
- there exists a G_δ set in $\mathcal{M} \setminus \mathcal{L}^*$ (see [FH]).

Now, from Corollary 1.1 we derive

PROPOSITION 1.3. (a) *There exists a Bernstein set relatively to a perfect nowhere dense Lebesgue null set, which is not in $\mathcal{B}\triangle\mathcal{M}$. Consequently, $\mathcal{L}^* \setminus (\mathcal{B}\triangle\mathcal{M}) \neq \emptyset$ and $\mathcal{B}\triangle\mathcal{M} \subsetneq \mathcal{B}\triangle(\mathcal{K} \cap \mathcal{L})$.*

(b) *There exists a Bernstein set relatively to a G_δ nowhere dense Lebesgue null set, which is not in $\mathcal{B}\triangle\mathcal{L}^*$. Consequently, $\mathcal{B}\triangle\mathcal{L}^* \subsetneq \mathcal{B}\triangle(\mathcal{K} \cap \mathcal{L})$.*

(c) *There exists a Bernstein set relatively to a G_δ σ-porous set, which is not in $\mathcal{B}\triangle\mathcal{L}^*$. Consequently, $\mathcal{M} \setminus (\mathcal{B}\triangle\mathcal{L}^*) \neq \emptyset$.* ∎

Remark. We have $\mathcal{B}\triangle(\mathcal{K} \cap \mathcal{L}) = (\mathcal{B}\triangle\mathcal{K}) \cap (\mathcal{B}\triangle\mathcal{L})$ (see [B2]). Similarly,

$$\mathcal{B}\triangle(I \cap J) = (\mathcal{B}\triangle I) \cap (\mathcal{B}\triangle J),$$

if I and J are Borel supported σ-ideals (see [BHWW]).

For two families $\mathcal{F}_1, \mathcal{F}_2$ of subsets of X we write

$$\mathcal{F}_1 \oplus \mathcal{F}_2 = \{E \subseteq X : (\exists E_1 \in \mathcal{F}_1)(\exists E_2 \in \mathcal{F}_2)(E \subseteq E_1 \cup E_2)\}.$$

Let $[A]^{<c}$ denote the family of all sets $E \subseteq A$ with $|E| < c$.

EXAMPLES. (a) The assertion of Corollary 1.1 can hold for some σ-ideals which are not Borel supported. Let s_0 denote the σ-ideal of Marczewski null sets, namely $E \in s_0$ if each perfect set has a perfect part disjoint from E (cf. [Sz], [Mi]). Obviously, $s_0 \subseteq TI$ and s_0 is not Borel supported since each uncountable s_0 set (which exists, cf. [Mi]) cannot be covered by a Borel s_0 set. For each perfect set A, we have $A \notin s_0$ and, for each $E \in s_0$, $A \setminus E$ contains a perfect set. Thus, by Proposition 1.2, no Bernstein set relatively to A is in $\mathcal{B}\triangle I$.

(b) Let I consist of sets $E \subseteq \mathbb{R}^2$ such that $E \subseteq D \cup H$ for some D of plane measure zero and $H \in s_0$ (in $\mathbb{R}^2$). Observe that, if $A \subseteq \mathbb{R}^2$ is of positive inner measure, and if $E \in I$ and D, H are as above, then $A \setminus D$ has positive inner measure, so it contains a perfect set P in which we can pick a perfect part disjoint from H. Consequently, $A \setminus E$ contains a perfect set and, by Proposition 1.2, no Bernstein set relatively to A is in $\mathcal{B}\triangle I$. Observe that I is not Borel supported. It follows from the fact that a nonmeasurable s_0 set (which exists, see [W2]) cannot be covered by a Borel set from I (see [B1, Proposition 2]). A similar construction holds for the category case.

(c) There are interesting σ-ideals on $\mathbb{R}$ containing Bernstein sets (relatively to $\mathbb{R}$). Namely, if $\mathcal{F}$ is a fixed family of one-to-one functions $f : \mathbb{R} \to \mathbb{R}$ and $|\mathcal{F}| \leq c$ then there exists a Bernstein set B such that $|B\triangle f[B]| < c$ for each $f \in \mathcal{F}$ ([B3], cf. also [S] and [Mo, Th.23, p.168]). Thus

$$I = \{A \subseteq \mathbb{R} : A \subseteq B \cup E \text{ for some } E \in [\mathbb{R}]^{<c}\}$$

forms a proper $\mathcal{F}$-invariant σ-ideal which is not Borel supported (the same holds for the σ-ideals $I \oplus \mathcal{K}$ and $I \oplus \mathcal{L}$, provided that $\mathcal{K}$ and $\mathcal{L}$ are $\mathcal{F}$-invariant; cf. [B3]). Observe that, if B is replaced by $\mathbb{R} \setminus B$ in the definition of I, we get another $\mathcal{F}$-invariant σ-ideal I^*. Obviously both B and $\mathbb{R} \setminus B$ belong to $(\mathcal{B}\triangle I) \cap (\mathcal{B}\triangle I^*)$. In Section 2, we will show that $\mathcal{B}\triangle I \neq \mathcal{B}\triangle I^*$.

2. Comparing the sizes of quotient σ-algebras and some iteration process

We are going to show, how one can decide whether the implication

$$I \neq J \Rightarrow \mathcal{B}\triangle I \neq \mathcal{B}\triangle J$$

is true or false, by comparing the sizes of the respective Boolean algebras and by the use of special axioms of set theory. In fact we follow the methods applied in [P, Th. 3] where a related problem concerning the equality $\mathcal{B}\triangle I = \mathcal{P}(X)$ is studied ($\mathcal{P}(A)$ stands for the power set of A) for invariant ideals on $X = \mathbb{R}$.

By $\mathcal{B}(A)$ we denote the family of all Borel sets relatively to A. Let $\mathcal{B}(A)\triangle I$ abbreviate $\mathcal{B}(A)\triangle(I \cap \mathcal{P}(A))$.

Proposition 2.1. (a) *Let I and J be σ-ideals of subsets of X. If there is $A \in I \setminus J$ with $|A| = c$, and $[A]^{<c} \subseteq J$ then $(\mathcal{B}\triangle I) \setminus (\mathcal{B}\triangle J) \neq \emptyset$.*

(b) *If MA+¬CH holds, there are σ-ideals I and J of subsets of X such that $I \setminus J \neq \emptyset$, $J \setminus I \neq \emptyset$ and $\mathcal{B}\triangle I = \mathcal{B}\triangle J$.*

Proof. (a) Since $[A]^{<c} \subseteq J$ and $|A| = c$, we have $|\mathcal{P}(A)/J| > c$, by the result of A. Taylor [T]. From $A \in I$ it obviously follows that $\mathcal{B}(A)\triangle I = \mathcal{P}(A)$. Hence

$$(*) \qquad |(\mathcal{B}(A)\triangle I)/J| > c \geq |(\mathcal{B}(A)\triangle J)/J|.$$

Thus we infer that $\mathcal{B}(A)\triangle J \subsetneq \mathcal{B}(A)\triangle I$. Suppose now that $\mathcal{B}\triangle I \subseteq \mathcal{B}\triangle J$. Hence $\mathcal{B}(A)\triangle I \subseteq \mathcal{B}(A)\triangle J$ which contradicts $(*)$.

(b) Consider disjoint sets $A, E \subseteq X$ of cardinality ω_1. Let I be the σ-ideal generated by $X \setminus A$ and all singletons. Analogously we define J replacing A by E. Then $X \setminus A \in I \setminus J$ and $X \setminus E \in J \setminus I$. By Silver's lemma (see [MS]), MA and $|A| = |E| < c$ imply $\mathcal{P}(A) = \mathcal{B}(A)$ and $\mathcal{P}(E) = \mathcal{B}(E)$. That easily gives $\mathcal{B}\triangle I = \mathcal{P}(X) = \mathcal{B}\triangle J$. ∎

Remarks. (i) If CH holds, the assumptions $|A| = c$ and $[A]^{<c} \subseteq J$ in Proposition 2.1(a) evidently result from $A \notin J$. The version of Proposition 2.1(a) in which CH is supposed has a proof analogous to that given in [P, Th. 3].

(ii) From Proposition 2.1(a) it follows that

$$(\mathcal{B}\triangle I) \setminus (\mathcal{B}\triangle I^*) \neq \emptyset \text{ and } (\mathcal{B}\triangle I^*) \setminus (\mathcal{B}\triangle I) \neq \emptyset$$

for the σ-ideals I and I^* defined in Example (c) of Section 1.

(iii) If $X = \mathbb{R}$, one can choose I and J in Proposition 2.1(b) being invariant with respect to all translations by rationals (cf. [P, Th. 3]).

Further we will assume that $\mathcal{S}$ is a proper σ-algebra of subsets of X (i.e. $\mathcal{S} \neq \mathcal{P}(X)$). Define $H(\mathcal{S})$ as the family of all *hereditary $\mathcal{S}$-measurable* sets, that is

$$H(\mathcal{S}) = \{E \subseteq X : (\forall A \subseteq E)(A \in \mathcal{S})\}.$$

Then $H(\mathcal{S})$ is the largest σ-ideal in $\mathcal{P}(X)$ contained in $\mathcal{S}$. It is obvious that the operation H is monotonic with respect to inclusion.

LEMMA 2.1. (a) *If $\mathcal{S}$ is a σ-algebra of subsets of X, and $\mathcal{B} \subseteq \mathcal{S}$, then $\mathcal{B}\triangle H(\mathcal{S}) \subseteq \mathcal{S}$ and $H(\mathcal{S}) = H(\mathcal{B}\triangle H(\mathcal{S}))$.*

(b) *If I is a σ-ideal of subsets of X then $I \subseteq H(\mathcal{B}\triangle I)$ and $\mathcal{B}\triangle I = \mathcal{B}\triangle H(\mathcal{B}\triangle I)$.*

We omit an easy proof. ∎

We see that, using the operations $\mathcal{B}\triangle(.)$ and $H(.)$ alternately, the iteration process starting from a σ-ideal I or from a σ-algebra $\mathcal{S} \supseteq \mathcal{B}$ stabilizes.

Now, let us discuss some cases when the inclusions $\mathcal{B}\triangle H(\mathcal{S}) \subseteq \mathcal{S}$ and $I \subseteq H(\mathcal{B}\triangle I)$ can be (or cannot) be reversed.

EXAMPLE. A natural σ-algebra containing $\mathcal{B}$, associated with the σ-ideal s_0, consists of s-sets defined as follows (see [Sz]). We say that $E \subseteq X$ is an s-set (or that $E \in s$) if each perfect set has a perfect part which is contained in E or is disjoint from E. It is known that $H(s^0) = s$ (see [Sz]). Walsh proved in [W1, Th. 2.4] that for $X = \mathbb{R}^2$ there exists a family $\mathcal{F} \subseteq s$ such that $|\mathcal{F}| = 2^c$ and $A \triangle B \notin s_0$ for any distinct $A, B \in \mathcal{F}$. Consequently, $|s/s_0| = 2^c > c = |(\mathcal{B}\triangle s_0)/s_0|$ and thus $\mathcal{B}\triangle s_0 \neq s$. Hence the inclusion $\mathcal{B}\triangle H(s) \subseteq s$ cannot be reversed.

Remark. In general, if $\mathcal{S} \supseteq \mathcal{B}$ is a σ-algebra such that $|\mathcal{S}/H(\mathcal{S})| > c$ then $\mathcal{B}\triangle H(\mathcal{S}) \subsetneq \mathcal{S}$. In that case, $\mathcal{S}$ cannot be of the form $\mathcal{B}\triangle I$ for any σ-ideal I since if $\mathcal{S} = \mathcal{B}\triangle I$ then $\mathcal{B}\triangle H(\mathcal{S}) = \mathcal{S}$ (see Lemma 2.1(b)).

Let us turn to the question about $H(\mathcal{B}\triangle I) \subseteq I$.

EXAMPLE. Since each set of positive outer Lebesgue measure contains a nonmeasurable set [O, Th. 5.5], we have $H(\mathcal{B}\triangle\mathcal{L}) = \mathcal{L}$. Similarly $H(\mathcal{B}\triangle\mathcal{K}) = \mathcal{K}$.

PROPOSITION 2.2. *Let I be a σ-ideal of subsets of X.*

(a) *Each of the following conditions is sufficient for $H(\mathcal{B}\triangle I) = I$:*

1^0 *each set from $(\mathcal{B}\triangle I)\setminus I$ contains a perfect set,*

2^0 $[X]^{<c} \subseteq I$.

(b) *If MA+¬CH holds and there is $A \notin I$ with $|A| = \omega_1$ then $I \subsetneq H(\mathcal{B}\triangle I)$ (in fact, $A \in H(\mathcal{B}\triangle I)\setminus I$).*

Proof. (a) It suffices to show that $H(\mathcal{B}\triangle I) \subseteq I$.

Assume 1^0. Suppose that there exists $A \in H(\mathcal{B}\triangle I)\setminus I$. Let $E \in I$. Since $A \in H(\mathcal{B}\triangle I)$, we have $A \setminus E \in (\mathcal{B}\triangle I)\setminus I$. Thus by 1^0, $A\setminus E$ contains a perfect set. By Proposition 1.2, a Bernstein set relatively to A is not in $\mathcal{B}\triangle I$. That contradicts $A \in H(\mathcal{B}\triangle I)$.

Assume 2^0. Suppose that $A \in H(\mathcal{B}\triangle I)\setminus I$. Then $|A| = c$ and following the proof of Proposition 2.1(a) we infer that $\mathcal{B}(A)\triangle I \subsetneq \mathcal{P}(A)$ which contradicts $A \in H(\mathcal{B}\triangle I)$.

(b) By Silver's lemma (compare the proof of Proposition 2.1(b)), we have $\mathcal{P}(A) = \mathcal{B}(A)$ which implies that $A \in H(\mathcal{B}\triangle I)$. ∎

Remarks. It follows from 1^0 that $H(\mathcal{B}\triangle I) = I$ holds for all Borel supported σ-ideals I. A verification of 2^0 for some σ-ideals is unconvenient since 2^0 can depend on special axioms of set theory (e.g. for $\mathcal{K}$ and $\mathcal{L}$). For other σ-ideals, 2^0 can be clear (e.g. for the σ-ideal I given in Example (c) of Section 1, or for the σ-ideal s_0; see [W1, Th. 2.1]). Of course, 2^0 is always true, under CH. For certain σ-ideals I the statement $H(\mathcal{B}\triangle I) =$

I is independent of ZFC. Indeed, let I be the σ-ideal generated by a Bernstein set B in $\mathbb{R}$ and by all countable subsets of $\mathbb{R}$. Assume CH. Then from 2^0, we get $H(\mathcal{B}\triangle I) = I$. If MA+¬CH is assumed, let us choose a set $A \subseteq \mathbb{R} \setminus B$ with $|A| = \omega_1$. Then $A \notin I$. Hence $I \subsetneq H(\mathcal{B}\triangle I)$, by Proposition 2.2(b).

Acknowledgements. I would like to thank A.B. Kharazishvili who has pointed out an error in the previous version of Proposition 2.1(b).

References

[B1] M. Balcerzak, *A classification of σ-ideals on the real line*, Problemy Mat. 7 (1985), 51–61.

[B2] M. Balcerzak, *Classification of σ-ideals*, Math. Slovaca 37 (1987), 63–70.

[B3] M. Balcerzak, *Some Bernstein-type construction and its applications*, Acta Univ. Lodz., Folia Math. 2 (1987), 3–8.

[BBH] M. Balcerzak, J. E. Baumgartner, J. Hejduk, *On certain σ-ideals of sets*, Real Anal. Exchange 14 (1988–89), 447–453.

[BHWW] M. Balcerzak, J. Hejduk, W. Wilczyński, S. Wroński, *Why only measure and category*? Scientific Bulletin of Łódź Technical University, 26 (1994), 89–94.

[C] P. Corazza, *Ramsey sets, the Ramsey ideal and other classes over* $\mathbb{R}$, J. Symbolic Logic 57 (1992), 1441–1467.

[FH] J. Foran, P. D. Humke, *Some set-theoretic properties of σ-porous sets*, Real Anal. Exchange 6 (1980–81), 114–119.

[I] A. Iwanik, *Solution of P* 649, Colloq. Math. 34 (1975), 143.

[K] K. Kuratowski, *Topology I*. Academic Press, New York, 1966.

[MS] D. A. Martin, R. M. Solovay, *Internal Cohen extension*, Ann. Math. Logic 2 (1970), 143–178.

[Mi] A. W. Miller, *Special subsets of the real line*, in: Handbook of Set Theoretic Topology , K. Kunen and J. Vaughan (eds.) North-Holland, Amsterdam, 1984.

[Mo] J. C. Morgan II, *Point Set Theory*. Marcel Dekker Inc., New York, 1990.

[O] J. C. Oxtoby, *Measure and Category*. Springer, New York, 1971.

[P] A. Pelc, *Invariant ideals and Borel sets*, Proc. Amer. Math. Soc. 97 (1986), 503–506.

[S] W. Sierpiński, *Sur les translations des ensembles linéaires*, Fund. Math. 19 (1932), 22–28.

[Sz] E. Szpilrajn-Marczewski, *Sur une classe de fonctions de M. Sierpiński et la class correspondante d'ensembles*, Fund. Math. 24 (1935), 17–34.

[T] A. Taylor, *The cardinality of reduced power set algebras*, Proc. Amer. Math. Soc. 103 (1988), 277–280.

[W1] J. T. Walsh, *Marczewski sets, measure and the Baire property*, Fund. Math. 129 (1988), 83–89.

[W2] J. T. Walsh, *Marczewski sets, measure and the Baire property II*, Proc. Amer. Math. Soc. 106 (1989), 1027–1030.

[Z] L. Zajiček, *Porosity and σ-porosity*, Real Anal. Exchange 13 (1987-88), 314–350.

INSTITUTE OF MATHEMATICS
ŁÓDŹ TECHNICAL UNIVERSITY
Al. Politechniki 11, I-2
90-924 ŁÓDŹ, POLAND

Received June 28, 1994.