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IDEMPOTENT AND DISTRIBUTIVE GROUP RELATED GROUPOIDS, II

In a former paper [E4] we were concerned with the basic properties of certain idempotent and distributive resp. merely idempotent groupoids $(\mathcal{R}, \bullet)$ ($SGRID$- resp. $SGRI$-groupoids), the binary operation of which is influenced by the structure of an abelian group $(G, +)$: We assume $\mathcal{R}$ to be a subset of G with $0 \in \mathcal{R}$, and $\bullet$ being given by

$$x \bullet y = \sigma(x) + \tau(y) \quad (x, y \in \mathcal{R})$$

where $\sigma, \tau : \mathcal{R} \to G$ are mappings. Group related groupoids are of a certain interest for topological purposes, $SGRID$-groupoids turned out to form groupoid modes ([E4], (2.1)) and are linked with the point of view presented in ([JK], (3,3.9)) (see introduction of [E4]).

Throughout this second half of a series of papers on group related groupoids we use without any further explanation the notation, notions, and conventions of the first part [E4].

In chapter 1 we turn to homomorphisms of $SGRI$-groupoids and, as a characterization for homomorphisms between canonically monoid splitting $SGRID$-groupoids, we find a kind of natural decomposition of the given homomorphism into a monoid homomorphism and a homomorphism of quotients of the $SGRID$-groupoids under consideration.

As a guiding line for chapter 2 we use a result in ([E2],(2.17)), where it is shown for group related (2–)symmetric groupoids that the equivalence relation $\mathfrak{L}$ coming from successive left translation coincides with the one induced by a certain subgroup of the underlying group, which is closely related to the given (2–)symmetric groupoid. After some general considerations concerning $\mathfrak{L}$ on a $SGRID$-groupoid $\mathcal{R}$, we develop necessary and sufficient conditions for the coincidence of $\mathfrak{L}$ with the equivalence relation given by a subgroup of the underlying group, which is assigned to $\mathcal{R}$ in a way analogous to the (2–)symmetric case.

In the final chapter we deal with the transfer of algebraic identities and the property of being a *SGRID*-groupoid from a groupoid $(\mathcal{R}, \star)$ to its derived groupoids, i.e. to groupoids with the same underlying set and a binary operation, which is made up of iterated applications of $\star$ and the trivial binary operations on $\mathcal{R}$. In particular, we turn our attention to the k-symmetry law and describe *SGRID*-groupoids satisfying this identity.

For describing mappings or homomorphisms of *SGRI*-groupoids, depending on the situation we sometimes only need their counterparts restricted in domain, or range, or domain and range, respectively (see also text after ([E4],(1.5))). Since what is meant always comes clear from the context, and since we don't want to exaggerate with precision in notation, which would make a mess of the formulation of some results, for a mapping and corresponding restrictions we use the same symbol.

1. Homomorphisms of *SGRID*-groupoids

Now we discuss some aspects concerning homomorphisms between idempotent group related groupoids. Any such groupoids $(\mathcal{R}', \bullet)$ and $(\mathcal{S}', \bullet)$ and a homomorphism $\eta : \mathcal{R}' \to \mathcal{S}'$ being given, applying ([E4],(1.5)) we first find a strictly group related copy $\mathcal{R}$ of $\mathcal{R}'$ and an isomorphism $\imath : (\mathcal{R}, \bullet) \to (\mathcal{R}', \bullet)$, and again by ([E4],(1.5)) (via the translation $d_{-\eta\circ\imath(0)}$ in the underlying group of $\mathcal{S}'$) a strictly group related copy $\mathcal{S}$ of $\mathcal{S}'$ and an isomorphism $\jmath : (\mathcal{S}', \bullet) \to (\mathcal{S}, \bullet)$ such that $(\jmath \circ \eta \circ \imath)(0) = 0$. Hence we can restrict ourselves to the consideration of homomorphisms between *SGRI*-groupoids preserving the neutral elements of the respective underlying groups. We call such homomorphisms distinguished.

In addition and parallel to the convention after ([E4],(1.5)) concerning *SGRI*-groupoids written by $(\mathcal{R}, \bullet)$, we note that throughout this chapter, for a *SGRI*-groupoid written by $(\mathcal{S}, \bullet)$, the pair of describing mappings will be denoted by (ζ, ϑ), and its underlying group by H.

1.1. Proposition. *Let* $(\mathcal{R}, \bullet)$, $(\mathcal{S}, \bullet)$ *be SGRI-groupoids* , $\lambda : \mathcal{R} \to \mathcal{S}$ *a mapping such that* $\lambda(0) = 0$. *The following are equivalent*:

(1) λ *is a distinguished groupoid homomorphism*,

(2) $\lambda \in \mathrm{Part}_{\mathcal{R}^\top, \mathcal{R}^\perp}(\mathcal{R}, H)$, $\lambda \circ \sigma = \zeta \circ \lambda$,

(3) $\lambda \in \mathrm{Part}_{\mathcal{R}^\top, \mathcal{R}^\perp}(\mathcal{R}, H)$, $\lambda \circ \tau = \vartheta \circ \lambda$.

Proof. (1) $\Rightarrow$ (2),(3). For $r, s \in \mathcal{R}$, λ being a groupoid homomorphism is equivalent to

$$\lambda(\sigma(r) + \tau(s)) = \zeta(\lambda(r)) + \vartheta(\lambda(s)).$$

If we put by turns $s := 0$ resp. $r := 0$, we get $\lambda(\sigma(r)) = \zeta(\lambda(r))$, $\lambda(\tau(s)) = \vartheta(\lambda(s))$, hence $\lambda\circ\sigma = \zeta\circ\lambda$ and $\lambda\circ\tau = \vartheta\circ\lambda$. Therefore $\lambda \in \mathrm{Part}_{\mathcal{R}^{\top},\mathcal{R}^{\perp}}(\mathcal{R}, H)$.

(2) $\Rightarrow$ (3). For $r \in \mathcal{R}$ we conclude

$$\begin{aligned}\lambda(\sigma(r)) = \zeta(\lambda(r)) &\Rightarrow \lambda(\sigma(r)) + \lambda(\tau(r)) - \lambda(\tau(r)) = \zeta(\lambda(r))\\ &\Rightarrow \lambda(\sigma(r) + \tau(r)) - \lambda(\tau(r)) = \lambda(r) - \vartheta(\lambda(r)),\\ &\Rightarrow \lambda(r) - \lambda(\tau(r)) = \lambda(r) - \vartheta(\lambda(r));\end{aligned}$$

consequently, $\lambda \circ \tau = \vartheta \circ \lambda$.

(3) $\Rightarrow$ (1). For $r, s \in \mathcal{R}$ we calculate

$$\begin{aligned}\lambda(r \bullet s) &= \lambda(\sigma(r) + \tau(s))\\ &= \lambda(\sigma(r)) + \lambda(\tau(s)), \quad \text{since } \lambda \in \mathrm{Part}_{\mathcal{R}^{\top},\mathcal{R}^{\perp}}(\mathcal{R}, H),\\ &= \lambda(\sigma(r)) + \vartheta(\lambda(s)), \quad \text{since } \lambda \circ \tau = \vartheta \circ \lambda,\\ &= \lambda(\sigma(r)) + \lambda(\tau(r)) - \lambda(\tau(r)) + \vartheta(\lambda(s))\\ &= \lambda(\sigma(r) + \tau(r)) - \vartheta(\lambda(r)) + \vartheta(\lambda(s)),\end{aligned}$$

since $\lambda \in \mathrm{Part}_{\mathcal{R}^{\top},\mathcal{R}^{\perp}}(\mathcal{S}, H)$ and $\lambda \circ \tau = \vartheta \circ \lambda$,

$$= \lambda(r) - \vartheta(\lambda(r)) + \vartheta(\lambda(s)) = \lambda(r) \bullet \lambda(s).$$ ∎

Now we are in a situation to present a second example of an $SGRID$–groupoid which is not isomorphic to a subreduct of an affine space (cf. text before ([E4],(2.3))).

1.2. EXAMPLE. We denote by $\mathbb{Z}^\infty$ the direct sum of infinitely many copies of $\mathbb{Z}$; and for $\nu \in \mathbb{N}$ by $\mathrm{t}_\nu : \mathbb{Z}^\infty \to \mathbb{Z}^\infty$ we mean the ν–th shift operator given by the assignment

$$(x_\iota)_{1\le\iota<\infty} \mapsto (x_\iota)_{\nu\le\iota<\infty}.$$

One can see immediately that both t_ν and $\mathbf{1} - \mathrm{t}_\nu$ are group epimorphisms, where $\mathbf{1}$ stands for the identity map of $\mathbb{Z}^\infty$. According to example ([E4], (2.3),(a)), by t_ν we get a binary operation $\bullet_\nu$ on $\mathbb{Z}^\infty$, which makes $(\mathbb{Z}^\infty, \bullet_\nu)$ a $SGRID$-groupoid. Furthermore, we can consider $\mathbb{Z}^\infty$ in a canonical way as K–module for rings $K \in \{\mathbb{Z}, \mathbb{Z}^\infty\}$. For $a \in K$ let $t_a : \mathbb{Z}^\infty \to \mathbb{Z}^\infty$ be the homomorphism defined by component wise multiplication $\mathfrak{x} \mapsto a\mathfrak{x}$. Also according to ([E4],(2.3),(a)), we equip $\mathbb{Z}^\infty$ with the binary operation $\bullet_a$ coming from t_a.

Now we show that for any choice of ν and a, the groupoid $(\mathbb{Z}^\infty, \bullet_\nu)$ is never isomorphic to a subalgebra of $(\mathbb{Z}^\infty, \bullet_a)$. Assume the contrary and let $k : (\mathbb{Z}^\infty, \bullet_\nu) \to (\mathbb{Z}^\infty, \bullet_a)$ be a distinguished (cf. text before (1.1)) algebra monomorphism. Applying (1.1) we find $k \in \mathrm{Part}_{\mathbb{Z}^\infty,\mathbb{Z}^\infty}(\mathbb{Z}^\infty, \mathbb{Z}^\infty)$ is even a group monomorphism, the diagrams

$$\begin{array}{ccc} \mathbf{Z}^\infty & \xrightarrow{k} & \mathbf{Z}^\infty \\ {\scriptstyle t_\nu}\downarrow & & \downarrow{\scriptstyle t_a} \\ \mathbf{Z}^\infty & \xrightarrow[k]{} & \mathbf{Z}^\infty \end{array} \qquad\qquad \begin{array}{ccc} \mathbf{Z}^\infty & \xrightarrow{k} & \mathbf{Z}^\infty \\ {\scriptstyle 1-t_\nu}\downarrow & & \downarrow{\scriptstyle 1-t_a} \\ \mathbf{Z}^\infty & \xrightarrow[k]{} & \mathbf{Z}^\infty \end{array}$$

commute, and $k(\mathbf{Z}^\infty)$ is not trivial. Since both t_ν and $1 - t_\nu$ are onto, we get $k(\mathbf{Z}^\infty) = t_a(k(\mathbf{Z}^\infty)) = (1 - t_a)(k(\mathbf{Z}^\infty))$, which contradicts non-triviality of $k(\mathbf{Z}^\infty)$ and the fact that each element of $\mathbf{Z}$ has a unique decomposition in prime numbers.

Now we prepare for a characterization of homomorphisms between canonically monoid splitting $SGRID$-groupoids. To this end we introduce two mappings on quotients by monoids.

1.3. Definition. Let $(\mathcal{R}, \bullet)$ be a canonically monoid splitting $SGRID$-groupoid. Denote by $\sigma_\natural : \mathcal{R}/\mathcal{R}^\perp_\succ \to \mathcal{R}^\perp_\succ/\sigma(\mathcal{R}^\perp_\curlyvee)$ the mapping given by $x + \mathcal{R}^\perp_\succ \mapsto \sigma(x) + \sigma(\mathcal{R}^\perp_\curlyvee)$.

$\sigma_\natural$ is well defined, since for $x, x' \in \mathcal{R}$ and $x + \mathcal{R}^\perp_\succ = x' + \mathcal{R}^\perp_\succ$ there is $n \in \mathcal{R}^\perp_\curlyvee$ such that $x - x' = n$. By $\sigma \in \mathrm{Part}_{\mathcal{R},\mathcal{R}^\perp_\succ}(\mathcal{R}, G)$ we get $\sigma(x) = \sigma(x' + x - x') = \sigma(x') + \sigma(x - x')$, consequently

$$\sigma(x - x') = \sigma(x) - \sigma(x') = \sigma(n) \in \sigma(\mathcal{R}^\perp_\curlyvee).$$

Again by partiality of σ follows that $\sigma|\mathcal{R}^\perp_\succ$ is a monoid homomorphism and therefore, $\sigma(\mathcal{R}^\perp_\curlyvee)$ is a subgroup of $\mathcal{R}^\perp_\succ$. From the above, $\sigma(x) + \sigma(\mathcal{R}^\perp_\curlyvee) = \sigma(x') + \sigma(\mathcal{R}^\perp_\curlyvee)$.

1.4. Definition. Let $\mathcal{A}, \mathcal{B}$ be submonoids of abelian groups, $\alpha : \mathcal{A} \to \mathcal{A}$, $\beta : \mathcal{B} \to \mathcal{B}$, $k : \mathcal{A} \to \mathcal{B}$ be monoid homomorphisms such that $k \circ \alpha = \beta \circ k$. By $k^* : \mathcal{A}/\alpha(\mathcal{A}_\curlyvee) \to \mathcal{B}/\beta(\mathcal{B}_\curlyvee)$ we denote the mapping given by the assignment

$$a + \alpha(\mathcal{A}_\curlyvee) \mapsto k(a) + \beta(\mathcal{B}_\curlyvee).$$

k^* is well defined: Since $\mathcal{A}_\curlyvee < \mathcal{A}$ implies $\alpha(\mathcal{A}_\curlyvee) < \mathcal{A}_\curlyvee$, and for $a, a' \in \mathcal{A}$ satisfying $a + \alpha(\mathcal{A}_\curlyvee) = a' + \alpha(\mathcal{A}_\curlyvee)$ equivalently holds $a - a' \in \alpha(\mathcal{A}_\curlyvee)$, there exists $u \in \mathcal{A}_\curlyvee$ such that $a - a' = \alpha(u)$, and

$$k(a) - k(a') = k(a - a') = k(\alpha(u)) = \beta(k(u)).$$

But $k(u) \in \mathcal{B}_\curlyvee$, for $u \in \mathcal{A}$ implies $k(u) \in \mathcal{B}$ and $-u \in \mathcal{A}$ yields $-k(u) = k(-u) \in \mathcal{B}$, thus $k(a) - k(a') \in \beta(\mathcal{B}_\curlyvee)$, where $\beta(\mathcal{B}_\curlyvee) < \mathcal{B}_\curlyvee$, and we conclude $k(a) + \beta(\mathcal{B}_\curlyvee) = k(a') + \beta(\mathcal{B}_\curlyvee)$. The next result shows that homomorphisms between canonically monoid splitting $SGRID$-groupoids are made up in a natural way of two components, a monoid homomorphism and a groupoid homomorphism of respective quotients of the considered $SGRID$-groupoids. Moreover, the properties of groupoid homomorphisms shown in (1.1), i.e. partiality and commutativity with describing mappings, are transferred to these components in an analogous way.

1.5. THEOREM. *Let $(\mathcal{R}, \bullet)$, $(\mathcal{S}, \bullet)$ be $SGRID$-groupoids and let $\mathcal{R}$ be canonically monoid splitting. The following are equivalent.*

(1) *There exists a distinguished homomorphism $\lambda : (\mathcal{R}, \bullet) \to (\mathcal{S}, \bullet)$.*

(2) *There is a canonically monoid splitting $SGRID$-groupoid $\mathcal{T} \subseteq \mathcal{S}$, a monoid homomorphism $l : \mathcal{R}^{\perp}_{\succ} \to \mathcal{T}^{\perp}_{\succ}$ commuting with σ and ζ, and a homomorphism $\Lambda : (\mathcal{R}/\mathcal{R}^{\perp}_{\succ}, \blacksquare) \to (\mathcal{T}/\mathcal{T}^{\perp}_{\succ}, \blacksquare)$, satisfying for $A, B \in \mathcal{R}/\mathcal{R}^{\perp}_{\succ}$, $m, n \in \mathcal{R}^{\perp}_{\succ}$*

(i) $$A + m = B + n \Rightarrow \Lambda(A) + l(m) = \Lambda(B) + l(n),$$

(ii) $$l^*(\sigma_\natural(A \blacksquare B)) = \zeta_\natural(\Lambda(A) \blacksquare \Lambda(B)).$$

(3) *There is a canonically monoid splitting $SGRID$-groupoid $\mathcal{T} \subseteq \mathcal{S}$, a monoid homomorphism $l : \mathcal{R}^{\perp}_{\succ} \to \mathcal{T}^{\perp}_{\succ}$ commuting with σ and ζ, and a mapping $\Lambda : \mathcal{R}/\mathcal{R}^{\perp}_{\succ} \to \mathcal{T}/\mathcal{T}^{\perp}_{\succ}$ satisfying for $A, B \in \mathcal{R}/\mathcal{R}^{\perp}_{\succ}$, $m, n \in \mathcal{R}^{\perp}_{\succ}$*

(i) $$A + m = B + n \Rightarrow \Lambda(A) + l(m) = \Lambda(B) + l(n),$$

(ii) $$l^*(\sigma_\natural(A)) = \zeta_\natural(\Lambda(A)).$$

Proof. (1) $\Rightarrow$ (2). By (1.1), $\lambda \in \mathrm{Part}_{\mathcal{R},\mathcal{R}^{\perp}_{\succ}}(\mathcal{R}, H)$, hence $\lambda|\mathcal{R}^{\perp}_{\succ}$ is a monoid homomorphism, and λ commutes with σ and ζ.

Put $\mathcal{T} := \lambda(\mathcal{R})$. Clearly, $(\mathcal{T}, \bullet)$ is a $SGRID$-groupoid, and $\mathcal{T}$ is canonically monoid splitting, since $\mathcal{T}^{\perp} = \lambda(\mathcal{R}^{\perp})$ implies

$$\mathcal{T}^{\perp}_{\succ} = \bigcup_{k \in \mathbb{N}} k\mathcal{T}^{\perp} = \bigcup_{k \in \mathbb{N}} k\lambda(\mathcal{R}^{\perp}) = \lambda\Big(\bigcup_{k \in \mathbb{N}} k\mathcal{R}^{\perp}\Big) = \lambda(\mathcal{R}^{\perp}_{\succ}),$$

and consequently,

$$\mathcal{T} + \mathcal{T}^{\perp}_{\succ} = \lambda(\mathcal{R}) + \lambda(\mathcal{R}^{\perp}_{\succ}) = \lambda(\mathcal{R} + \mathcal{R}^{\perp}_{\succ}) \subseteq \lambda(\mathcal{R}) = \mathcal{T}.$$

Moreover, for $t \in \mathcal{T}$, $u \in \mathcal{T}^{\perp}_{\succ}$ and $r \in \mathcal{R}$, $n \in \mathcal{R}^{\perp}_{\succ}$ such that $\lambda(r) = t$, $\lambda(n) = u$ we calculate by commutativity of λ with σ and ζ and partiality of the mappings under consideration

$$\begin{aligned}\zeta(t + u) &= \zeta(\lambda(r) + \lambda(n)) = \zeta(\lambda(r + n)) \\ &= \lambda(\sigma(r + n)) = \lambda(\sigma(r)) + \lambda(\sigma(n)) \\ &= \zeta(\lambda(r)) + \zeta(\lambda(n)) = \zeta(t) + \zeta(u).\end{aligned}$$

Let $l : \mathcal{R}^{\perp}_{\succ} \to \mathcal{T}^{\perp}_{\succ}$ be the restriction of λ in both domain and range. Obviously, l is a monoid homomorphism commuting with σ and ζ. Now we define $\Lambda : \mathcal{R}/\mathcal{R}^{\perp}_{\succ} \to \mathcal{T}/\mathcal{T}^{\perp}_{\succ}$ assigning $x + \mathcal{R}^{\perp}_{\succ} \mapsto \lambda(x) + \mathcal{T}^{\perp}_{\succ}$. The mapping Λ is well defined by partiality of λ and definition of $\mathcal{T}$. Λ satisfies (i), since for $x, x' \in \mathcal{R}$, $m, n \in \mathcal{R}^{\perp}_{\succ}$ such that $x + \mathcal{R}^{\perp}_{\succ} + m = x' + \mathcal{R}^{\perp}_{\succ} + n$ we get the desired implication applying λ on both sides of the equation above. Λ satisfies (ii), since for $a, b \in \mathcal{R}$ and $A := a + \mathcal{R}^{\perp}_{\succ}$, $B := b + \mathcal{R}^{\perp}_{\succ}$ we calculate

$$\begin{aligned} l^*(\sigma_\natural(A \blacksquare B)) &= l^*(\sigma(a \bullet b) + \sigma(\mathcal{R}^{\perp}_{\curlyvee})) = \lambda(\sigma(a \bullet b)) + \zeta(\mathcal{T}^{\perp}_{\curlyvee}) \\ &= \zeta(\lambda(a \bullet b)) + \zeta(\mathcal{T}^{\perp}_{\curlyvee}) = \zeta_\natural(\lambda(a) \bullet \lambda(b) + \mathcal{T}^{\perp}_{\succ}) \\ &= \zeta_\natural((\lambda(a) + \mathcal{T}^{\perp}_{\succ}) \bullet (\lambda(b) + \mathcal{T}^{\perp}_{\succ})) = \zeta_\natural(\Lambda(A) \blacksquare \Lambda(B)). \end{aligned}$$

Finally, by calculations similar to the above one can show that Λ is a groupoid homomorphism.

(2) $\Rightarrow$ (3). By idempotency of $\blacksquare$, (3),(ii) follows from (2),(ii).

(3) $\Rightarrow$ (1). Parallel to the proof of ([E4],(3.1)), we take an index set J with $0 \in J$ and a subset $\{x_j \mid j \in J\}$ of $\mathcal{R}$ with $x_0 := 0$ satisfying

$$\begin{gathered} \forall\, A \in \mathcal{R}/\mathcal{R}^{\perp}_{\succ}\ \exists\, j \in J : x_j + \mathcal{R}^{\perp}_{\curlywedge} \supseteq A, \\ \forall\, j, k \in J, j \neq k : x_j + \mathcal{R}^{\perp}_{\curlywedge} \cap x_k + \mathcal{R}^{\perp}_{\curlywedge} = \emptyset. \end{gathered}$$

Now we define a groupoid homomorphism λ. Put $\lambda(x_0) := 0$, and for $0 \neq j \in J$ let $w'_j \in \mathcal{T}$ such that $\Lambda(x_j + \mathcal{R}^{\perp}_{\succ}) = w'_j + \mathcal{T}^{\perp}_{\succ}$. By virtue of (ii),

$$\begin{aligned} \zeta(w'_j) + \zeta(\mathcal{T}^{\perp}_{\curlyvee}) &= \zeta_\natural(w'_j + \mathcal{T}^{\perp}_{\succ}) = \zeta_\natural(\Lambda(x_j + \mathcal{R}^{\perp}_{\succ})) \\ &= l^*(\sigma_\natural(x_j + \mathcal{R}^{\perp}_{\succ})) = l^*(\sigma(x_j) + \sigma(\mathcal{R}^{\perp}_{\succ} \cap -\mathcal{R}^{\perp}_{\succ})) \\ &= l(\sigma(x_j)) + \zeta(\mathcal{T}^{\perp}_{\curlyvee}); \end{aligned}$$

hence there exists $p \in \mathcal{T}^{\perp}_{\curlyvee}$ such that $l(\sigma(x_j)) = \zeta(w'_j) + \zeta(p)$. Put $\lambda(x_j) := w_j := w'_j + p$; consequently $\lambda(\sigma(x_j)) = \zeta(\lambda(x_j))$, and for $x \in \mathcal{R}$ and $m, n \in \mathcal{R}^{\perp}_{\succ}$ with $x_j + m = x + n$ we define

$$\lambda(x) := \lambda(x_j) + l(m) - l(n).$$

λ is well defined, since for $x, x_j \in \mathcal{R}$ and $m, n, m', n' \in \mathcal{R}^{\perp}_{\succ}$ satisfying $x_j + m = x + n$, $x_j + m' = x + n'$ we deduce calculating in G that $m + n' = m' + n$, which implies $l(m) - l(n) = l(m') - l(n')$, for l is a monoid homomorphism. Therefore $\lambda(x_j) + l(m) - l(n)$ and $\lambda(x_j) + l(m') - l(n')$ coincide. Obviously by definition, λ and l are the same on $\mathcal{R}^{\perp}_{\succ}$. $\lambda(x) \in \mathcal{T}$, since for $x, x_j \in \mathcal{R}$ and $m, n \in \mathcal{R}^{\perp}_{\succ}$ such that $x_j + m = x + n$ we get $x_j + \mathcal{R}^{\perp}_{\succ} + m = x + \mathcal{R}^{\perp}_{\succ} + n$, and by (i), $\Lambda(x_j + \mathcal{R}^{\perp}_{\succ}) + l(m) = \Lambda(x + \mathcal{R}^{\perp}_{\succ}) + l(n)$, from which we deduce

$$w'_j + \mathcal{T}^{\perp}_{\succ} + l(m) - l(n) = \Lambda(x + \mathcal{R}^{\perp}_{\succ}) \subseteq \mathcal{T},$$

consequently, $\lambda(x) = \lambda(x_j) + l(m) - l(n) \in \mathcal{T}$. $\lambda \in \mathrm{Part}_{\mathcal{R}, \mathcal{R}^{\perp}_{\succ}}(\mathcal{R}, H)$, since for $x, x_j \in \mathcal{R}$, $m, n, m_0 \in \mathcal{R}^{\perp}_{\succ}$ and $x_j + m + m_0 = x + n + m_0$ we calculate

$$\lambda(x + m_0) = \lambda(x_j) + \lambda(m + m_0) - \lambda(n) = \lambda(x) + \lambda(m_0).$$

Finally we show that λ commutes with σ and ζ. For $x \in \mathcal{R}$, $m, n \in \mathcal{R}^{\perp}_{\succ}$ and $x + n = x_j + m$ we deduce using the definition of λ as well as $\zeta \in \mathrm{Part}_{\mathcal{T},\mathcal{T}^{\perp}_{\succ}}(\mathcal{T}, H)$, $l \circ \sigma = \zeta \circ l$ and $\lambda(\sigma(x_j)) = \zeta(\lambda(x_j))$,

$$\begin{aligned} \zeta(\lambda(x)) &= \zeta(\lambda(x_j)) + \zeta(l(m)) - \zeta(l(n)) \\ &= \lambda(\sigma(x_j)) + l(\sigma(m)) - l(\sigma(n)) \\ &= l(\sigma(x_j)) + l(\sigma(m)) - l(\sigma(n)) \\ &= l(\sigma(x_j) + \sigma(m) - \sigma(n)), \quad \text{for } l \text{ is a monoid homomorphism,} \\ &= l(\sigma(x_j + m - n)), \quad \text{for } \sigma \in \mathrm{Part}_{\mathcal{R},\mathcal{R}^{\perp}_{\succ}}(\mathcal{R}, G), \\ &= \lambda(\sigma(x)). \end{aligned}$$

∎

In particular cases, monomorphisms between canonically monoid splitting $SGRID$-groupoids can be characterized using describing mappings of the respective quotient structure. Using the notation $\tau_\natural$ for the mapping $\tau_{\mathcal{N}}$ from ([E4],(1.18)) with $\mathcal{N} := \mathcal{R}^{\perp}_{\succ}$ we can state

1.6. THEOREM. *Let $(\mathcal{R}, \bullet)$, $(\mathcal{S}, \bullet)$ be $SGRID$-groupoids, let $(\mathcal{R}, \bullet)$ be canonically monoid splitting and $\sigma(\mathcal{R}^{\perp}_{\curlyvee}) = \mathcal{R}^{\perp}_{\curlyvee}$. The following are equivalent:*

(1) *There exists a distinguished monomorphism $\lambda : (\mathcal{R}, \bullet) \to (\mathcal{S}, \bullet)$.*

(2) *There is a canonically monoid splitting $SGRID$-groupoid $\mathcal{T} \subseteq \mathcal{S}$ such that $\zeta(\mathcal{T}^{\perp}_{\curlyvee}) = \mathcal{T}^{\perp}_{\curlyvee}$, a monoid monomorphism $l : \mathcal{R}^{\perp}_{\succ} \to \mathcal{T}^{\perp}_{\succ}$, which commutes with τ and ϑ, and a homomorphism $\Lambda : (\mathcal{R}/\mathcal{R}^{\perp}_{\succ}, \blacksquare) \to (\mathcal{T}/\mathcal{T}^{\perp}_{\succ}, \blacksquare)$ such that for $A, B \in \mathcal{R}/\mathcal{R}^{\perp}_{\succ}$, $m, n \in \mathcal{R}^{\perp}_{\succ}$ holds*

(i) $$A + m = B + n \iff \Lambda(A) + l(m) = \Lambda(B) + l(n),$$

(ii) $$\Lambda(\tau_\natural(A \blacksquare B)) = \vartheta_\natural(\Lambda(A) \blacksquare \Lambda(B)).$$

(3) *There is a canonically monoid splitting $SGRID$-groupoid $\mathcal{T} \subseteq \mathcal{S}$ such that $\zeta(\mathcal{T}^{\perp}_{\curlyvee}) = \mathcal{T}^{\perp}_{\curlyvee}$, a monoid monomorphism $l : \mathcal{R}^{\perp}_{\succ} \to \mathcal{T}^{\perp}_{\succ}$, which commutes with τ and ϑ, and a mapping $\Lambda : \mathcal{R}/\mathcal{R}^{\perp}_{\succ} \to \mathcal{T}/\mathcal{T}^{\perp}_{\succ}$ such that for $A, B \in \mathcal{R}/\mathcal{R}^{\perp}_{\succ}$, $m, n \in \mathcal{R}^{\perp}_{\succ}$ holds*

(i) $$A + m = B + n \iff \Lambda(A) + l(m) = \Lambda(B) + l(n),$$

(ii) $$\Lambda(\tau_\natural(A)) = \vartheta_\natural(\Lambda(A)).$$

Proof. By an argumentation similar to that in the proof of (1.5). ∎

2. Canonical relations on *SGRID*-groupoids

In this chapter we are mainly concerned with the relation $\mathfrak{L}$ on *SGRID*-groupoids $(\mathcal{R}, \bullet)$ given by successively applying left translations. One gets corresponding results for the respective relation arising from successive right translations by dualizing, i.e. considering $\mathfrak{L}$ on the opposite groupoid $(\mathcal{R}, \bar{\bullet})$ (cf. ([E4],(1.2))). In order to simplify notation for paranthetical expressions we introduce

CONVENTION. For a groupoid $(\mathcal{R}, \star)$, a natural k and $a_1, \ldots a_k \in \mathcal{R}$ we agree to write

$$a_k \star \ldots \star a_1 := a_k \star (a_{k-1} \star (\ldots \star (a_2 \star a_1) \ldots))$$

and call such terms bracket free.

2.1. DEFINITION/REMARK. Let $(\mathcal{R}, \star)$ be a groupoid, and for $k \in \mathbb{N}_0$ let $\mathfrak{L}_k \subseteq \mathcal{R} \times \mathcal{R}$ the relation defined by

$$(a, b) \in \mathfrak{L}_k :\Longleftrightarrow \exists\ a_1, \ldots a_k \in \mathcal{R} : b = a_k \star \ldots \star a_1 \star a,$$

and $\mathfrak{L} := \bigcup_{k \in \mathbb{N}_0} \mathfrak{L}_k$. Clearly, $\mathfrak{L}$ is reflexive and transitive, but not necessarily an equivalence relation.

The next result generalizes part of ([R],(2.3)).

2.2. PROPOSITION. *For a groupoid $(\mathcal{R}, \star)$ satisfying the idempotency law, $\mathfrak{L}_k$ is reflexive and $\mathfrak{L}_k \subseteq \mathfrak{L}_{k+1}$ for all $k \in \mathbb{N}_0$. If $(\mathcal{R}, \star)$ even forms a groupoid mode, all relations $\mathfrak{K} \in \{\mathfrak{L}\} \cup \{\mathfrak{L}_k |\ k \in \mathbb{N}_0\}$ are congruences with*

$$\mathfrak{K}[x \star y] = \mathfrak{K}[x] \star \mathfrak{K}[y] \quad \forall\ x, y \in \mathcal{R}.$$

As a consequence, $\mathfrak{K}$, $\mathcal{R}/\mathfrak{K}$ as well as $\mathfrak{K}[x]$ $(x \in \mathcal{R})$ with the respective canonically given binary operations are groupoid modes.

Proof. The statements concerning idempotency are obvious. In order to show the property of being a congruence we first note that for a merely entropic groupoid $(\mathcal{R}, \star)$, $\ell \in \mathbb{N}$ and $\alpha_1, \ldots \alpha_\ell$, $\beta_1, \ldots \beta_\ell \in \mathcal{R}$ always holds

$$(\alpha_\ell \star \ldots \star \alpha_1) \star (\beta_\ell \star \ldots \star \beta_1) = (\alpha_\ell \star \beta_\ell) \star \ldots \star (\alpha_1 \star \beta_1).$$

As an instance, we now turn to the relation $\mathfrak{L}$. "$\subseteq$" For $z \in \mathfrak{L}[x \star y]$, let $m \in \mathbb{N}_0$, $d_1, \ldots d_m \in \mathcal{R}$ such that $z = d_m \star \ldots \star d_1 \star x \star y$. Idempotency yields

$$z = (d_m \star d_m) \star \ldots \star (d_1 \star d_1) \star (x \star y),$$

hence $z \in \mathfrak{L}[x] \star \mathfrak{L}[y]$ by the above remark. "$\supseteq$" Let $m, n \in \mathbb{N}_0$, $m \leq n$, and x, y, x', y', $a_1, \ldots a_m$, $b_1, \ldots b_n \in \mathcal{R}$ such that

$$x' = a_m \star \ldots \star a_1 \star x, \quad y' = b_n \star \ldots \star b_1 \star y.$$

By idempotency of $(\mathcal{R},\star)$ we get $x' = a_m \star \ldots \star a_1 \star \underbrace{x \star \ldots \star x}_{n-m} \star x$, hence $x' = c_n \star \ldots \star c_1 \star x$ with $c_1, \ldots c_n \in \mathcal{R}$ appropriate. Now the formula above shows $(x \star y, x' \star y') \in \mathfrak{L}$. The remaining assertions are immediate consequences of $\mathfrak{K}$ being congruences. ∎

For a $SGRID$-groupoid $(\mathcal{R},\bullet)$, the relations $\mathfrak{K} \in \{\mathfrak{L}\} \cup \{\mathfrak{L}_k \mid k \in \mathbb{N}_0\}$ are $SGRID$-groupoids with underlying group $G \times G$ and describing map $\tau \times \tau$, and $\mathfrak{K}[x]$ $(x \in \mathcal{R})$ is isomorphic to a $SGRID$-groupoid via translation (cf. ([E4],(1.3), (1.6),(a))).

Now we develop a product formula, which among others is useful for the investigation of $\mathfrak{L}$ in particular cases.

2.3. Proposition. *Let $(\mathcal{R},\bullet)$ be a $SGRID$-groupoid. Then we have the formula*

$\forall\, k \in \mathbb{N}\ \forall\, a_1, \ldots a_k \in \mathcal{R}:$

$$a_k \bullet \ldots \bullet a_1 = \Big(\sum_{i=1}^{k-1} \tau^{i-1}(a_{k-i+1}) - \tau^i(a_{k-i+1})\Big) + \tau^{k-1}(a_1).$$

In particular, for $a, b \in \mathcal{R}$ and $\ell \in \mathbb{N}$ holds

$$\underbrace{a \bullet \ldots \bullet a}_{\ell} \bullet b = a - \tau^\ell(a) + \tau^\ell(b).$$

Proof. The proof is done inductively. Clearly the equation holds for $k = 1$. For the inductive step we calculate, since τ is a homomorphism with respect to $\bullet$,

$$\begin{aligned} a_{k+1} \bullet a_k \bullet \ldots \bullet a_1 &= a_{k+1} - \tau(a_{k+1}) + \tau(a_k) \bullet \ldots \bullet \tau(a_1) \\ &\underset{\text{IND}}{=} a_{k+1} - \tau(a_{k+1}) + \Big(\sum_{i=2}^{k} \tau^{i-1}(a_{k+1-i+1}) - \tau^i(a_{k+1-i+1})\Big) + \tau^k(a_1) \\ &= \Big(\sum_{i=1}^{k} \tau^{i-1}(a_{(k+1)-i+1}) - \tau^i(a_{(k+1)-i+1})\Big) + \tau^k(a_1). \end{aligned}$$ ∎

2.4. Lemma. *For a $SGRID$-groupoid $(\mathcal{R},\bullet)$ and $x, y \in \mathcal{R}$ holds*

$$(x, y) \in \mathfrak{L} \iff \exists\, k \in \mathbb{N}_0 : y - \tau^k(x) \in \sum_{i=0}^{k-1} \tau^i(\mathcal{R}^\perp),$$

where we put $\sum_{i=0}^{k-1} \tau^i(\mathcal{R}^\perp) := 0$ *for* $k = 0$. *In particular,*

$$\mathfrak{L}[x] = \bigcup_{k\in\mathbb{N}_0} \Big(\tau^k(x) + \sum_{i=0}^{k-1} \tau^i(\mathcal{R}^\perp)\Big) \quad \forall\, x \in \mathcal{R}.$$

Proof. Since for a $SGRID$-groupoid, σ and τ commute by ([E4],(2.1)), we deduce using (2.3) for $k \in \mathbb{N}_0$, x, y, $a_1, \ldots a_k \in \mathcal{R}$

$$y = a_k \bullet \ldots \bullet a_1 \bullet x \iff y = \Big(\sum_{i=1}^{k} \underbrace{\tau^{i-1}(a_{k-i+1}) - \tau(\tau^{i-1}(a_{k-i+1}))}_{\in \tau^{i-1}(\mathcal{R}^\perp)}\Big) + \tau^k(x).$$

■

For subsequent parts of this paper we remember that by ([E4],(2.5),(a)), $SGRID$-groupoids $(\mathcal{R}, \bullet)$ with $\mathcal{R}^\top = \mathcal{R}$ are canonically monoid splitting and satisfy $\tau(\mathcal{R}^\perp) = \mathcal{R}^\perp$.

2.5. Proposition. *For a* $SGRID$*-groupoid* $(\mathcal{R}, \bullet)$ *we have the inclusions* $(x, y \in \mathcal{R})$

(i) $$\mathfrak{L}[x \bullet y] \subseteq \mathfrak{L}[y],$$

(ii) $$\tau(\mathfrak{L}[x]) \subseteq \mathfrak{L}[\tau(x)] \subseteq \mathfrak{L}[x],$$

(iii) $$\sigma(\mathfrak{L}[x]) \subseteq \mathfrak{L}[\sigma(x)].$$

In addition, if $(\mathcal{R}, \bullet)$ *satisfies* $\mathcal{R}^\top = \mathcal{R}$, *the first inclusion of (ii) becomes equality and* $(m \in \mathcal{R}^\perp_\succ)$

(iv) $$\mathfrak{L}[x] \subseteq x + \mathcal{R}^\perp_\curlywedge,$$

(v) $$\mathfrak{L}[m] \subseteq \mathcal{R}^\perp_\succ, \text{ in particular, } \mathfrak{L}[0] = \mathcal{R}^\perp_\succ,$$

(vi) $$\mathfrak{L}[x + m] \subseteq \mathfrak{L}[x] + \mathfrak{L}[m].$$

(2.2) and (v) imply that $\mathcal{R}^\perp_\succ$ is a union of an increasing sequence of $SGRID$-subgroupoids, namely $(\mathfrak{L}_k[0])_{k\in\mathbb{N}_0}$.

Proof. The inclusions are shown mainly using the representation of classes of $\mathfrak{L}$ from (2.4).

(i) $$\begin{aligned}\mathfrak{L}[x \bullet y] &= \bigcup_{k\in\mathbb{N}_0} (\tau^k(x \bullet y) + \sum_{i=0}^{k-1} \tau^i(\mathcal{R}^\perp)) \\ &\underset{([E4],(2.1))}{=} \bigcup_{k\in\mathbb{N}_0} (\tau^k(x) \bullet \tau^k(y) + \sum_{i=0}^{k-1} \tau^i(\mathcal{R}^\perp)) \\ &= \bigcup_{k\in\mathbb{N}_0} (\tau^{k+1}(y) + \tau^k(\sigma(x)) + \sum_{i=0}^{k-1} \tau^i(\mathcal{R}^\perp))\end{aligned}$$

$$\subseteq \bigcup_{k\in\mathbb{N}_0} (\tau^{k+1}(y) + \sum_{i=0}^{k} \tau^i(\mathcal{R}^\perp))$$

$$\subseteq \bigcup_{k\in\mathbb{N}_0} (\tau^{k}(y) + \sum_{i=0}^{k-1} \tau^i(\mathcal{R}^\perp)) = \mathfrak{L}[y].$$

(ii)
$$\tau(\mathfrak{L}[x]) = \tau(\bigcup_{k\in\mathbb{N}_0} (\tau^{k}(x) + \sum_{i=0}^{k-1} \tau^i(\mathcal{R}^\perp)))$$

$$= \bigcup_{k\in\mathbb{N}_0} \tau(\tau^{k}(x) + \sum_{i=0}^{k-1} \tau^i(\mathcal{R}^\perp))$$

$$= \bigcup_{k\in\mathbb{N}_0} (\tau^{k+1}(x) + \sum_{i=0}^{k-1} \tau^{i+1}(\mathcal{R}^\perp)),$$

since $\tau \in \mathrm{Part}_{\mathcal{R},\mathcal{R}^\perp_\succ}(\mathcal{R}, G)$,

$$\subseteq \bigcup_{k\in\mathbb{N}_0} (\tau^{k+1}(x) + \sum_{i=0}^{k-1} \tau^{i}(\mathcal{R}^\perp)),$$

since $\tau^{j+1}(\mathcal{R}^\perp) \subseteq \tau^j(\mathcal{R}^\perp)$, $j \in \mathbb{N}_0$,

$$= \mathfrak{L}[\tau(x)] = \mathfrak{L}[0 \bullet x] \underset{(i)}{\subseteq} \mathfrak{L}[x].$$

If $\mathcal{R}^\top = \mathcal{R}$, parallel to the calculations above by ([E4],(2.5),(a)) we conclude $\tau(\mathfrak{L}[x]) = \mathfrak{L}[\tau(x)]$.

(iii) is shown in analogy to (ii).

(iv) For $x \in \mathcal{R}$ and $j \in \mathbb{N}_0$ obviously $x - \tau^j(x) \in j\mathcal{R}^\perp$ resp. $\tau^j(x) \in x + j(-\mathcal{R}^\perp)$. Therefore we conclude

$$\mathfrak{L}[x] = \bigcup_{k\in\mathbb{N}_0} (\tau^k(x) + k\mathcal{R}^\perp) \subseteq \bigcup_{k\in\mathbb{N}_0} (x + k(-\mathcal{R}^\perp) + k\mathcal{R}^\perp)$$

$$= x + \bigcup_{k\in\mathbb{N}_0} (k\mathcal{R}^\perp + k(-\mathcal{R}^\perp)) \subseteq x + \bigcup_{k\in\mathbb{N}_0} k\mathcal{R}^\perp + \bigcup_{k\in\mathbb{N}_0} k(-\mathcal{R}^\perp)$$

$$= x + \mathcal{R}^\perp_\succ + (-\mathcal{R}^\perp_\succ) = x + \mathcal{R}^\perp_\curlywedge.$$

(v) is obvious; (vi) is proved similarly to the above. ∎

To a certain extent, $(\mathcal{R}/\mathfrak{L}, \bullet)$ reminds of a *SGRID*-groupoid, although in general there is no canonically given underlying group, nor there is a describing mapping. For the following interpretation, cf. (1.1).

2.6. Remark. In the situation of (2.5), let denote the canonical projection $\mathcal{R} \to \mathcal{R}/\mathfrak{L}$ by $q_{\mathfrak{L}}$. Then the inclusions (ii),(iii),(vi) of (2.5) become

$$\begin{aligned} \tau(q_{\mathfrak{L}}(x)) &\subseteq q_{\mathfrak{L}}(\tau(x)), \\ \sigma(q_{\mathfrak{L}}(x)) &\subseteq q_{\mathfrak{L}}(\sigma(x)), \\ q_{\mathfrak{L}}(x+m) &\subseteq q_{\mathfrak{L}}(x) + q_{\mathfrak{L}}(m). \end{aligned}$$

Roughly speaking, the groupoid homomorphism $q_{\mathfrak{L}}$ behaves similar to a homomorphism between $SGRID$-groupoids.

In ([E2],(2.17)) it was shown for group related (2–)symmetric groupoids that the relations $\mathfrak{L}$ and

$$\mathfrak{M} := R^{\mathcal{R}}_{\mathcal{R}^{\perp}_{\succ}}$$

(cf. ([E4],(1.9)) for the definition) coincide. The coincidence of classes of both relations has far-reaching consequences for the $SGRID$-groupoids under consideration.

2.7. THEOREM. *Let $(\mathcal{R}, \bullet)$ be a $SGRID$-groupoid with $\mathcal{R}^{\top} = \mathcal{R}$ and suppose*

$$\forall\, x \in \mathcal{R}\ \exists\, y \in \mathcal{R} : \mathfrak{M}[x] = \mathfrak{L}[y].$$

Then $\mathcal{R}^{\perp}_{\succ}$ is a group, and $\mathfrak{L} \subseteq \mathfrak{M}$ (where the latter is implied independently by both the condition for the classes of $\mathfrak{M}$ and $\mathfrak{L}$, and $\mathcal{R}^{\perp}_{\succ}$ being a group).

Proof. For $x \in \mathcal{R}$, let $y \in \mathcal{R}$ such that $\mathfrak{M}[x] = \mathfrak{L}[y]$. Using (2.5) we conclude

$$\mathfrak{M}[\tau(x)] = \tau(\mathfrak{M}[x]) = \tau(\mathfrak{L}[y]) = \mathfrak{L}[\tau(y)],$$

as well as

$$\begin{array}{ccc} \mathfrak{M}[x] & \subseteq & \mathfrak{M}[\tau(x)] \\ \| & & \| \\ \mathfrak{L}[y] & \supseteq & \mathfrak{L}[\tau(y)], \end{array}$$

whence $\mathfrak{M}[x] = \mathfrak{M}[\tau(x)]$. Thus for all $x \in \mathcal{R}$ holds $x - \tau(x) \in \mathcal{R}^{\perp}_{\succ} \cap (-\mathcal{R}^{\perp}_{\succ})$, therefore $\mathcal{R}^{\perp}_{\succ} \subseteq -\mathcal{R}^{\perp}_{\succ}$, and $\mathcal{R}^{\perp}_{\succ}$ turns out to be a group. Now (2.5),(iv) yields $\mathfrak{L}[x] \subseteq \mathfrak{M}[x]$ for all $x \in \mathcal{R}$ (or independently from $\mathcal{R}^{\perp}_{\succ}$ being a group, $x \in \mathfrak{M}[x] = \mathfrak{L}[y]$ implies $\mathfrak{L}[x] \subseteq (\mathfrak{L} \circ \mathfrak{L})[y] = \mathfrak{L}[y] = \mathfrak{M}[x]$ by transitivity of $\mathfrak{L}$). ∎

As a consequence we get

2.8. LEMMA. *Let $(\mathcal{R}, \bullet)$ be a $SGRID$-groupoid with $\mathcal{R}^{\top} = \mathcal{R}$ and $\mathfrak{L} = \mathfrak{M}$. Then $(\mathcal{R}/\mathfrak{L}, \bullet)$ is a right zero band.*

The notion of a right zero band can be found in ([RS1],p. 28) et al. . In zero bands, the binary operation is given by projection onto the first resp. second factor.

Proof. Since $\mathfrak{L} = \mathfrak{M}$, by (2.7) $\mathcal{R}_{\succ}^{\perp}$ is a group, i.e. $\mathcal{R}_{\succ}^{\perp} = \mathcal{R}_{\curlywedge}^{\perp}$, and $\mathfrak{L}[z] = z + \mathcal{R}_{\curlywedge}^{\perp}$ ($z \in \mathcal{R}$). Hence we get for $x, y \in \mathcal{R}$ by (2.2) and (2.5)(i)

$$\mathfrak{L}[x] \bullet \mathfrak{L}[y] = \mathfrak{L}[x \bullet y] = \mathfrak{L}[y]. \qquad \blacksquare$$

For the rest of this paper, among others we investigate a certain class of *SGRID*-groupoids which covers k-symmetric *SGRID*-groupoids (see [E3] and chapter 3 of this paper).

2.9. PROPOSITION/DEFINITION. *For a SGRID-groupoid* $(\mathcal{R}, \bullet)$ *we assume* τ *satisfying*

$$\text{(K)} \qquad \forall\, x \in \mathcal{R}\ \exists\, k \in \mathbf{N} : \tau^k(x) = x.$$

SGRID-groupoids $\mathcal{R}$ *with condition* (K) *have the properties*

(1) $(\mathcal{R}, \bullet)$ *is left cancellative.*

(2) *For any* $a, b \in \mathcal{R}$ *being given, the equation* $a \bullet x = b$ *has a unique solution.*

(3) *Every term in* $\mathcal{R}$ *can be represented in bracket free notation.*

Proof. Throughout the whole proof, for $a, b \in \mathcal{R}$ let α, β be naturals such that according to (K), $\tau^\alpha(a) = a$, $\tau^\beta(b) = b$, and let $\gamma \in \mathbf{N}$ be a common multiple of α and β.

(1) For $x \in \mathcal{R}$ we conclude using (2.3)

$$\begin{aligned} x \bullet a = x \bullet b &\Rightarrow \underbrace{x \bullet \ldots \bullet x}_{\gamma} \bullet a = \underbrace{x \bullet \ldots \bullet x}_{\gamma} \bullet b \\ &\Rightarrow x - \tau^\gamma(x) + \tau^\gamma(a) = x - \tau^\gamma(x) + \tau^\gamma(b) \\ &\Rightarrow a = b. \end{aligned}$$

(2) Put $x := \underbrace{a \bullet \ldots \bullet a}_{\gamma - 1} \bullet b$. By (2.3) we deduce

$$a \bullet \underbrace{a \bullet \ldots \bullet a}_{\gamma - 1} \bullet b = a - \tau^\gamma(a) + \tau^\gamma(b) = b,$$

hence x is a solution. Uniqueness now follows by (1).

(3) Let $c \in \mathcal{R}$. It is sufficient to show that $(a \bullet c) \bullet b$ can be written without brackets. Since $\underbrace{a \bullet \ldots \bullet a}_{\gamma} \bullet b = b$, and $\bullet$ is left distributive by ([E4],(2.1)),

we conclude

$$(a \bullet c) \bullet b = (a \bullet c) \bullet (\underbrace{a \bullet \ldots \bullet a}_{\gamma} \bullet b) = a \bullet c \bullet \underbrace{a \bullet \ldots \bullet a}_{\gamma-1} \bullet b. \qquad \blacksquare$$

As a kind of converse of (2.7) we prove

2.10. PROPOSITION. *Let $(\mathcal{R}, \bullet)$ be a SGRID-groupoid satisfying condition (K) and $\mathcal{R}^{\top} = \mathcal{R}$. Then $\mathcal{R}^{\perp}_{\succ}$ is a group, and the relations $\mathfrak{M}$ and $\mathfrak{L}$ coincide. In particular, if $\mathcal{R}$ is finite, $\mathcal{R}$ is always a union of cosets by a subgroup of the underlying group.*

Proof. In case that τ satisfies condition (K), we show $-\mathcal{R}^{\perp} \subseteq \mathcal{R}^{\perp}_{\succ}$, hence $\mathcal{R}^{\perp}_{\succ}$ is a group. To this end, let $x \in \mathcal{R}$, $k \in \mathbb{N}$ such that $\tau^k(x) = x$, and put $p := x - \tau(x)$. Then we get

$$\begin{aligned} -p &= \tau(x) - x = \tau(x) - \tau^k(x) \\ &= \tau(x) - \sum_{\ell=2}^{k-1} \tau^{\ell}(x) + \sum_{\ell=2}^{k-1} \tau^{\ell}(x) - \tau^k(x) = \sum_{\ell=2}^{k} \underbrace{(\tau^{\ell-1}(x) - \tau^{\ell}(x))}_{\in \mathcal{R}^{\perp}} \in \mathcal{R}^{\perp}_{\succ}. \end{aligned}$$

Now let $y \in \mathfrak{M}[x] = x + \bigcup_{\ell \in \mathbb{N}_0} \ell \mathcal{R}^{\perp}$, i.e. $y = x + m_1 + \ldots + m_j$, with $j \in \mathbb{N}_0$, $m_1, \ldots m_j \in \mathcal{R}^{\perp}$ appropriate. We can find $\mu \in \mathbb{N}$ such that $\mu k \geq j$ and conclude

$$y - x = y - \tau^{\mu k}(x) \in j\mathcal{R}^{\perp} \subseteq (\mu k)\mathcal{R}^{\perp},$$

thus $y \in \mathfrak{L}[x]$ by (2.4) and since $\tau(\mathcal{R}^{\perp}) = \mathcal{R}^{\perp}$. Applying the last part of (2.7) completes the proof. ■

2.11. PROPOSITION. *Let $(\mathcal{R}, \bullet)$ be a SGRID-groupoid with $\mathcal{R}^{\top} = \mathcal{R}$, let $\mathcal{R}^{\perp}_{\succ}$ be a group, and assume the existence of $\ell \in \mathbb{N}$ such that $\mathcal{R}^{\perp}_{\succ} = \ell \mathcal{R}^{\perp}$. Then $\mathfrak{L} = \mathfrak{M}$.*

Proof. Since $\mathcal{R}^{\perp}_{\succ}$ is a group, for $x \in \mathcal{R}$ by (2.4) and $\tau(\mathcal{R}^{\perp}) = \mathcal{R}^{\perp}$ we conclude

$$\mathfrak{L}[x] = \bigcup_{k \in \mathbb{N}} \tau^k(x) + k\mathcal{R}^{\perp} = \tau^{\ell}(x) + \ell\mathcal{R}^{\perp} = \tau^{\ell}(x) + \mathcal{R}^{\perp}_{\succ} = x + \mathcal{R}^{\perp}_{\succ} = \mathfrak{M}[x]. \qquad \blacksquare$$

3. Derived groupoids and k–symmetry

For a given groupoid $(\mathcal{R}, \star)$, by a derived groupoid we understand $\mathcal{R}$ equipped with a binary operation formed by compositions of $\star$ and the canonical projections from $\mathcal{R} \times \mathcal{R}$ to $\mathcal{R}$, to be more precise

3.1. DEFINITION. Let $(\mathcal{R}, \star)$ be a groupoid.

(a) Denote by Ω the minimal subset of $\mathcal{R}^{\mathcal{R} \times \mathcal{R}}$ such that

(i) the canonical projections π_i onto the i–th factor ($i = 1, 2$) are elements of Ω,

(ii) $v, w \in \Omega \Rightarrow v \star w \in \Omega$.

(b) Moreover, let $\Psi := \{\psi_\ell : \mathcal{R} \times \mathcal{R} \to \mathcal{R} | \ \ell \in \mathbb{N}\}$, where we define inductively

$$\psi_1(x, y) := x \star y, \quad \psi_{\ell+1}(x, y) := x \star \psi_\ell(x, y).$$

By definition, $\Psi \subseteq \Omega$. We call Ω the set of derived binary operations on $\mathcal{R}$. For the binary operation on $\mathcal{R}$ given by $\omega \in \Omega$ we shall sometimes write $\star_\omega$. Furthermore, $\omega \in \Omega$ can be represented merely using π_1, π_2 and $\star$. If m is the minimal number of occurences of $\star$ over all such representations for ω, we call $m + 1$ the length of ω.

Ω is the clone of terms of the groupoid $(R, \star)$.–

For the concepts used in the following, cf. the introductory part of [RS1]. The subsequent result was already shown in ([RS2], p. 249), coming from a slightly different viewpoint. We shall give a direct proof.

3.2. THEOREM. *If $(\mathcal{R}, \star)$ is an idempotent resp. entropic groupoid, then $(\mathcal{R}, \Omega)$ is an idempotent resp. entropic algebra.*

Proof. The assertion concerning idempotency is obvious. In order to prove the entropicity part we introduce

$$\Omega_n := \{\omega \in \Omega | \text{ length of } \omega \leq n\} \quad (n \in \mathbb{N})$$

and show inductively that $(\mathcal{R}, \Omega_n)$ is an entropic algebra for all $n \in \mathbb{N}$, in other words,

$$\forall \ v, w \in \Omega_n \forall \ a, b, c, d \in \mathcal{R} : v(w(a, b), w(c, d)) = w(v(a, c), v(b, d)).$$

For $\Omega_1 = \{\pi_1, \pi_2\}$ and $\Omega_2 = \{\pi_1, \pi_2, \star, \bar{\star}\}$ one verifies immediately that $(\mathcal{R}, \Omega_i)$, $(i = 1, 2)$ is an entropic algebra.

Now let $a, b \in \mathcal{R}$ and write $\mu(a, b) := a \star b$. For the inductive step from n to $n + 1$ let $v, w \in \Omega_{n+1}$ and $v_1, v_2, w_1, w_2 \in \Omega_n$ such that

$$v(a, b) = \mu(v_1(a, b), v_2(a, b)), w(a, b) = \mu(w_1(a, b), w_2(a, b)).$$

With another $c, d \in \mathcal{R}$ we calculate using entropicity of μ

$$\begin{aligned} v(w(a, b), w(c, d)) &= \mu(v_1(\mu(w_1(a, b), w_2(a, b)), \mu(w_1(c, d), w_2(c, d))), \\ &\qquad v_2(\mu(w_1(a, b), w_2(a, b)), \mu(w_1(c, d), w_2(c, d)))) \\ &\underset{\text{IND}}{=} \mu(\mu(v_1(w_1(a, b), w_1(c, d)), v_1(w_2(a, b), w_2(c, d))), \\ &\qquad \mu(v_2(w_1(a, b), w_1(c, d)), v_2(w_2(a, b), w_2(c, d)))) \end{aligned}$$

$$
\begin{aligned}
&\underset{\text{IND}}{=} \mu(\mu(w_1(v_1(a,c),v_1(b,d)),w_2(v_1(a,c),v_1(b,d))),\\
&\qquad \mu(w_1(v_2(a,c),v_2(b,d)),w_2(v_2(a,c),v_2(b,d))))\\
&= \mu(\mu(w_1(v_1(a,c),v_1(b,d)),w_1(v_2(a,c),v_2(b,d))),\\
&\qquad \mu(w_2(v_1(a,c),v_1(b,d)),w_2(v_2(a,c),v_2(b,d))))\\
&\underset{\text{IND}}{=} \mu(w_1(\mu(v_1(a,c),v_2(a,c)),\mu(v_1(b,d),v_2(b,d))),\\
&\qquad w_2(\mu(v_1(a,c),v_2(a,c)),\mu(v_1(b,d),v_2(b,d))))\\
&= w(v(a,c),v(b,d)). \qquad \blacksquare
\end{aligned}
$$

If we denote by $*_w$ the binary operation on $\mathcal{R}^2$ induced by $\star_w$ ($w \in \Omega$) in a canonical manner, for an entropic groupoid $(\mathcal{R},\star)$ and another $v \in \Omega$, (3.2) means that

$$v(a \star_w b, c \star_w d) = v((a,c) *_w (b,d)) = v(a,c) \star_w v(b,d),$$

or in other words, $v : (\mathcal{R}^2, *_w) \to (\mathcal{R}, \star_w)$ is a homomorphism.

The second half of the subsequent theorem to a certain extent generalizes ([E4],(1.4)).

3.3. THEOREM. *For a SGRID-groupoid $(\mathcal{R},\bullet)$ and $w \in \Omega$, the groupoid $(\mathcal{R},\bullet_w)$ is a SGRID-groupoid with underlying group G and describing map given by $x \mapsto w(x,0)$, or in other words $(x,y \in \mathcal{R})$*

(i) $$w(x,y) = w(x,0) + w(0,y);$$

and as a consequence,

(ii) $$w(x,y) + w(y,x) = x + y.$$

With another $v \in \Omega$ we get general balancedness, i.e.

(iii) $$v(x,y) = w(x,y) \iff v(y,x) = w(y,x),$$

which yields for $n \in \mathbb{N}$

$$\underbrace{x \bullet \ldots \bullet x}_{n} \bullet y = y \iff \underbrace{y \bullet \ldots \bullet y}_{n} \bullet x = x,$$
$$\underbrace{x \bullet \ldots \bullet x}_{n} \bullet y = x \iff \underbrace{y \bullet \ldots \bullet y}_{n} \bullet x = y.$$

Proof. (i) By induction. For the canonical projections the statement is immediate. Now let w be a binary operation of length ≥ 2 and $w = w_1 \bullet w_2$, where $w_i \in \Omega$, $(i = 1,2)$. With $x,y \in \mathcal{R}$ we calculate using ([E4],(2.1))

$$\begin{aligned} w(x,y) &= w_1(x,y) \bullet w_2(x,y) \\ &= \sigma(w_1(x,y)) + \tau(w_2(x,y)) \\ &= w_1(\sigma(x),\sigma(y)) + w_2(\tau(x),\tau(y)) \\ &\underset{\text{IND}}{=} w_1(\sigma(x),\sigma(0)) + w_1(\sigma(0),\sigma(y)) + w_2(\tau(x),\tau(0)) + w_2(\tau(0),\tau(y)) \\ &= \sigma(w_1(x,0)) + \tau(w_2(x,0)) + \sigma(w_1(0,y)) + \tau(w_2(0,y)) \\ &= w_1(x,0) \bullet w_2(x,0) + w_1(0,y) \bullet w_2(0,y) \\ &= w(x,0) + w(0,y). \end{aligned}$$

(ii) $w(x,y) + w(y,x) \underset{\text{(i)}}{=} w(x,0) + w(0,x) + w(y,0) + w(0,y)$

$\underset{\text{(i)}}{=} w(x,x) + w(y,y) = x + y$ by idempotency.

(iii) By (ii) we deduce

$$w(x,y) - v(x,y) = v(y,x) - w(y,x),$$

from which we conclude the desired equivalence. ∎

We add the note that on a $SGRID$-groupoid $(\mathcal{R},\bullet)$, two elements $v, w \in \Omega$ define a congruence relation on $\mathcal{R}$ by

$$\mathfrak{G}_{v,w} := \{(x,y) \in \mathcal{R} \times \mathcal{R} |\ v(x,y) = w(x,y)\}.$$

These relations will be considered in a later paper in combination with uniform structures on $SGRID$-groupoids [E5].– The next result generalizes ([E2],(2.17)) and ([E1], Prop. 6).

3.4. THEOREM. *Let $(\mathcal{R},\bullet)$ be a $SGRID$-groupoid, $k \in \mathbb{N}$. The following are equivalent*

(1) $$\tau^k = \mathrm{id}_{\mathcal{R}},$$

(2) $$\forall\ x,y \in \mathcal{R} : \underbrace{x \bullet \ldots \bullet x}_{k} \bullet y = y \quad (k\text{-}symmetry).$$

k-symmetric $SGRID$-groupoids are canonically monoid splitting and $\mathcal{R}^{\perp}_{\succ} = \mathcal{R}^{\perp}_{\curlyvee} = \mathcal{R}^{\perp}_{\curlywedge}$. The relations $\mathfrak{M}$ and $\mathfrak{L}$ coincide, and $(\mathcal{R}/\mathfrak{L},\bullet)$ is a right zero band.

Proof. The equivalence of (1) and (2) follows by (2.3) and $\tau(0) = 0$. Since $\tau^k = \mathrm{id}_{\mathcal{R}}$ implies $\mathcal{R}^{\top} = \mathcal{R}$, and condition (K) is a consequence of the k-symmetry law, we get that $(\mathcal{R},\bullet)$ is canonically monoid splitting by ([E4],(2.5)), conclude $\mathcal{R}^{\perp}_{\succ} = \mathcal{R}^{\perp}_{\curlywedge} = \mathcal{R}^{\perp}_{\curlyvee}$ and $\mathfrak{L} = \mathfrak{M}$ by (2.10), and the last assertion by (2.8). ∎

By a simple calculation one shows immediately

3.5. PROPOSITION. *If $(\mathcal{R}, \star)$ is a groupoid satisfying the k-symmetric law, then $(\mathcal{R}, \star_{\psi_\ell})$ satisfies the $\frac{k}{gcd(k,\ell)}$-symmetric law.* ■

(3.5) together with (3.3) and (2.3) implies

3.6. PROPOSITION. *If $(\mathcal{R}, \bullet)$ is a k-symmetric $SGRID$-groupoid, then $(\mathcal{R}, \bullet_{\psi_\ell})$ is a $\frac{k}{gcd(k,\ell)}$-symmetric $SGRID$-groupoid with describing map τ^ℓ.* ■

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