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IDEMPOTENT AND DISTRIBUTIVE GROUP RELATED GROUPOIDS, I

Preliminaries

The present treatise forms the first half of a series of papers on groupoids, the binary operation of which is strongly influenced by the structure of an abelian group. The concept of group relatedness (1.1) has already been considered with respect to symmetric groupoids [E1]. The methods used there will by applied below to a more general situation, where the symmetry law and by parts also (left) distributivity is dropped.

Group related (symmetric) groupoids turned out to be of high value for the description of homotopy sets $[S^p \times S^q; S^n]$, $p, q, n \in \mathbb{N}$ (where S^m denotes the m–dimensional sphere for a natural m), which themselves are non group related symmetric groupoids with the binary operation induced by point reflection (cf. [E1],[E2]) on spheres.

Further examples of group related groupoids have been treated in ([S], chapter 3) and ([SS], chapter 3), where the binary operation is named a linear multiplication. Their significance is shown e.g. in ([SS],(3.9)), where the authors prove that for a locally compact connected abelian group, each multiplication (i.e. continuous binary operation) preserving the neutral element 0 is homotopic *rel* $\{(0,0)\}$ to a linear multiplication.

After some inevitable trivial observations in the beginning, in chapter 1 we are mainly concerned with group related groupoids merely satisfying the idempotency law. It is a consequence of an analysis of the situation given in [E1] that in particular groupoids, which contain a whole monoid, are suitable objects for investigation. (This standpoint comes clear the more in the middle of chapter 2, where parallels of [E1] and our present subjects of interest are pointed out.) The philosophy above is closely related to the notion of partial homomorphisms on subsets of abelian groups, which prove to be a useful tool throughout the whole paper; and both together lead to the concept of a monoid splitting group related idempotent groupoid. For such groupoids we can form quotients and extended groupoids, which inherit algebraic identities as well as group relatedness ((1.18),(1.19) and also (2.2)).

In chapter 2, to idempotency we add one sided distributivity and get plenty of equivalent conditions (2.1), among others entropicity. Therefore, group related idempotent distributive groupoids in particular are groupoid modes [RS], i.e. groupoids with an idempotent and entropic binary operation. In (2.3),(c), as an answer to a question posed by A. Romanowska, we provide an example of a group related groupoid mode not being isomorphic to a subreduct of an affine space. The respective variety of algebras is of central interest in ([RS], section 2.5). Moreover, (2.1) forms a certain contrast to ([JK], (3.3.9)), where it is shown that each idempotent entropic groupoid is related (in a sense analogous to (1.1)) to a commutative semigroup with a pair (f, g) of semigroup automorphisms as describing mappings and $f(x)+g(x) = x$, where x denotes an element of the semigroup.– In the second part of the chapter we point out circumstances, under which group related idempotent distributive groupoids are monoid splitting in a canonical sense and discuss several aspects around this property.

The last chapter is dedicated to the description of the structure of canonically monoid splitting group related idempotent distributive groupoids from various points of view.

Notation and terminology. Different from common usage, for a groupoid $(\mathcal{R}, \star)$, a subset $R \subseteq \mathcal{R} \times \mathcal{R}$ is called a congruence, if R is a subgroupoid of $(\mathcal{R}, \star) \times (\mathcal{R}, \star)$.

By the symbol $\circ\!<$ we denote the property of being a submonoid.

In order to simplify notation, we sometimes omit operation symbols and denote algebras only by their underlying sets. For instance, we speak of an abelian group G instead of writing $(G, +)$.

1. Idempotent groupoids

1.1. DEFINITION. Let $(G, +)$ be an abelian group, $\mathcal{R} \subseteq G$ and $\bullet : \mathcal{R}\times\mathcal{R} \rightarrow \mathcal{R}$ a binary operation. The groupoid $(\mathcal{R}, \bullet)$ is called a groupoid related to G, or simply group related without specifying the respective group, if there is a pair (σ, τ) of mappings from $\mathcal{R}$ to G such that

$$\forall\, x, y \in \mathcal{R} : x \bullet y = \sigma(x) + \tau(y).$$

We call σ and τ mappings describing the binary operation $\bullet$, and $(G, +)$ an underlying group of $(\mathcal{R}, \bullet)$.

The following point of view plays a role later in chapter 2 and in a subsequent paper [E3].

1.2. Remark. For a groupoid $(\mathcal{R},\bullet)$ we define $\bar{\bullet}:\mathcal{R}\times\mathcal{R}\to\mathcal{R}$ assigning $(a,b)\mapsto b\bullet a$, and call $(\mathcal{R},\bar{\bullet})$ the dual or opposite groupoid. If $(\mathcal{R},\bullet)$ is group related with a pair (σ,τ) of describing mappings, then $(\mathcal{R},\bar{\bullet})$ is also group related with the pair $(\bar{\sigma},\bar{\tau})$ of describing mappings, where $\bar{\sigma}=\tau,\bar{\tau}=\sigma$.

Throughout this paper, for an abelian group $(G,+)$ and $h\in G$, by $d_h:G\to G$ we denote translation by h. By means of translations we can show that without loss of generality, for group related groupoids $\mathcal{R}$ we can always assume $0\in\mathcal{R}$, where 0 denotes the neutral element of its respective underlying abelian group, as well as some properties for the pair of describing mappings. This is shown by

1.3. PROPOSITION. *Let $(\mathcal{R},\bullet)$ be related to $(G,+)$ with a pair (σ,τ) of describing mappings.*

(a) *Fix $g\in G$ and put $\mathcal{R}':=\mathcal{R}+g$. By means of the bijection $d'_{-g}:\mathcal{R}'\to\mathcal{R}$ (where d'_{-g} denotes the restriction of d_{-g} in both domain and range) we get a binary operation $\bullet'$ on $\mathcal{R}'$ assigning*

$$(x,y)\mapsto {d'}^{-1}_{-g}(d'_{-g}(x)\bullet d'_{-g}(y)).$$

Clearly, d'_{-g} is an isomorphism between the groupoids $(\mathcal{R},\bullet)$ and $(\mathcal{R}',\bullet')$, and $(\mathcal{R}',\bullet')$ is related to $(G,+)$.

(b) *For any $g\in G$, the pair $(d_g\circ\sigma,d_{-g}\circ\tau)$ describes $\bullet$ as well. If $0\in\mathcal{R}$, there is a describing pair (σ_1,τ_1) (resp. (σ_2,τ_2)) such that $\sigma_1(0)=0$, $\tau_1(\mathcal{R})\subseteq\mathcal{R}$ (or $\tau_2(0)=0$, $\sigma_2(\mathcal{R})\subseteq\mathcal{R}$, respectively).*

Proof. (a) We get describing maps for $\bullet'$ by

$$\sigma':\mathcal{R}'\to G,\quad x\mapsto\sigma(x-g)+g,\qquad \tau':\mathcal{R}'\to G,\quad x\mapsto\tau(x-g).$$

(b) The statement concerning $(d_g\circ\sigma,d_{-g}\circ\tau)$ is obvious. Now let $0\in\mathcal{R}$, put $h:=\sigma(0)$, and $\sigma_1:=d_{-h}\circ\sigma$, $\tau_1:=d_h\circ\tau$. Then $\sigma_1(0)=0$, and $\tau_1(x)=\sigma_1(0)+\tau_1(x)=0\bullet x$ for $x\in\mathcal{R}$, hence $\tau_1(\mathcal{R})\subseteq\mathcal{R}$. In a similar way we prove the existence of σ_2 and τ_2. ∎

1.4. PROPOSITION. *Let $(\mathcal{R},\bullet)$ be related to $(G,+)$ with describing mappings $\sigma,\tau:\mathcal{R}\to G$. The following are equivalent:*

(1) $\forall\, x\in\mathcal{R}: x\bullet x=x$ *(idempotency).*

(2) $\sigma=\mathrm{id}'_{\mathcal{R}}-\tau$,

where $\mathrm{id}'_{\mathcal{R}}$ *denotes the canonical injection from $\mathcal{R}$ to G.* (1) *or* (2) *implies*

(3) $\forall\, x,y\in\mathcal{R}: x\bullet y+y\bullet x=x+y$,

and from (3) *follows balancedness, i.e.*

$$\text{(B)} \qquad \forall\, x, y \in \mathcal{R} : x \bullet y = y \iff y \bullet x = x.$$

Moreover, if G has no elements of order 2, (1), (2) *and* (3) *are equivalent.*

Proof. Assuming that there is no element of order 2 in G, starting from (3) we obtain $2(x - \tau(x) - \sigma(x)) = 0$, hence (2) holds. All other statements can be proved by simple calculations similar to the above. ∎

Because of (1.4), instead of a pair (σ, τ), one mapping is enough in order to describe an idempotent group related groupoid $(\mathcal{R}, \bullet)$. In this situation we prefer to call τ the describing map of $(\mathcal{R}, \bullet)$. Using the symbol σ with respect to a given idempotent group related groupoid $(\mathcal{R}, \bullet)$, in the sequel we always mean the mapping defined by $\mathrm{id}'_{\mathcal{R}} - \tau$.

1.5. PROPOSITION/DEFINITION. *Let $(\mathcal{R}', \bullet')$ be an idempotent groupoid which is related to $(G, +)$. Then there is a group related idempotent groupoid $(\mathcal{R}, \bullet)$, isomorphic to $(\mathcal{R}', \bullet')$, with a unique map τ describing $\bullet$ such that*

$$(\star) \qquad 0 \in \mathcal{R}, \tau(0) = 0,$$

and $\tau(\mathcal{R}) \subseteq \mathcal{R}$ as well as $\sigma(\mathcal{R}) \subseteq \mathcal{R}$, $\sigma(0) = 0$ as a consequence. Group related idempotent groupoids $(\mathcal{R}, \bullet)$ with describing map τ satisfying $(\star)$ are called strictly group related idempotent groupoids (or $SGRI$-groupoids for short) with underlying group $(G, +)$.

Proof. As in (1.3),(a), via translation by $-g$ $(g \in \mathcal{R}')$ we get a groupoid $(\mathcal{R}, \bullet)$ with $0 \in \mathcal{R}$, which is isomorphic to $(\mathcal{R}', \bullet')$. Applying (1.3),(b) we find a describing map τ for $\bullet$ satisfying $\tau(0) = 0$, hence $\sigma(0) = 0$ by (1.4). Moreover, $\tau(x) = 0 \bullet x$ resp. $\sigma(x) = x \bullet 0$ for $x \in \mathcal{R}$ yield $\tau(\mathcal{R}) \subseteq \mathcal{R}$ and $\sigma(\mathcal{R}) \subseteq \mathcal{R}$.

Now let $\tau' : \mathcal{R} \to G$ be another describing map with $\tau'(0) = 0$. By $x - \tau(x) + \tau(y) = x - \tau'(x) + \tau'(y)$ $(x, y \in \mathcal{R})$ and $\tau(0) = 0 = \tau'(0)$ we conclude $\tau = \tau'$. ∎

Because of (1.5), the mappings σ, τ belonging to a $SGRI$-groupoid $(\mathcal{R}, \bullet)$ can as well be interpreted as mappings with range $\mathcal{R}$. This point of view will play a role in later parts of the paper, whenever compositions with these mappings are considered. In order to avoid cumbersome formulation, we agree to the following

CONVENTION. Troughout the whole paper, if not specified otherwise, whenever a $SGRI$-groupoid is written by $(\mathcal{R}, \bullet)$, we always denote the respective underlying group by $(G, +)$ and its describing map by τ.

The property of being a $SGRI$-groupoid is transferred to subgroupoids and products in a canonical manner.

1.6. Remark. (a) Let $(\mathcal{R}, \bullet)$ be a $SGRI$–groupoid, $0 \in \mathcal{S} \subseteq \mathcal{R}$, and $(\mathcal{S}, \bullet)$ a subgroupoid. Then $(\mathcal{S}, \bullet)$ is a $SGRI$–groupoid (with G as underlying group and $\tau|\mathcal{S}$ as describing map).

(b) Given an index set I and $\iota \in I$, let $(\mathcal{R}_\iota, \bullet_\iota)$ be a $SGRI$–groupoid with underlying group G_ι and describing map τ_ι. Then $\prod_{\iota\in I}\mathcal{R}_\iota$ is a $SGRI$–groupoid with underlying group $\prod_{\iota\in I} G_\iota$ and describing map given by the assignment

$$\prod_{\iota\in I}\mathcal{R}_\iota \to \prod_{\iota\in I} G_\iota, \quad (r_\iota)_{\iota\in I} \mapsto (\tau_\iota(r_\iota))_{\iota\in I}.$$

The transfer to quotients is discussed later in (1.18).

1.7. Remark. Let $(\mathcal{R}, \bullet)$ be a $SGRI$–groupoid, and denote by $Z(\mathcal{R}) := \{x \in \mathcal{R} \mid \forall\, y \in \mathcal{R} : x \bullet y = y \bullet x\}$ the centre of $\mathcal{R}$. The following are equivalent:

(a) $(\mathcal{R}, \bullet)$ is commutative.

(b) $Z(\mathcal{R}) \neq \emptyset$.

(c) $\forall\, x \in \mathcal{R} : 2\tau(x) = x$.

In particular, commutative $SGRI$–groupoids consist only of elements which can be halved in the underlying group.

Proof. (a) $\Rightarrow$ (b) is obvious.

(b) $\Rightarrow$ (c). Since $Z(\mathcal{R}) \neq \emptyset$, there is $x \in \mathcal{R}$ such that for all $y \in \mathcal{R}$ holds

$$x - \tau(x) + \tau(y) = y - \tau(y) + \tau(x);$$

so for $y := 0$ we get $2\tau(x) = x$. Inserting this into the above equation yields (c).

(c) $\Rightarrow$ (a). By calculating

$$x \bullet y = x - \tau(x) + \tau(y) = \tau(x) + \tau(y) = y - \tau(y) + \tau(x) = y \bullet x. \qquad \blacksquare$$

It can easily be seen that commutative $SGRI$–groupoids come under the concept discussed later in (2.1).

1.8. Definition. Let $(G, +)$ be an abelian group, $B \subseteq G$, write $0 \cdot B := 0$, and $m \cdot B := B + (m-1)B$ for $m \in \mathbb{N}$. Using $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ we denote by

$B_\succ := \bigcup_{n\in\mathbb{N}_0} n \cdot B$ *the submonoid of G generated by B,*

$B_\curlywedge := B_\succ + (-B_\succ)$ *the subgroup of G generated by B resp. $B_\succ$,*

$B_\curlyvee := B_\succ \cap (-B_\succ)$ *the maximal subgroup contained in $B_\succ$.*

1.9. Proposition/Definition. *For an abelian group $(G,+)$, let $B, A \subseteq G$. We put*

$$R_B^A := \{(x,y) \in A \times A \mid y - x \in B\},$$
$$R_B^A[x] := \{y \in A \mid (x,y) \in R_B^A\} \quad (x \in A),$$
$$A/B := \{R_B^A[x] \mid x \in A\}.$$

If $(B,+)$ is a submonoid of $(G,+)$, the relation R_B^A is a reflexive and transitive relation on A. If B forms a group, R_B^A is an equivalence relation. For a monoid B and $A + B \subseteq A$ we have $R_B^A[x] = x + B$ for $x \in A$, and get a bijective map $c : A/B \to A/B_\curlyvee$ by the assignment $x + B \mapsto x + B_\curlyvee$, or in other words, $x + B = y + B \iff x + B_\curlyvee = y + B_\curlyvee$ for all $x, y \in A$. If A is also a monoid, the relation R_B^A is a congruence w.r.t. $+$, and A/B carries the monoid structure which is canonically given by $A/B_\curlyvee$.

Proof. Let B be a monoid. Then $\Delta_A \subseteq R_B^A$, since $0 \in B$, and for $(x,y),(y,z) \in R_B^A$ there is $m, n \in B$ such that $y - x = m$, $z - y = n$, consequently, $z - x = m + n \in B$, thus $(z,x) \in R_B^A$, and R_B^A is shown to be transitive. For a group B, reflexivity of R_B^A is a consequence of $-(y - x) \in B \iff x - y \in B$. The inclusion $R_B^A[x] \subseteq x + B$ always holds. Assuming $A + B \subseteq A$ we get $x + B \subseteq A + B \subseteq A$ for $x \in A$, hence $x + B \subseteq R_B^A[x]$. For $x, x' \in A$ we have

$$\begin{aligned} x + B = x' + B &\iff \exists\, m, m' \in B : x = x' + m, x' = x + m' \\ &\iff x - x', x' - x \in B \\ &\iff x - x' \in B_\curlyvee, \end{aligned}$$

which shows that c is well defined and injective; the surjectivity of c is immediate. For a monoid A and $(x,y),(x',y') \in R_B^A$, clearly $x+x', y+y' \in A$ and $y-x+y'-x' \in B$; hence $(x+x', y+y') \in R_B^A$, which shows that R_B^A is a congruence. Finally, the binary operation on A/B, given by the assignment

$$(x + B, y + B) \mapsto (x + y) + B$$

coincides with the one defined by $c^{-1}(c(x + B) + c(y + B))$. ∎

The following assertions are proved straightforward.

1.10. Proposition/Definition. *Let $(G,+),(H,+)$ be abelian groups, $\emptyset \neq A, B \subseteq C \subseteq G$ such that $A + B \subseteq C$. A mapping $\varphi : C \to H$ is called a partial homomorphism with respect to A and B, if*

$$\forall\, a \in A\ \forall\, b \in B : \varphi(a + b) = \varphi(a) + \varphi(b).$$

The set $\mathrm{Part}_{A,B}(C,H)$ of partial homomorphisms from C to H w.r.t. A and B forms an abelian group with component wise defined addition and *the constant map to* $0 \in H$ *as neutral element. If* $G = H$, *then* $\mathrm{id}'_C - \varphi \in \mathrm{Part}_{A,B}(C,G)$ *for* $\varphi \in \mathrm{Part}_{A,B}(C,G)$, *where* id'_C *denotes the canonical injection from* C *to* G. ∎

1.11. PROPOSITION. *Let* G *be an abelian group and* $A, B \subseteq G$. *If* $A+B \subseteq A$, *then* $A + B_{\succ} \subseteq A$; *and* $0 \in A$ *implies* $B_{\succ} \subseteq A$.

Proof. Trivially $A + 0 \subseteq A$, and inductively one can show $A + nB \subseteq A$ for all $n \in \mathbb{N}$:

$$A + (n+1)B = A + B + nB \subseteq A + nB \underset{\mathrm{IND}}{\subseteq} A.$$

Therefore, $A + \bigcup_{n \in \mathbb{N}_0} nB = A + B_{\succ} \subseteq A$. ∎

1.12. Remark. Let G, H be abelian groups, $\emptyset \neq A, B \subseteq C \subseteq G$, $0 \in A \cup B$, $A + B \subseteq C$ and $\varphi \in \mathrm{Part}_{A,B}(C,H)$. Then $\varphi(0) = 0$.

Proof. If $0 \in A$, for $b \in B$ we conclude

$$\varphi(b) = \varphi(0+b) = \varphi(0) + \varphi(b) \Rightarrow \varphi(0) = 0. \ ∎$$

1.13. PROPOSITION. *For abelian groups* G, H, *let* $\emptyset \neq A, B \subseteq C \subseteq G$, $A + B \subseteq A$ *and* $\varphi \in \mathrm{Part}_{A,B}(C,H)$. *If* $0 \in A$, *then* $\varphi \in \mathrm{Part}_{A,B_{\succ}}(C,H)$, *and* $\varphi | B_{\succ}$ *is a monoid homomorphism.*

Proof. Let $a \in A$, $0 \neq b \in B_{\succ}$, i.e. there exists $k \in \mathbb{N}$ with $b = b_1 + \ldots + b_k$, $(b_1, \ldots b_k \in B)$.

$$\begin{aligned}
&\varphi(\underbrace{a + b_1 + \ldots + b_{k-1}}_{\in A + B_{\succ} \subseteq A} + b_k) \\
&\quad = \varphi(a + b_1 + \ldots + b_{k-1}) + \varphi(b_k) \quad \text{since } \varphi \in \mathrm{Part}_{A,B}(C,G), \\
&\quad \underset{\mathrm{IND}}{=} \varphi(a) + \varphi(b_1) + \ldots + \varphi(b_k) \\
&\quad \underset{B_{\succ} \subseteq A}{=} \varphi(a) + \varphi(b_1 + b_2) + \ldots + \varphi(b_k) \\
&\quad \underset{\mathrm{IND}}{=} \varphi(a) + \varphi(b_1 + \ldots + b_k).
\end{aligned}$$

For $a = 0$ the above calculation shows that $\varphi | B_{\succ}$ is a monoid homomorphism. ∎

1.14. Remark. For abelian groups G, H, let $\emptyset \neq B \subseteq A \subseteq G$, $0 \in A$. Then the following are equivalent:

(i) $A + B_{\succ} \subseteq A$, $\varphi \in \mathrm{Part}_{A,B_{\succ}}(A,H)$.
(ii) $A + B \subseteq A$, $\varphi \in \mathrm{Part}_{A,B}(A,H)$.

Proof. (ii) $\Rightarrow$ (i) is implied by (1.11) and (1.13), and (i) $\Rightarrow$ (ii) is obvious. ∎

1.15. PROPOSITION. *Let* $(G,+),(H,+)$ *be abelian groups,* $0 \in A \subseteq G$, *and* $B \circ\!< G$ *be a submonoid such that* $A+B \subseteq A$. *For* $A^{\curlywedge} := A+B_{\curlywedge}$ *holds the inclusion* $A^{\curlywedge}+B_{\curlywedge} \subseteq A^{\curlywedge}$ *and* $x+B \cap y+B \neq \emptyset \iff x+B_{\curlywedge} = y+B_{\curlywedge}$ *for all* $x,y \in A$. *Moreover,* $\varphi \in \mathrm{Part}_{A,B}(A,H)$ *has a unique extension* $\varphi_{\curlywedge} \in \mathrm{Part}_{A^{\curlywedge},B_{\curlywedge}}(A^{\curlywedge},H)$.

Proof. Clearly, $A^{\curlywedge}+B_{\curlywedge} \subseteq A^{\curlywedge}$, and the equivalence follows by an easy calculation. As for the extension of φ, let $x \in A^{\curlywedge}$, $x = r+a-b$, $r \in A$, $a,b \in B$, and put

$$\varphi_{\curlywedge}(x) := \varphi(r)+\varphi(a)-\varphi(b).$$

In order to show that $\varphi_{\curlywedge}$ is well defined, let $s \in A$, $a',b' \in B$ and $x = r+a-b = s+a'-b'$. Then

$$\begin{aligned} r+a+b' &= s+a'+b \Rightarrow \\ \varphi(r+a+b') &= \varphi(s+a'+b) \Rightarrow \\ \varphi(r)+\varphi(a)+\varphi(b') &= \varphi(s)+\varphi(a')+\varphi(b) \Rightarrow \\ \varphi(r)+\varphi(a)-\varphi(b) &= \varphi(s)+\varphi(a')-\varphi(b') = \varphi_{\curlywedge}(x). \end{aligned}$$

$\varphi_{\curlywedge} \in \mathrm{Part}_{A^{\curlywedge},B_{\curlywedge}}(A^{\curlywedge},G)$, since for $x \in A^{\curlywedge}$, $h \in B_{\curlywedge}$ and $r \in A$, $a,a',b,b' \in B$ with $x = r+a-b$ and $h = a'-b'$ we get

$$\begin{aligned} \varphi_{\curlywedge}(x+h) &= \varphi_{\curlywedge}(r+(a+a')-(b+b')) = \varphi(r)+\varphi(a+a')-\varphi(b+b') \\ &= \varphi(r)+\varphi(a)-\varphi(b)+\varphi(a')-\varphi(b') = \varphi_{\curlywedge}(x)+\varphi_{\curlywedge}(h). \end{aligned}$$

Now let $\psi \in \mathrm{Part}_{A^{\curlywedge},B_{\curlywedge}}(A^{\curlywedge},G)$, $\psi|A = \varphi$. In particular, $\psi|B_{\curlywedge}$ is a group homomorphism, and for $x \in A^{\curlywedge}$, $a,b \in B$, $r \in A$ such that $x = r+a-b$ we get

$$\begin{aligned} \psi(r+a-b) &= \psi(r)+\psi(a-b) = \psi(r)+\psi(a)-\psi(b) \\ &= \varphi(r)+\varphi(a)-\varphi(b) = \varphi_{\curlywedge}(x). \end{aligned}$$

■

1.16. DEFINITION. For a *SGRI*-groupoid $(\mathcal{R},\bullet)$ we agree to write

$$\mathcal{R}^{\top} := \tau(\mathcal{R}), \quad \mathcal{R}^{\perp} := \sigma(\mathcal{R}).$$

Obviously, $\mathcal{R} = \mathcal{R}^{\top}+\mathcal{R}^{\perp}$.

1.17. DEFINITION/REMARK. Let $(\mathcal{R},\bullet)$ be a *SGRI*-groupoid and $\mathcal{N} \circ\!< G$. The groupoid $(\mathcal{R},\bullet)$ is said to be $\mathcal{N}$–splitting, or monoid splitting with respect to $\mathcal{N}$, if

$$\mathcal{R}+\mathcal{N} \subseteq \mathcal{R}, \quad \tau \in \mathrm{Part}_{\mathcal{R},\mathcal{N}}(\mathcal{R},G).$$

Clearly, if $(\mathcal{R},\bullet)$ is $\mathcal{N}$–splitting, $\tau|\mathcal{N}$ is a monoid homomorphism (cf. (1.13)). $(\mathcal{R},\bullet)$ is called canonically monoid splitting, if it is monoid splitting with respect to $\mathcal{R}^{\perp}_{\succ}$.†

† Since no ambiguity can arise, we write $\mathcal{R}^{\perp}_{\succ}$ instead of $(\mathcal{R}^{\perp})_{\succ}$.

1.18. THEOREM. *Let $(\mathcal{R}, \bullet)$ be a SGRI-groupoid which is $\mathcal{N}$-splitting for $\mathcal{N} \circ\!< G$. Then we have the formula*

$$\forall\ r, s \in \mathcal{R}\ \forall\ p, q \in \mathcal{N} : (r + p) \bullet (s + q) = r \bullet s + p \bullet q.$$

If $(\mathcal{N}, \bullet)$ is a subgroupoid (thus a SGRI-groupoid by virtue of (1.6)), then $(R^{\mathcal{R}}_{\mathcal{N}}[x], \bullet)$ is an idempotent subgroupoid for all $x \in \mathcal{R}$, the relation $R^{\mathcal{R}}_{\mathcal{N}}$ is a congruence w.r.t. $\bullet$, by the assignment

$$\mathcal{R}/\mathcal{N} \times \mathcal{R}/\mathcal{N} \ni (x + \mathcal{N}, y + \mathcal{N}) \mapsto x \bullet y + \mathcal{N} \in \mathcal{R}/\mathcal{N}$$

we get a binary operation $\blacksquare$ on $\mathcal{R}/\mathcal{N}$, and the canonical projection

$$q_{\mathcal{N}} : \mathcal{R} \to \mathcal{R}/\mathcal{N}, \quad x \mapsto R^{\mathcal{R}}_{\mathcal{N}}[x] = x + \mathcal{N},$$

is a homomorphism w.r.t. $\bullet$ and $\blacksquare$. The groupoid $(\mathcal{R}/\mathcal{N}, \blacksquare)$ is a SGRI-groupoid with underlying group $G/\mathcal{N}$, and describing map given by

$$\tau_{\mathcal{N}} : \mathcal{R}/\mathcal{N} \to G/\mathcal{N}, \quad r + \mathcal{N} \mapsto \tau(r) + \mathcal{N}.$$

Proof. For $p, q \in \mathcal{N}$, $r, s \in \mathcal{R}$ we calculate

$$\begin{aligned}(r + p) \bullet (s + q) &= r + p - \tau(r + p) + \tau(s + q)\\ &= r + p - \tau(r) - \tau(p) + \tau(s) + \tau(q) \quad \text{for } \tau \in \mathrm{Part}_{\mathcal{R},\mathcal{N}}(\mathcal{R}, G),\\ &= r - \tau(r) + \tau(s) + p - \tau(p) + \tau(q)\\ &= r \bullet s + p \bullet q.\end{aligned}$$

By this formula we conclude for $x, y \in \mathcal{R}$ and a subgroupoid $(\mathcal{N}, \bullet)$

$$x \bullet y + \mathcal{N} = x \bullet y + \mathcal{N} \bullet \mathcal{N} = (x + \mathcal{N}) \bullet (y + \mathcal{N}),$$

hence $(R^{\mathcal{R}}_{\mathcal{N}}[x], \bullet)$ is a subgroupoid, $R^{\mathcal{R}}_{\mathcal{N}}$ is a congruence, and trivially, $q_{\mathcal{N}}$ is a homomorphism; thus $(\mathcal{R}/\mathcal{N}, \blacksquare)$ is idempotent, since $(\mathcal{R}, \bullet)$ is an idempotent groupoid. The mapping $\tau_{\mathcal{N}}$ is well defined, since for $x, x' \in \mathcal{R}$ we get

$$\begin{aligned}x + \mathcal{N} = x' + \mathcal{N} &\iff x + \mathcal{N}_{\Upsilon} = x' + \mathcal{N}_{\Upsilon}\\ &\Rightarrow \tau(x + \mathcal{N}_{\Upsilon}) = \tau(x' + \mathcal{N}_{\Upsilon})\\ &\Rightarrow \tau(x) + \tau(\mathcal{N}_{\Upsilon}) = \tau(x') + \tau(\mathcal{N}_{\Upsilon}), \quad \text{since } \tau \in \mathrm{Part}_{\mathcal{R},\mathcal{N}}(\mathcal{R}, G),\\ &\Rightarrow \tau(x) + \mathcal{N}_{\Upsilon} = \tau(x') + \mathcal{N}_{\Upsilon}, \quad \text{since } \tau(\mathcal{N}_{\Upsilon}) < \mathcal{N}_{\Upsilon},\\ &\iff \tau(x) + \mathcal{N} = \tau(x') + \mathcal{N}.\end{aligned}$$

$\tau(0) = 0$ implies $\tau_{\mathcal{N}}(\mathcal{N}) = \mathcal{N}$, and $\tau_{\mathcal{N}}$ describes $\blacksquare$, since

$$\begin{aligned}(x+\mathcal{N})\blacksquare(y+\mathcal{N}) &= x\bullet y+\mathcal{N}\\ &= x-\tau(x)+\tau(y)+\mathcal{N}\\ &= (x+\mathcal{N})+(-\tau(x)+\mathcal{N})+(\tau(y)+\mathcal{N})\\ &= (x+\mathcal{N})-\tau_{\mathcal{N}}(x+\mathcal{N})+\tau_{\mathcal{N}}(y+\mathcal{N}),\end{aligned}$$

by (1.9), for $G/\mathcal{N}$ inherits the group structure from $G/\mathcal{N}_{\Upsilon}$. ∎

1.19. PROPOSITION/DEFINITION. *Let $(\mathcal{R},\bullet)$ be a SGRI-groupoid which is $\mathcal{N}$-splitting for $\mathcal{N} \circ\!< G$, and suppose $\tau(\mathcal{N}) \circ\!< \mathcal{N}$. For $\mathcal{R}^{\curlywedge} := \mathcal{R}+\mathcal{N}_{\curlywedge}$, denote by $\tau_{\curlywedge}$ the extension of τ given by (1.15). Then $(\mathcal{R}^{\curlywedge},\bullet_{\curlywedge})$ with*

$$\bullet_{\curlywedge} : \mathcal{R}^{\curlywedge}\times\mathcal{R}^{\curlywedge}\to\mathcal{R}^{\curlywedge},\quad (r,s)\mapsto r-\tau_{\curlywedge}(r)+\tau_{\curlywedge}(s),$$

is a SGRI-groupoid (with G as underlying group and describing map $\tau_{\curlywedge}$) which is $\mathcal{N}_{\curlywedge}$-splitting, and $\tau_{\curlywedge}(\mathcal{N}_{\curlywedge}) < \mathcal{N}_{\curlywedge}$. We call $(\mathcal{R}^{\curlywedge},\bullet_{\curlywedge})$ the extended groupoid of $(\mathcal{R},\bullet)$ by $\mathcal{N}$.

Proof. $\mathcal{R}^{\curlywedge}\bullet\mathcal{R}^{\curlywedge}\subseteq\mathcal{R}^{\curlywedge}$, since for $r,s\in\mathcal{R}$, $a,a',b,b'\in\mathcal{N}$ and $x := r+a-b$, $y := s+a'-b'$ holds

$$\begin{aligned}x\bullet_{\curlywedge} y &= r+a-b-(\tau(r)+\tau(a)-\tau(b))+\tau(s)+\tau(a')-\tau(b')\\ &= \underbrace{r-\tau(r)+\tau(s)}_{\in\mathcal{R}}+\underbrace{a+\tau(b)+\tau(a')}_{\in\mathcal{N}}-(\underbrace{b+\tau(a)+\tau(b')}_{\in\mathcal{N}}).\end{aligned}$$

By $\tau_{\curlywedge}(0)=\tau(0)=0$ follows that $(\mathcal{R}^{\curlywedge},\bullet_{\curlywedge})$ is a *SGRI*-groupoid. (1.15) implies that $(\mathcal{R}^{\curlywedge},\bullet_{\curlywedge})$ is $\mathcal{N}_{\curlywedge}$-splitting, and $\tau_{\curlywedge}(\mathcal{N}_{\curlywedge}) < \mathcal{N}_{\curlywedge}$ is a consequence of $\tau(\mathcal{N}) \circ\!< \mathcal{N}$ and $\tau_{\curlywedge}$ being a group homomorphism (cf.(1.17)). ∎

2. Idempotent and distributive groupoids

For groupoid modes [RS] it is immediate that idempotency and entropicity imply both left and right distributivity. On the other hand, idempotent and even from both sides distributive groupoids are not necessarily entropic. In case of *SGRI*-groupoids, entropicity turns out to be equivalent to left and right distributivity.

2.1. THEOREM/DEFINITION. *For a SGRI-groupoid $(\mathcal{R},\bullet)$ the following are equivalent:*

(LD) $\forall\, x,y,z\in\mathcal{R} : (x\bullet y)\bullet(x\bullet z) = x\bullet(y\bullet z)$ (*left distributivity*),
(RD) $\forall\, x,y,z\in\mathcal{R} : (x\bullet z)\bullet(y\bullet z) = (x\bullet y)\bullet z$ (*right distributivity*),
(E) $\forall\, w,x,y,z\in\mathcal{R} : (x\bullet y)\bullet(w\bullet z) = (x\bullet w)\bullet(y\bullet z)$ (*entropicity*),
(Part′) $\sigma\in \mathrm{Part}_{\mathcal{R}^{\top},\mathcal{R}^{\perp}}(\mathcal{R},G)$,
(Part) $\tau\in \mathrm{Part}_{\mathcal{R}^{\top},\mathcal{R}^{\perp}}(\mathcal{R},G)$,
(Hom′) $\forall\, x,y\in\mathcal{R} : \sigma(x\bullet y) = \sigma(x)\bullet\sigma(y)$,
(Hom) $\forall\, x,y\in\mathcal{R} : \tau(x\bullet y) = \tau(x)\bullet\tau(y)$,

where (Hom) *can also be expressed by the formula*

$$\forall\ x, y \in \mathcal{R} : \tau(x - \tau(x) + \tau(y)) = \tau(x) - \tau^2(x) + \tau^2(y).$$

If one of the above conditions is satisfied, $\mathcal{R}$ *is flexible, i.e.*

$$\text{(F)} \qquad \forall\ x, y \in \mathcal{R} : (x \bullet y) \bullet x = x \bullet (y \bullet x),$$

and for all $x \in \mathcal{R}$ *holds* $\sigma(\tau(x)) = \tau(\sigma(x))$, *or equivalently*, $\tau(x) - \tau^2(x) = \tau(x - \tau(x))$. *We call a SGRI-groupoid* $(\mathcal{R}, \bullet)$, *satisfying one of the equivalent conditions above, a strictly group related idempotent distributive groupoid* (*or SGRID-groupoid for short*).

Proof. (Hom) $\Rightarrow$ (E). By calculation, making use of the formula below (Hom).

(E) $\Rightarrow$ (RD). Put $w = z$ in (E). Then

$$(x \bullet y) \bullet z = (x \bullet y) \bullet (z \bullet z) = (x \bullet z) \bullet (y \bullet z).$$

(RD) $\Rightarrow$ (Hom). From right distributivity we get the equation $(x, y, z \in \mathcal{R})$

$$\tau(y - \tau(y) + \tau(z)) = \tau(y) - \tau(x - \tau(x) + \tau(y)) + \tau(x - \tau(x) + \tau(z)),$$

which yields for $x := 0$

$$\tau(y \bullet z) = \tau(y) - \tau^2(y) + \tau^2(z) = \tau(y) \bullet \tau(z).$$

(LD) $\iff$ (Hom) is shown by calculations similar to the above.

(Hom) $\Rightarrow$ (Part). (Hom) implies

$$\forall\ x, y \in \mathcal{R} : \tau(x - \tau(x) + \tau(y)) = \tau(x - \tau(x)) + \tau^2(y),$$

thus $\tau \in \mathrm{Part}_{\mathcal{R}^\top, \mathcal{R}^\perp}(\mathcal{R}, G)$.

(Part) $\Rightarrow$ (Part'). Let $r \in \mathcal{R}$, $m \in \mathcal{R}^\perp$. Then

$$\begin{aligned}\sigma(\tau(r) + m) &= \tau(r) + m - \tau(\tau(r) + m)\\ &= \tau(r) + m - \tau^2(r) - \tau(m), \quad \text{since } \tau \in \mathrm{Part}_{\mathcal{R}^\top, \mathcal{R}^\perp}(\mathcal{R}, G)\\ &= \tau(r) - \tau^2(r) + m - \tau(m)\\ &= \sigma(\tau(r)) + \sigma(m).\end{aligned}$$

(Part') $\Rightarrow$ (Hom'). First we note that for $\sigma \in \mathrm{Part}_{\mathcal{R}^\top, \mathcal{R}^\perp}(\mathcal{R}, G)$ and $r \in \mathcal{R}$ holds

$$\sigma(r) = \sigma(\underbrace{r - \sigma(r)}_{=\tau(r)} + \sigma(r)) = \sigma(r - \sigma(r)) + \sigma^2(r),$$

hence $\tau(\sigma(r)) = \sigma(r) - \sigma^2(r) = \sigma(r - \sigma(r)) = \sigma(\tau(r))$, and by partiality of σ and commutativity of σ and τ we calculate for $r, s \in \mathcal{R}$

$$\sigma(r \bullet s) = \sigma(\sigma(r) + \tau(s)) = \sigma^2(r) + \sigma(\tau(s)) = \sigma^2(r) + \tau(\sigma(s)) = \sigma(r) \bullet \sigma(s).$$

(Hom') $\Rightarrow$ (Hom). First note that (Hom') is equivalent to the formula

$$\forall\, x, y \in \mathcal{R} : \tau(x - \tau(x) + \tau(y)) = \tau(x - \tau(x)) + \tau(y) - \tau(y - \tau(y)).$$

This yields for $x := 0$

$$\forall\, y \in \mathcal{R} : \tau(y - \tau(y)) = \tau(y) - \tau^2(y),$$

consequently,

$$\tau(x - \tau(x) + \tau(y)) = \tau(x) - \tau^2(x) + \tau^2(y).$$

(F) follows, since by (RD) we get $(x \bullet y) \bullet x = (x \bullet x) \bullet (y \bullet x) = x \bullet (y \bullet x)$. ∎

Some authors prefer to call (E) medial law and (F) diassociativity. However, we note that in our situation, the notion of diassociativity is completely different from the one used with respect to loops, where diassociativity means that any subloop generated by two elements is associative, i.e. a group (cf. [B], p.87).

2.2. Remark. (a) Canonically monoid splitting $SGRI$-groupoids are $SGRID$-groupoids.

(b) If in the situation of (1.18), $(\mathcal{R}, \bullet)$ is a $SGRID$-groupoid, then also $(\mathcal{R}/\mathcal{N}, \blacksquare)$ is a $SGRID$-groupoid. Moreover, $(\mathcal{N}, +, \bullet)$ is an entropic algebra ([RS],(127)).

(c) If in the situation of (1.19), $(\mathcal{R}, \bullet)$ is a $SGRID$-groupoid, then also the extended groupoid $(\mathcal{R}^{\curlywedge}, \bullet_{\curlywedge})$ is a $SGRID$-groupoid.

Proof. (b) Since $q_{\mathcal{N}}$ is a homomorphism, $(\mathcal{R}/\mathcal{N}, \blacksquare)$ is distributive. $(\mathcal{N}, +, \bullet)$ is entropic by the formula in (1.18), by commutativity of $+$ and entropicity of $(\mathcal{N}, \bullet)$.

(c) By calculation, using (Hom) and the definition of $\tau_{\curlywedge}$. ∎

Examples of $SGRID$-groupoids can be found among affine spaces, which have been widely considered in ([RS], p.3, p.39ff). For the reader's convenience we discuss these examples anew below in (2.3),(b). Preparing for this, for a module W over a ring L with unit and $r \in L$, by $t_r : W \to W$ we agree to denote the mapping given by $w \mapsto r \cdot w$. Strictly speaking, in the following we sometimes consider only suitable restrictions of mappings t_r (cf. (1.3)).

All subreducts of affine spaces are isomorphic by translation to subgroupoids of $SGRID$-groupoids, but not necessarily vice versa, at least not in a canonical manner. We shall illustrate this by one of the following examples and in a subsequent paper ([E3],(1.2)).

2.3. EXAMPLE. (a) Let G be an abelian group, $H < G$, $h : G \to G$ a homomorphism such that $h(H) < H$. Then H, equipped with the binary operation

$$(x, y) \mapsto x - h(x) + h(y)$$

forms a $SGRID$-groupoid. In particular, any group homomorphism induces on G the structure of a $SGRID$-groupoid.

(b) Let L be a ring with unit, W an L-module and V a submodule of W. For any $r \in L$, on V we get the structure of an $SGRID$-groupoid with the binary operation $\bullet_r$ given by

$$(x, y) \mapsto x - t_r(x) + t_r(y) = (1 - r)x + ry.$$

Moreover, if $L \in \{\mathbf{R}, \mathbf{C}, \mathbf{H}\}$, and $r \in [0, 1]$, each convex subset C of L with $0 \in C$ becomes a $SGRID$-groupoid by $\bullet_r$.

(c) Let L be a ring with unit, W an L-module and $g : W \to W$ a linear map, $g \neq \mathrm{id}_W$, having a fixed point $x_0 \neq 0$, and denote by $\bullet$ the binary operation on W given by

$$(x, y) \mapsto x - g(x) + g(y)$$

(cf. Ex. (a)). For a field K and a K-vector space V, for $r \in K$ we consider on V the binary operation $\bullet_r$ given by t_r (cf. Ex. (b)). Then for all $r \in K$, there is no algebra monomorphism from $(W, \bullet)$ to $(V, \bullet_r)$. In particular for a field L, the $SGRID$-groupoid $(W, \bullet)$ is not isomorphic to a subreduct of an affine subspace of any vector space.

Proof. (a) Idempotency is obvious. Since h is a group homomorphism, condition (Hom) of (2.1) is satisfied.

(b) Similar to the proof of (a), using the fact that t_r is a group homomorphism on $(W, +)$.

(c) For an algebra homomorphism $k : (W, \bullet) \to (V, \bullet_r)$ and $x, y \in W$ we get

$$k(x - g(x) + g(y)) = (1 - r)k(x) + rk(y),$$

and $g(0) = 0$ implies

$$k(x - g(x)) = (1 - r)k(x) + rk(0).$$

For the fixed point $0 \neq x_0$ of g we conclude

$$k(0) = (1 - r)k(x_0) + rk(0), \text{ i.e.}$$
$$(1 - r)(k(x_0) - k(0)) = 0.$$

In case $r \neq 1$ we get $k(x_0) = k(0)$, since V is a vector space, hence injectivity fails. For $r = 1$, the binary operation $\bullet_1$ is trivial, but $\bullet$ is not, since $g \neq \mathrm{id}_W$; consequently there are $x, y \in W$ such that $x \bullet y \neq y$, but $k(y) = k(x) \bullet k(y) = k(x \bullet y)$, which again shows that k is not injective. ∎

As was already mentioned in the preliminaries, in our investigations we orient ourselves from [E1], where group related symmetric groupoids have been discussed. Such groupoids in particular are *SGRID*-groupoids, which different from the general case (see (2.6)), are canonically monoid splitting with $\mathcal{R}^{\perp}_{\succ} = \mathcal{R}^{\perp}_{\curlyvee}$, and $\mathcal{R}^{\top} = \mathcal{R}$, because τ for symmetric groupoids is even bijective. Since by means of these conditions, in generalization of the methods applied in [E1], we can get far-reaching results for *SGRID*-groupoids, we now study circumstances of their validity for our situation.

According to (1.6), underlying groups in the next proposition are all the same, and describing maps are given by restrictions of τ.

2.4. PROPOSITION. *For a SGRID-groupoid* $(\mathcal{R}, \bullet)$, *also* $(\mathcal{R}^{\perp}, \bullet)$ *is a SGRID-groupoid. In addition, if* $\mathcal{R}^{\top} + \mathcal{R}^{\perp}_{\succ} \subseteq \mathcal{R}$ *and* $\tau|\mathcal{R}^{\perp}_{\succ}$ *is a monoid homomorphism, then* $(\mathcal{R}^{\perp}_{\succ}, \bullet)$ *and* $(\mathcal{R}^{\perp}_{\curlyvee}, \bullet)$ *are SGRID-groupoids as well, and* $\mathcal{R}^{\perp}_{\succ}$ *is a left ideal in* $\mathcal{R}$ *w.r.t.* $\bullet$, *i.e.* $\mathcal{R} \bullet \mathcal{R}^{\perp}_{\succ} \subseteq \mathcal{R}^{\perp}_{\succ}$.

Proof. The first assertion follows by (2.1),(Hom') and (1.6). Now let $\mathcal{R}^{\top} + \mathcal{R}^{\perp}_{\succ} \subseteq \mathcal{R}$, and $\tau|\mathcal{R}^{\perp}_{\succ}$ be a monoid homomorphism. $\tau(\mathcal{R}^{\perp}) \subseteq \mathcal{R}^{\perp}$ implies $\tau(\mathcal{R}^{\perp}_{\succ}) \subseteq \mathcal{R}^{\perp}_{\succ}$, thus $\mathcal{R} \bullet \mathcal{R}^{\perp}_{\succ} \subseteq \mathcal{R}^{\perp} + \tau(\mathcal{R}^{\perp}_{\succ}) \subseteq \mathcal{R}^{\perp}_{\succ}$ and trivially, $\mathcal{R}^{\perp}_{\succ} \bullet \mathcal{R}^{\perp}_{\succ} \subseteq \mathcal{R}^{\perp}_{\succ}$. Hence $\mathcal{R}^{\perp}_{\succ}$ is a left ideal as well as a *SGRID*-groupoid. From $\tau(\mathcal{R}^{\perp}_{\succ}) \subseteq \mathcal{R}^{\perp}_{\succ}$ and $\tau|\mathcal{R}^{\perp}_{\succ}$ being a monoid homomorphism we conclude $\tau(\mathcal{R}^{\perp}_{\curlyvee}) \subseteq \mathcal{R}^{\perp}_{\curlyvee}$, therefore

$$\mathcal{R}^{\perp}_{\curlyvee} \bullet \mathcal{R}^{\perp}_{\curlyvee} \subseteq \sigma(\mathcal{R}^{\perp}_{\curlyvee}) + \tau(\mathcal{R}^{\perp}_{\curlyvee}) \subseteq \mathcal{R}^{\perp}_{\curlyvee} - \tau(\mathcal{R}^{\perp}_{\curlyvee}) + \tau(\mathcal{R}^{\perp}_{\curlyvee}) \subseteq 3\mathcal{R}^{\perp}_{\curlyvee} \subseteq \mathcal{R}^{\perp}_{\curlyvee},$$

which shows that $\mathcal{R}^{\perp}_{\curlyvee}$ is a *SGRID*-groupoid. ∎

Corresponding results for $\mathcal{R}^{\top}$, $\mathcal{R}^{\top}_{\succ}$ and $\mathcal{R}^{\top}_{\curlyvee}$ instead of $\mathcal{R}^{\perp}$, $\mathcal{R}^{\perp}_{\succ}$ and $\mathcal{R}^{\perp}_{\curlyvee}$ can be obtained by dualizing applying (1.2).

2.5. PROPOSITION. *Let* $(\mathcal{R}, \bullet)$ *be a SGRI-groupoid.*

(a) *The following are equivalent:*

(1) $(\mathcal{R}, \bullet)$ *is a SGRID-groupoid, and* $\mathcal{R}^{\top} = \mathcal{R}$.

(2) $\mathcal{R}$ *is canonically monoid splitting, and* $\tau(\mathcal{R}^{\perp}) = \mathcal{R}^{\perp}$.

(b) *If* $\mathcal{R}^{\top} = \mathcal{R}$, *then* $\mathcal{R} + \mathcal{R}^{\perp} \subseteq \mathcal{R}$ *and consequently,* $\mathcal{R} + \mathcal{R}^{\perp}_{\succ} \subseteq \mathcal{R}$, *and the following are equivalent:*

(1) $(\mathcal{R}, \bullet)$ *is a SGRID-groupoid,*

(2) $\tau \in \mathrm{Part}_{\mathcal{R},\mathcal{R}^{\perp}_{\succ}}(\mathcal{R}, G)$,

(3) $\sigma \in \mathrm{Part}_{\mathcal{R},\mathcal{R}^{\perp}_{\succ}}(\mathcal{R}, G)$,

(4) *there is* $\mathcal{N} \circ\!< G$ *such that* $\mathcal{R}$ *is* $\mathcal{N}$*-splitting, and* $\mathcal{R}^\perp \subseteq \mathcal{N}$,

(5) *there is* $\mathcal{N} \circ\!< G$ *such that* $\mathcal{R} + \mathcal{N} \subseteq \mathcal{R}$, $\sigma \in \mathrm{Part}_{\mathcal{R},\mathcal{N}}(\mathcal{R}, G)$, *and* $\mathcal{R}^\perp \subseteq \mathcal{N}$.

If one of the equivalent conditions of (a) resp. (b) is satisfied, $\tau(\mathcal{R}^\perp_\succ) = \mathcal{R}^\perp_\succ$.

Proof. (a), (1) $\Rightarrow$ (2). By (2.1) we conclude $\tau(\mathcal{R}^\perp) = \tau(\sigma(\mathcal{R})) = \sigma(\mathcal{R}^\top) = \sigma(\mathcal{R}) = \mathcal{R}^\perp$. Trivially, $\mathcal{R} + \mathcal{R}^\perp \subseteq \mathcal{R}$, and (2.1) implies $\tau \in \mathrm{Part}_{\mathcal{R},\mathcal{R}^\perp}(\mathcal{R}, G)$. Thus by (1.14), $\mathcal{R}$ is canonically monoid splitting.

(2) $\Rightarrow$ (1). For $r \in \mathcal{R}$, by definition $r - \tau(r) =: m \in \mathcal{R}^\perp$. $\tau(\mathcal{R}^\perp) = \mathcal{R}^\perp$ yields m' such that $\tau(m') = m$, and $\tau \in \mathrm{Part}_{\mathcal{R},\mathcal{R}^\perp}(\mathcal{R}, G)$ gives $r = \tau(r) + \tau(m') = \tau(r + m')$. By (2.2),(a), $(\mathcal{R}, \bullet)$ is a $SGRID$-groupoid.

(b) cf. (a), (2.1), (1.14) and (1.10).

The last assertion can be easily seen using (a),(2). ■

Now we turn to a necessary condition for canonically monoid splitting $SGRID$-groupoids, namely $\mathcal{R}^\perp_\succ \subseteq \mathcal{R}$. Among the following examples there are $SGRID$-groupoids with $\mathcal{R}^\perp_\succ \not\subseteq \mathcal{R}$ and consequently, $\mathcal{R}^\perp \neq \mathcal{R}^\perp_\succ$. In particular, these examples are not canonically monoid splitting.

2.6. Example. Let $k \in \mathbb{N}, k \geq 2$, put $\mathcal{L} := \{kz \mid z \in \mathbb{Z}\}, \mathcal{R} := \mathcal{L} \cup (1+\mathcal{L}) \subseteq \mathbb{Z}$, and define $\tau : \mathcal{R} \to \mathbb{Z}$ by $r \mapsto kr$. Clearly $\tau(0) = 0$, and $\mathcal{R}^\top \subseteq \mathcal{L} \subseteq \mathcal{R}$. Assigning $(a, b) \mapsto a - \tau(a) + \tau(b)$ we get a binary operation $\bullet$ on $\mathcal{R}$, since for $\alpha, \beta \in \mathbb{Z}$, $\gamma, \delta \in \{0, 1\}$ and $a := \gamma + k\alpha$, $b := \delta + k\beta$ we calculate

$$\begin{aligned} a - \tau(a) + \tau(b) &= \gamma + k\alpha - k(\gamma + k\alpha) + k(\delta + k\beta) \\ &= \gamma + k(\alpha - \gamma - k\alpha + \delta + k\beta) \in \mathcal{R}. \end{aligned}$$

Obviously, the binary operation is idempotent, and $\bullet$ is even left distributive, since

$$\tau(a \bullet b) = k(a - ka + kb) = ka - k^2a + k^2b = \tau(a) \bullet \tau(b).$$

Hence, $(\mathcal{R}, \bullet)$ is a $SGRID$-groupoid. For $k = 2$ we have $\mathcal{R} = \mathbb{Z}$, $\sigma(r) = -r$; consequently, $\mathcal{R}^\perp = \mathcal{R}^\perp_\succ = \mathcal{R}^\perp_\curlyvee = \mathcal{R}$. For $k > 2$ we get

$$\begin{aligned} \mathcal{R}^\perp = \sigma(\mathcal{R}) &= \sigma(\mathcal{L} \cup (1 + \mathcal{L})) = \sigma(\mathcal{L}) \cup \sigma(1 + \mathcal{L}) \\ &= \{(1-k)kz \mid z \in \mathbb{Z}\} \cup \{(1-k)(1+kz) \mid z \in \mathbb{Z}\} \\ &= \{(1-k)kz \mid z \in \mathbb{Z}\} \cup \{1 + k(z - 1 - kz) \mid z \in \mathbb{Z}\}, \end{aligned}$$

which shows $\mathcal{R}^\perp_\succ \not\subseteq \mathcal{R}$: For $z \in \mathbb{Z}$ and $m := 1 + k(z - 1 - kz) \in \mathcal{R}^\perp$, obviously $2m \notin \mathcal{R}$.

For the following we remember that for $SGRID$-groupoids, $\mathcal{R}^\top = \mathcal{R}$ implies $\tau(\mathcal{R}^\perp) = \mathcal{R}^\perp$, hence (2.7) covers the situation given in (2.5),(a).

2.7. PROPOSITION. *Let $(\mathcal{R}, \bullet)$ be a SGRID-groupoid and $\tau(\mathcal{R}^\perp) = \mathcal{R}^\perp$. Then $\mathcal{R}^\perp_\succ \subseteq \mathcal{R}$.*

Proof. (2.1) implies that $\tau \in \mathrm{Part}_{\mathcal{R}^\top, \mathcal{R}^\perp}(\mathcal{R}, G)$. Trivially always holds $\tau^{m+1}(\mathcal{R}) \subseteq \tau^m(\mathcal{R})$, and $\tau(\mathcal{R}^\perp) = \mathcal{R}^\perp$ implies $\tau^m(\mathcal{R}^\perp) = \mathcal{R}^\perp$ $(m \in \mathbb{N}_0)$. First we show inductively $\tau^k \in \mathrm{Part}_{\mathcal{R}^\top, \mathcal{R}^\perp}(\mathcal{R}, G)$ for $k \in \mathbb{N}$. For the inductive step from k to $k+1$, let $s \in \mathcal{R}^\top$ and $m \in \mathcal{R}^\perp$. Then

$$\begin{aligned} \tau^{k+1}(s+m) &= \tau(\tau^k(s+m)) \\ &\underset{\text{IND}}{=} \tau(\tau^k(s) + \tau^k(m)) \\ &= \tau^{k+1}(s) + \tau^{k+1}(m), \end{aligned}$$

since $\tau \in \mathrm{Part}_{\mathcal{R}^\top, \mathcal{R}^\perp}(\mathcal{R}, G)$, $\tau^k(s) \in \mathcal{R}^\top$ and $\tau^k(m) \in \mathcal{R}^\perp$.

Now by induction as well, we show $\mathcal{R} = \tau^k(\mathcal{R}) + k\mathcal{R}^\perp$ $(k \in \mathbb{N}_0)$. For $k = 0, 1$ the assertion is immediate (cf. (1.16)), and the inductive step looks like follows:

$$\begin{aligned} \mathcal{R} &\underset{\text{IND}}{=} \tau^k(\mathcal{R}) + k\mathcal{R}^\perp = \tau^k(\mathcal{R}^\top + \mathcal{R}^\perp) + k\mathcal{R}^\perp \\ &= \tau^{k+1}(\mathcal{R}) + \tau^k(\mathcal{R}^\perp) + k\mathcal{R}^\perp, \quad \text{since } \tau^k \in \mathrm{Part}_{\mathcal{R}^\top, \mathcal{R}^\perp}(\mathcal{R}, G)), \\ &= \tau^{k+1}(\mathcal{R}) + (k+1)\mathcal{R}^\perp, \quad \text{since } \tau^k(\mathcal{R}^\perp) = \mathcal{R}^\perp. \end{aligned}$$

Consequently, $\mathcal{R}^\perp_\succ = \bigcup_{k \in \mathbb{N}_0} k\mathcal{R}^\perp \subseteq \mathcal{R}$. ∎

3. Characterization of some idempotent and distributive groupoids

Now we prove a theorem for canonically monoid splitting *SGRID*–groupoids, which illuminates their structure from various standpoints: (2) and (3) take into account that on $\mathcal{R} + \mathcal{N}$ under consideration we have a partial order $\leq$ on the set of classes by means of inclusion. Proceeding from item to item we shall see that for the description of such *SGRID*–groupoids we can forget about much of both algebraic and order structure. Finally in (4), we come to an almost set theoretic statement.

To avoid possible misunderstandings, we note that group relatedness of course refers to the group G given in the prerequisites of the respective statements (cf. convention in chapter 1).

3.1. THEOREM. *For an abelian group $(G, +)$ and a subset $\mathcal{R} \subseteq G$ with $0 \in \mathcal{R}$ the following are equivalent:*

(1) *There is a binary operation $\bullet : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$, such that $(\mathcal{R}, \bullet)$ is a canonically monoid splitting SGRID-groupoid.*

(2) *There is a submonoid $\mathcal{N}$ of G with $\mathcal{R} + \mathcal{N} \subseteq \mathcal{R}$, a monoid homomorphism $t : \mathcal{N} \to G$ such that $t(\mathcal{N}) \subseteq \mathcal{N}$ and $u - t(u) \in \mathcal{N}$ for all*

$u \in \mathcal{N}$, an idempotent binary operation $\blacksquare$ on $\mathcal{R}/\mathcal{N}$ and a homomorphism $\Theta : (\mathcal{R}/\mathcal{N}, \blacksquare) \to (\mathcal{R}/\mathcal{N}, \blacksquare)$, such that for $A, B \in \mathcal{R}/\mathcal{N}$, $m, n \in \mathcal{N}$

(i) $A \blacksquare B \le \Theta(A) \blacksquare \Theta(B)$,

(ii) $A + m = B + n \Rightarrow \Theta(A) + t(m) = \Theta(B) + t(n)$.

(3) *There is a submonoid $\mathcal{N}$ of G with $\mathcal{R} + \mathcal{N} \subseteq \mathcal{R}$, a monoid homomorphism $t : \mathcal{N} \to G$ such that $t(\mathcal{N}) \subseteq \mathcal{N}$ and $u - t(u) \in \mathcal{N}$ for all $u \in \mathcal{N}$, as well as a mapping $\Theta : \mathcal{R}/\mathcal{N} \to \mathcal{R}/\mathcal{N}$ satisfying for $A, B \in \mathcal{R}/\mathcal{N}$, $m, n \in \mathcal{N}$*

(i) $A \le \Theta(A)$,

(ii) $A + m = B + n \Rightarrow \Theta(A) + t(m) = \Theta(B) + t(n)$.

(4) *There is a submonoid $\mathcal{N}$ of G and a monoid homomorphism $t : \mathcal{N} \to G$ such that $t(\mathcal{N}) \subseteq \mathcal{N}$ and $u - t(u) \in \mathcal{N}$ for all $u \in \mathcal{N}$, a family $(x_i)_{i \in I}$ of elements of $\mathcal{R}$ such that $x_0 = 0$ and $x_j - x_k \notin \mathcal{N}_\lambda := \mathcal{N} + (-\mathcal{N})$ for $j \neq k$, $j, k \in I$, subsets $K_i \subseteq \mathcal{N}_\lambda$ and elements $t_i \in \mathcal{R}$ $(i \in I)$ with $t_0 = 0$, such that*

(i) $\forall\, i \in I : x_i + K_i + \mathcal{N} \supseteq t_i + t_\lambda(K_i)$,

(ii) $\forall\, i \in I\ \forall k_\iota \in K_i : x_i - t_i + k_\iota - t_\lambda(k_\iota) \in \mathcal{N}$,

(iii) $\mathcal{R} = \bigcup_{i \in I} x_i + K_i + \mathcal{N}$,

where $t_\lambda : \mathcal{N}_\lambda \to G$ denotes the extension of t given by (1.19).

Proof. (1) $\Rightarrow$ (2). $\mathcal{N} := \mathcal{R}^\perp_\succ$ and $t := \tau|\mathcal{R}^\perp_\succ$ satisfy the conditions required, and by (1.18), $(\mathcal{R}/\mathcal{N}, \blacksquare)$ is a $SGRI$-groupoid with describing map $\tau_\mathcal{N} : r + \mathcal{N} \mapsto \tau(r) + \mathcal{N}$. Since $(\mathcal{R}/\mathcal{N}, \blacksquare)$ is even distributive by (2.2),(b), the restriction Θ of $\tau_\mathcal{N}$ to the range is a groupoid homomorphism by (2.1). From $x - \tau(x) \in \mathcal{N}$ for all $x \in \mathcal{R}$ we conclude $x + \mathcal{N} \subseteq \tau(x) + \mathcal{N}$, consequently

$$A \blacksquare B \le \Theta(A \blacksquare B) = \Theta(A) \blacksquare \Theta(B),$$

thus (2),(i) holds.– As for (2),(ii), we calculate for $r, s \in \mathcal{R}$, $m, n \in \mathcal{N}$

$$\begin{aligned} r + \mathcal{N} + m &= s + \mathcal{N} + n \\ \Rightarrow \tau(r + \mathcal{N} + m) &= \tau(s + \mathcal{N} + n) \\ \Rightarrow \tau(r) + \tau(m) + \tau(\mathcal{N}) &= \tau(s) + \tau(n) + \tau(\mathcal{N}) \end{aligned}$$

by partiality of τ; and since $\tau(\mathcal{N}) + \mathcal{N} = \mathcal{N}$ and by definition of Θ, t we get

$$\Theta(r + \mathcal{N}) + t(m) = \Theta(s + \mathcal{N}) + t(n).$$

(2) $\Rightarrow$ (3). Since $\blacksquare$ is idempotent, we get for $A \in \mathcal{R}/\mathcal{N}$

$$A = A \blacksquare A \le \Theta(A) \blacksquare \Theta(A) = \Theta(A).$$

(3) $\Rightarrow$ (4). Consider $\mathcal{R}^\lambda := \mathcal{R} + \mathcal{N}_\lambda$ and take a family $(x_i)_{i \in I}$ of elements of $\mathcal{R}$ satisfying $\mathcal{R}^\lambda = \bigcup_{i \in I} x_i + \mathcal{N}_\lambda$ (where we put $x_0 = 0$) and $x_i - x_j \notin \mathcal{N}_\lambda$ for $i, j \in I$, $i \neq j$. For $i \in I$, let $(y_\iota^{(i)})_{\iota \in J^{(i)}}$ be a complete family of representatives of the relation $R^\mathcal{R}_\mathcal{N}$ on $H_i := \{y \in \mathcal{R} \mid y + \mathcal{N} \subseteq x_i + \mathcal{N}_\lambda\}$, and

set $K_i := \{y_\iota^{(i)} - x_i |\ \iota \in J^{(i)}\}$. Now obviously, (4),(iii) holds. Define $t_0 := 0$, choose $t_i \in \mathcal{R}$ for any $i \in I \setminus \{0\}$ such that $\Theta(x_i + \mathcal{N}) = t_i + \mathcal{N}$. Then for $i \in I$, $k_\iota \in K_i$ and $m_\iota, n_\iota \in \mathcal{N}$ with $k_\iota = m_\iota - n_\iota$ we have

$$x_i + k_\iota + \mathcal{N} + n_\iota = x_i + \mathcal{N} + m_\iota,$$

consequently by (ii)

$$\Theta(x_i + k_\iota + \mathcal{N}) + t(n_\iota) = t_i + \mathcal{N} + t(m_\iota),$$

hence by (i)

$$x_i + k_\iota + \mathcal{N} \le \Theta(x_i + k_\iota + \mathcal{N}) = t_i + t(m_\iota) - t(n_\iota) + \mathcal{N} = t_i + t_\lambda(k_\iota) + \mathcal{N},$$

which shows (4),(i). $\Theta(x_i + k_\iota + \mathcal{N}) \in \mathcal{R}/\mathcal{N}$ implies $t_i + t_\lambda(k_\iota) \in \mathcal{R}$, thus by construction of K_i, there exists $k_\kappa \in K_i$ such that

$$t_i + t_\lambda(k_\iota) + \mathcal{N} = x_i + k_\kappa + \mathcal{N},$$

from which we conclude (4),(ii).

(4) $\Rightarrow$ (1). By (iii), $\mathcal{R} + \mathcal{N} \subseteq \mathcal{R}$. t_λ is a group homomorphism and $t_\lambda(\mathcal{N}_\lambda) < \mathcal{N}_\lambda$. We define $\tau : \mathcal{R} \to G$, assigning

$$x_i + k_i + u \mapsto t_i + t_\lambda(k_i + u), \quad x_i \in \mathcal{R}, k_i \in K_i, u \in \mathcal{N}.$$

The mapping τ is well defined, since for $x \in \mathcal{R}$ there is exactly one $i \in I$ such that $x \in x_i + \mathcal{N}_\lambda$, and for $k, k' \in K_i$ and $u, u' \in \mathcal{N}$ with $x = x_i + k + u = x_i + k' + u'$ we have $k + u = k' + u'$, and consequently $t_i + t_\lambda(k+u) = t_i + t_\lambda(k'+u')$.

The assignment $(x, y) \mapsto x - \tau(x) + \tau(y)$ defines a binary operation on $\mathcal{R}$. To show this, let $i, j \in I$, $k_i \in K_i, k_j \in K_j, u, v \in \mathcal{N}$, and $x := x_i + k_i + u$, $y := x_j + k_j + v$. Then

$$\begin{aligned} x - \tau(x) + \tau(y) &= x_i + k_i + u - t_i - t_\lambda(k_i + u) + t_j + t_\lambda(k_j + v) \\ &= \underbrace{x_i - t_i + k_i - t_\lambda(k_i)}_{\in \mathcal{N} \text{ by (ii)}} + \underbrace{u - t_\lambda(u)}_{\in \mathcal{N}} + \underbrace{t_j + t_\lambda(k_j)}_{\in \mathcal{R} \text{ by (i),(iii)}} + \underbrace{t_\lambda(v)}_{\in \mathcal{N}} \\ &\in \mathcal{R} + \mathcal{N} \subseteq \mathcal{R}. \end{aligned}$$

$t_0 = 0$ implies $\tau(0) = 0$, and since the binary operation is idempotent, $\mathcal{R}$ proves to be a $SGRI$–groupoid. The mapping τ is in $\mathrm{Part}_{\mathcal{R},\mathcal{N}}(\mathcal{R}, G)$, since for $v \in \mathcal{N}$, $x = x_i + k_i + u \in \mathcal{R}$ we have

$$\begin{aligned} \tau(x_i + v) = \tau(x_i + k_i + \underbrace{u + v}_{\in \mathcal{N}}) &= t_i + t_\lambda(k_i + u + v) \\ &= t_i + t_\lambda(k_i + u) + t_\lambda(v) = \tau(x_i) + \tau(v). \end{aligned}$$

Furthermore, with x as above we get

$$\begin{aligned} x - \tau(x) &= x_i + k_i + u - t_i - t_\lambda(k_i) - t_\lambda(u) \\ &= \underbrace{x_i - t_i + k_i - t_\lambda(k_i)}_{\in \mathcal{N} \text{ by (ii)}} + \underbrace{u - t_\lambda(u)}_{\in \mathcal{N}} \in \mathcal{N}, \end{aligned}$$

consequently, $\mathcal{R}^\perp \subseteq \mathcal{N}$, and $\mathcal{R}+\mathcal{N} \subseteq \mathcal{R}$, $\tau \in \mathrm{Part}_{\mathcal{R},\mathcal{N}}(\mathcal{R}, G)$ yield $\mathcal{R}+\mathcal{R}^\perp \subseteq \mathcal{R}$, $\tau \in \mathrm{Part}_{\mathcal{R},\mathcal{R}^\perp}(\mathcal{R}, G)$. Applying (1.14) completes the proof. ∎

In case of $\mathcal{N} = \mathcal{N}_\lambda$ and a family $(x_i)_{i \in I}$ with $x_0 = 0$ being given, (3.1),(4) means that by any choice of a family $(t_i)_{i \in I}$ such that $t_0 = 0$ and $x_i - t_i \in \mathcal{N}_\lambda$ $(i \in I)$ on $\mathcal{R} := \bigcup_{i \in I} x_i + \mathcal{N}_\lambda$ we get a $SGRID$-groupoid.

In general, (4),(i) and (ii) imply that

$$\forall\, i \in I : x_i + K_i + \mathcal{N} = t_i + t_\lambda(K_i) + \mathcal{N},$$

but not vice versa. By the following example we show that condition (4),(i) of (3.1) can not be replaced by

$$\forall\, i \in I\ \forall\, k_\iota \in K_i : x_i + k_\iota + \mathcal{N} \ni t_i + t_\lambda(k_\iota).$$

3.2. Example. Fix $\alpha \in]0, 1[$, let $G := \mathbb{R}^2$ be equipped with the usual vector addition, $\mathcal{N} := \mathbb{R}_0^+ \times \{0\}$, $t : \mathcal{N} \to \mathcal{N}$ be given by $u \mapsto \alpha u$. Obviously, $t(\mathcal{N}) \subseteq \mathcal{N}$ and $u - t(u) = (1-\alpha)u \in \mathcal{N}$ for all $u \in \mathcal{N}$. With the notation from (1.19), $\mathcal{N}_\lambda = \mathbb{R} \times \{0\}$ and $t_\lambda : \mathcal{N}_\lambda \to \mathcal{N}_\lambda$ is given by $k \mapsto \alpha k$.

Now let $\mathbb{R} \ni \beta < 0$, $x_1 := (\varepsilon_1, \varepsilon_2)$ with $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$, $\varepsilon_2 \neq 0$, and put $t_1 := (\varepsilon_1 + (1-\alpha)\beta, \varepsilon_2)$, $K_1 :=]\beta, 0] \times \{0\}$, and $K_0 := \{0\} \times \{0\}$. Clearly conditions (4),(i) and (4),(ii) of (3.1) are satisfied for $i = 0$; so it remains to check them for $i = 1$:

Condition (4),(i) holds, since

$$\begin{aligned}]\beta, 0] &\supseteq]\beta, (1-\alpha)\beta] \quad \Rightarrow \\ ]\beta, 0] + \mathbb{R}_0^+ &\supseteq]\beta, (1-\alpha)\beta] \quad \Longleftrightarrow \\ \varepsilon_1 +]\beta, 0] + \mathbb{R}_0^+ &\supseteq \varepsilon_1 + (1-\alpha)\beta + \alpha]\beta, 0], \end{aligned}$$

which implies $x_1 + K_1 + \mathcal{N} \supseteq t_1 + t_\lambda(K_1)$. (On the other hand, we have $\beta < \alpha\beta$, which is equivalent to $\varepsilon_1 + (1-\alpha)\beta + \frac{1}{2}\alpha\beta \notin \varepsilon_1 + \frac{1}{2}\beta + \mathbb{R}_0^+$, thus for $k := \beta/2$ we get $t_1 + t_\lambda(k) \notin x_1 + k + \mathcal{N}$.) Condition (4),(ii) for $i = 1$ is a consequence of

$$]0, -(1-\alpha)\beta] \subseteq \mathbb{R}_0^+ \iff \varepsilon_1 - (\varepsilon_1 + (1-\alpha)\beta) + (1-\alpha)]\beta, 0] \subseteq \mathbb{R}_0^+.$$

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