DEMONSTRATIO MATHEMATICA
Vol. XXIX No 2 1996

Ismat Beg, Rashid Ali

SOME PROPERTIES OF THE HAUSDORFF DISTANCE IN PROBABILISTIC METRIC SPACES

1. Introduction

Menger in [5] introduced the notion of the probabilistic metric spaces and the study of such spaces expanded rapidly starting with the pioneering works of Schweizer and Sklar [6, 7]. The theory of probabilistic metric spaces is of fundamental importance in probabilistic functional analysis. For detailed discussion of these spaces and their applications we refer to [1]–[4], [8] and [9]. Let Z be a subfamily of the family M of all nonempty and bounded subsets of a probabilistic metric space. For $A \in M$ define a distribution function $\mathfrak{H}_{\mathfrak{Z}}(A)$ as the probabilistic Hausdorff distance of A from the family $\mathfrak{Z}$. The function $\mathfrak{H}_{\mathfrak{Z}}$ is a kind of measure of noncompactness. In the paper we study properties of $\mathfrak{H}_{\mathfrak{Z}}$.

2. Preliminaries

Let $\mathbf{R}$ stands for the set of real numbers and $\mathbb{R}^+ = [0, \infty)$. A mapping $f : \mathbf{R} \to \mathbf{R}^+$ is called a distribution function if it is nondecreasing, left continuous with $\inf f(x) = 0$ and $\sup f(x) = 1$. We shall denote by $\mathcal{L}$ the set of all distribution functions on $\mathbb{R}$. Let us note that the Heaviside function

$$H(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ 1 & \text{if } t > 0, \end{cases}$$

is a distribution function.

DEFINITION 2.1. A probabilistic metric space is a pair $(X, \mathcal{F})$, where X is a nonempty set and $\mathcal{F}$ is mapping from $X \times X$ into $\mathcal{L}$.

Key words and phrases: probabilistic metric space, Hausdorff distance.
1991 *Mathematics Subject Classification*: 54E70.

We shall denote the distribution function $\mathcal{F}(x,y)$ by $\mathcal{F}_{x,y}$ and the value of $\mathcal{F}(x,y)$ at $t \in \mathbb{R}$ by $\mathcal{F}_{x,y}(t)$. The function $\mathcal{F}_{x,y}$ is assumed to satisfy the following conditions:

(P1) $\mathcal{F}_{x,y}(t) = H(t)$ for all $t \in \mathbb{R}$ if and only if $x = y$.
(P2) $\mathcal{F}_{x,y}(0) = 0$.
(P3) $\mathcal{F}_{x,y}(t) = \mathcal{F}_{y,x}(t)$.
(P4) If $\mathcal{F}_{x,y}(t_1) = 1$ and $\mathcal{F}_{y,z}(t_2) = 1$, then $\mathcal{F}_{x,y}(t_1 + t_2) = 1$ for every $x, y, z \in X$.

DEFINITION 2.2. A t-norm is a function $T : [0,1] \times [0,1] \to [0,1]$, which is associative, commutative, nondecreasing, $T(a,1) = a$ and $T(0,0) = 0$.

Let us notice that among a number of possible choices for the t-norm T mentioned by Schweizer and Sklar ([6]), "$T(a,b) = \min\{a,b\}$" is the strongest possible universal T and in this paper, we will always use this.

DEFINITION 2.3. A Menger probabilistic metric space, (shortly, a Menger PM-space) is a triple $(X, \mathcal{F}, T)$ where $(X, \mathcal{F})$ is a probabilistic metric space and T is a t-norm with the following condition:

(P5) $\mathcal{F}_{x,y}(t_1 + t_2) \geq T(\mathcal{F}_{x,z}(t_1), \mathcal{F}_{z,y}(t_2))$ for all $x, y, z \in X$, $t_1, t_2 \geq 0$.

Remark 2.4. Schweizer and Sklar [6] proved that if $(X, \mathcal{F}, T)$ is a Menger PM-space with the continuous t-norm T, then $(X, \mathcal{F}, T)$ is a Hausdorff space in the topology τ induced by the family,

$$\{U_x(\varepsilon, \lambda) : x \in X,\ \varepsilon > 0,\ \lambda \in (0,1)\}$$

of neighbourhoods $U_x(\varepsilon, \lambda)$, where

$$U_x(\varepsilon, \lambda) = \{y \in X : \mathcal{F}_{x,y}(\varepsilon) > 1 - \lambda\}.$$

DEFINITION 2.5. Let $(X, \mathcal{F})$ be a PM-space and A be a nonempty subset of X. The probabilistic diameter of A is a function D_A defined on $\mathbb{R}^+$ by

$$D_A(t) = \sup_{s<t} \inf_{p,q \in A} \mathcal{F}_{p,q}(s).$$

DEFINITION 2.6. Let $(X, \mathcal{F})$ be a PM-space. A subset A of X is said to be probabilistically

(i) bounded, if $\sup_{t>0} D_A(t) = 1$,
(ii) semibounded, if $0 < \sup_{t>0} D_A(t) < 1$,
(iii) unbounded, if $\sup_{t>0} D_A(t) = 0$.

DEFINITION 2.7. A nonempty subset A of a probabilistic metric space $(X, \mathcal{F})$ is said to be relatively compact if its closure is compact.

DEFINITION 2.8. Let $(X, \mathcal{F})$ be a probabilistic metric space and A be a nonempty subset of X. A finite subset B of X is said to be (ε, λ)-net for A

if for each $a \in A$, there is at least one $b \in B$ such that

$$\mathcal{F}_{a,b}(\varepsilon) > 1 - \lambda, \quad \varepsilon > 0 \text{ and } \lambda \in (0,1).$$

Let $(X, \mathcal{F}, T)$ be a complete Menger PM-space. Denote by M_x (or, briefly M) the family of all nonempty and probabilistically bounded subsets of X. Moreover, the family of all nonempty and relatively compact subsets of M will be denoted by N.

DEFINITION 2.9. If $x \in X$ and $\gamma > 0$, $\varepsilon \in (0,1)$, then we define the open balls centered at x by

$$K_\varepsilon(x,\gamma) = \{y \in X : \mathcal{F}_{x,y}(\gamma) > 1 - \varepsilon\}.$$

Similarly for $A \in M$, we define

$$K_\varepsilon(A,\gamma) = \bigcup_{x \in A} K_\varepsilon(x,\gamma).$$

By $\overline{A}$ we shall denote the closure of a subset $A \subset X$. Apart from this for an arbitrary family $\mathcal{U}$ of subsets $A \subset X$, we define

$$\mathcal{U}^c = \{A \in \mathcal{U} : A = \overline{A}\}.$$

Let $A, B \in M$ and denote by

$$d_{A,B}(t) = \sup\{\varepsilon \in [0,1] : A \subset K_\varepsilon(B,t)\},$$
$$D_{A.B}(t) = \sup_{r<t} T\{d_{A,B}(r), d_{A,B}(r)\}.$$

DEFINITION 2.10. The function $D_{A,B}(t)$ is called the Hausdorff distance between the sets A and B.

THEOREM 2.11 [4]. *If A and B are non-empty subsets of a Menger PM space X. Then*

$$D_{A,B} = H \quad \text{if and only if} \quad \overline{A} = \overline{B}.$$

NOTATION 2.12. Let $\mathcal{Z}$ be a nonempty subfamily of M. We will use the following notations:

$$D_{A,\mathfrak{z}} = \sup\{D_{A,B} : B \in \mathcal{Z}\},$$
$$d_{A,\mathfrak{z}} = \sup\{d_{A,B} : B \in \mathcal{Z}\}.$$

In what follows we will consider the function $\mathfrak{H}_\mathfrak{z} : M \to \mathcal{L}$, defined by

$$\mathfrak{H}_\mathfrak{z}(A) = D_{A,\mathfrak{z}}.$$

For simplicity, we will write $\mathfrak{H}(A)$ instead of $\mathfrak{H}_\mathfrak{z}(A)$.

3. The results

We begin with the following simple, but useful lemma:

LEMMA 3.1. *Let $A, B \in M$ and $r > 0$, $0 < \varepsilon < 1$. If $B \subset K_\varepsilon(A,r)$ then $A \cap K_\varepsilon(B,r) \neq \varphi$ and $B \subset K_\varepsilon(A \cap K_\varepsilon(B,r), r)$.*

Proof. Let b be an arbitrary element of B. Then by hypothesis there exists $a \in A$ such that $\mathcal{F}_{a,b}(r) > 1 - \varepsilon$. It implies that $a \in K_\varepsilon(b, r)$ and consequently $a \in K_\varepsilon(B, r)$. Hence $a \in A \cap K_\varepsilon(B, r)$. Thus $A \cap K_\varepsilon(B, r) \neq \varphi$. On the other hand, we have shown that for any $b \in B$ there is an $a \in A \cap K_\varepsilon(B, r)$, such that $\mathcal{F}_{a,b}(r) > 1 - \varepsilon$ which means that $b \in K_\varepsilon(A \cap K_\varepsilon(B, r), r)$. Hence $B \subset K_\varepsilon(A \cap K_\varepsilon(B, r), r)$.

THEOREM 3.2. *Let $\mathcal{Z}$ be a nonempty subfamily of M with the property:*

$$\text{(1)} \qquad \text{if } A \in \mathcal{Z},\ \varphi \neq B \subset A \quad \text{then} \quad B \in \mathcal{Z}.$$

Then for any $A \in M$, the following equality holds

$$d_{A,\mathfrak{Z}} = D_{A,\mathfrak{Z}}.$$

Proof. Since

$$D_{A,B}(t) = \sup_{r<t} T\{d_{A,B}(r), d_{A,B}(r)\}$$

therefore for all $t > 0$

$$\text{(2)} \qquad D_{A,\mathfrak{Z}}(t) \leq d_{A\mathfrak{Z}}(t).$$

To prove the reverse inequality let $\delta \in (0,1)$ be arbitrary but fixed and let $d_{A,\mathfrak{Z}} = \varepsilon$, i.e.

$$\sup\{d_{A,B}(t) : B \in \mathcal{Z}\} = \varepsilon.$$

Then there exists a $B \in \mathcal{Z}$ such that

$$d_{A,B}(t) > \varepsilon - \delta$$

Thus

$$A \subset K_{\varepsilon-\delta}(B, r).$$

This in view of Lemma (3.1) implies that

$$B \cap K_{\varepsilon-\delta}(A, r) \neq \varphi \quad \text{and} \quad A \subset K_{\varepsilon-\delta}(B \cap K_{\varepsilon-\delta}(A, r), r).$$

Consequently,

$$\text{(3)} \qquad d_{A,B\cap K_{\varepsilon-\delta}(A,r)}(r) \geq \varepsilon - \delta.$$

On the other hand, $B \cap K_{\varepsilon-\delta}(A, r) \subset K_{\varepsilon-\delta}(A, r)$. This allows us to infer that

$$\text{(4)} \qquad d_{B\cap K_{\varepsilon-\delta}(A,r),A}(r) \geq \varepsilon - \delta.$$

Combining (3) and (4), we get

$$\text{Min}\{d_{A,B\cap K_{\varepsilon-\delta}(A,r)}(r), d_{B\cap K_{\varepsilon-\delta}(A,r),A}(r)\} \geq \varepsilon - \delta.$$

Taking sup on $r < t$, we obtain

$$D_{A,B\cap K_{\varepsilon-\delta}(A,r)}(t) \geq \varepsilon - \delta.$$

But in view of condition (1) we have $B \cap K_{\varepsilon-\delta}(A,r) \in \mathfrak{Z}$. So the latter inequality implies that

$$D_{A,\mathfrak{Z}}(t) \geq \varepsilon - \delta.$$

Since δ is arbitrary, therefore

$$D_{A,\mathfrak{Z}}(t) \geq \varepsilon = d_{A,\mathfrak{Z}}(t). \tag{5}$$

From (2) and (5) we get

$$D_{A,\mathfrak{Z}}(t) = d_{A,\mathfrak{Z}}(t).$$

COROLLARY 3.3. *Let $\mathfrak{Z}$ be a nonempty subfamily of M satisfying the condition* (1), *then $D_{A,\mathfrak{Z}}(t) = d_{\mathfrak{Z},A}(t)$, where $d_{\mathfrak{Z},A}(t) = \sup\{d_{BA}(t) : B \in \mathfrak{Z}\}$.*

COROLLARY 3.4. *Let $\mathfrak{Z}$ be a subfamily of M satisfying the condition (1). If $A \subset B$ then $\mathfrak{H}(A) \geq \mathfrak{H}(B)$ (i.e. for all $t \geq 0$ $D_{A,\mathfrak{Z}}(t) \geq D_{B,\mathfrak{Z}}(t)$).*

Proof. From Theorem (3.2) we know that

$$D_{A,\mathfrak{Z}}(t) = d_{A,\mathfrak{Z}}(t) \quad \text{and} \quad D_{B,\mathfrak{Z}}(t) = d_{B,\mathfrak{Z}}(t).$$

So it suffices to show that $d_{A,\mathfrak{Z}}(t) \geq d_{B,\mathfrak{Z}}(t)$. Put

$$\varepsilon = d_{B,\mathfrak{Z}}(t) = \sup\{d_{B,C}(t) : C \in \}.$$

Then for any given $\delta > 0$, there exists a $C \in \mathfrak{Z}$ such that $d_{B,C} > \varepsilon - \delta$, what means that $B \subset K_{\varepsilon-\delta}(C,t)$, and consequently $A \subset K_{\varepsilon-\delta}(C,t)$. This implies that

$$d_{A,C}(t) \geq \varepsilon - \delta.$$

Since $C \in \mathfrak{Z}$ so $d_{A,\mathfrak{Z}}(t) \geq \varepsilon - \delta$. But δ was arbitrarily chosen so we get $d_{A,\mathfrak{Z}}(t) \geq \varepsilon = d_{B,\mathfrak{Z}}(t)$.

COROLLARY 3.5. (a) *If $A, B \in M$ then,* $\min\{\mathfrak{H}(A), \mathfrak{H}(B),\} \geq \mathfrak{H}(A \cup B)$.
(b) *If $A, B \in M$ and $A \cap B \neq \emptyset$, then*

$$\mathfrak{H}(A \cap B) \geq \max\{\mathfrak{H}(A), \mathfrak{H}(B)\}.$$

Proof. Since $A \subset A \cup B$ and $B \subset A \cup B$. So by using Corollary 3.4, we get

$$\min\{\mathfrak{H}(A), \mathfrak{H}(B) \geq \mathfrak{H}(A \cup B),$$

what showes (a). Also $A \cap B \subset A$ and $A \cap B \subset B$ and again by Corollary 3.4, we get

$$\mathfrak{H}(A \cap B) \geq \max(\mathfrak{H}(A), \mathfrak{H}(B)\}.$$

THEOREM 3.6. *If a family $\mathcal{Z}$ fulfils the condition* (1) *and also the following one*

$$A, B \in \mathcal{Z} \quad \text{implies} \quad A \cup B \in \mathcal{Z}.$$

Then

$$\mathfrak{H}(A \cup B) = \min\{\mathfrak{H}(A), \mathfrak{H}(B)\}.$$

Proof. Denote $\varepsilon = \min\{\mathfrak{H}(A), \mathfrak{H}(B)\}$ and take an arbitrary $\delta > 0$. Then there exist $C_1, C_2 \in \mathcal{Z}$ such that

$$d_{A,C_1}(t) > \varepsilon - \delta \quad \text{or} \quad d_{B,C_2}(t) > \varepsilon - \delta.$$

It implies that

$$A \subset K_{\varepsilon-\delta}(C_1, t) \quad \text{and} \quad B \subset K_{\varepsilon-\delta}(C_2, t).$$

Thus

$$A \cup B \subset K_{\varepsilon-\delta}(C_1, t) \cup K_{\varepsilon-\delta}(C_2, t) = K_{\varepsilon-\delta}(C_1 \cup C_2, t).$$

But by the given condition $C_1 \cup C_2 \in \mathcal{Z}$ and therefore

$$d_{A\cup B,\mathfrak{Z}}(t) \geq \varepsilon - \delta.$$

Since δ is arbitrary therefore, in view of Theorem 3.2, the last inequality implies that

$$D_{A\cup B,\mathfrak{Z}}(t) \geq \varepsilon = \min\{\mathfrak{H}(A), \mathfrak{H}(B)\}$$

and this together with Corollary 3.5 completes the proof.

THEOREM 3.7. *$\mathfrak{H}_{\mathfrak{Z}}(A) = H$ if and only if $\overline{A} \in \overline{\mathcal{Z}}$, the closure of $\mathcal{Z}$ in M^c with respect to the topology generated by D.*

Proof.

$$\begin{aligned}
\mathfrak{H}_{\mathfrak{Z}}(A) = H &\Leftrightarrow D_{A,\mathfrak{Z}} = H \\
&\Leftrightarrow \sup\{D_{A,B} : B \in \mathcal{Z}\} = H \\
&\Leftrightarrow D_{A,B} = H \quad \text{for some } B \in \mathcal{Z}. \\
&\Leftrightarrow \overline{A} = \overline{B},\ \overline{B} \in \overline{\mathcal{Z}} \text{ (by Theorem 2.11)} \\
&\Leftrightarrow \overline{A} \in \overline{\mathcal{Z}}.
\end{aligned}$$

Acknowledgment. We would like to thank the referee for helpful suggestions and comments in improving our paper.

References

[1] S. S. Chang, *On the theory of probabilistic metric spaces with applications*, Z. Wahrsch. Verw, Gebiete 67 (1984), 85–94.

[2] S. S. Chang, *On the theory of probabilistic metric space with applications*, Acta Math. Sinica, New Series 1 (4) (1985), 366–377.

[3] S. S. Chang, *Basic theory and applications of probabilistic metric spaces (II)*, Appl. Math. Mech. 9 (3) (1988), 213–225.

[4] R. T. Egbert, *Products and quotients of probabilistic metric spaces*, Pacific J. Math., 24 (1968), 437–455.

[5] K. Menger, *Statistical metrics*, Proc. Nat. Acad. Sci., U.S.A. 28 (1942), 535–537.

[6] B. Schweizer, A. Sklar, *Statistical metric spaces*, Pacific J. Math., 10 (1960), 313–344.

[7] B. Schweizer, A. Sklar, *Probabilistic Metric Spaces*. North-Holland, 1983.

[8] H. Sherwood, *Complete probabilistic metric spaces*, Z. Wahrsch. Verw, Gebiete 20 (1971), 117–128.

[9] N. X. Tan, *Generalized probabilistic metric spaces and fixed point theorems*, Math. Nachr. 129 (1986), 205–218.

I. Beg
DEPARTMENT OF MATHEMATICS
KUWAIT UNIVERSITY, KUWAIT
and
QUAID-I-AZAM UNIVERSITY
ISLAMABAD, PAKISTAN

R. Ali
DEPARTMENT OF MATHEMATICS
QUAID-I-AZAM UNIVERSITY
ISLAMABAD, PAKISTAN

Received January 13st, 1994.