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CEVA'S AND MENELAUS' THEOREMS FOR TETRAHEDRA (II)

The paper is a continuation of an earlier article [5] concerning spatial versions of the well known theorems of Ceva and Menelaus. A necessary and sufficient condition for six points lying on edges of a given tetrahedron to be coplanar is formulated. Similarly, a necessary and sufficient condition for six planes, each of them determined by an edge and the point on the opposite edge, to have a common point is stated.

There are many generalizations of the theorems of Ceva and Menelaus e.g. [1], [2], [3], [4], [5] and others. In particular, in [5] a tetrahedron $A_1A_2A_3A_4$ and four points B_1, B_2, B_3, B_4 on its face planes ($B_i \in A_{i+1}A_{i+2}A_{i+3}$-adding of indices mod4) were considered. The following results were given:

(1) A necessary and sufficient condition (NS) for points B_1, B_2, B_3, B_4 to be coplanar;

(2) NS for points B_1, B_2, B_3, B_4 to be collinear;

(3) NS for planes $A_iA_{i+1}B_i$, $i = 1,2,3,4$, to have a common point;

(4) NS for lines A_iB_i, $i = 1,2,3,4$, to have a common point.

Obviously, conditions (1), (2) are spatial versions of the theorem of Menelaus. Similarly, (3) and (4) are spatial versions of the theorem of Ceva.

In this paper we shall continue considerations from [5].

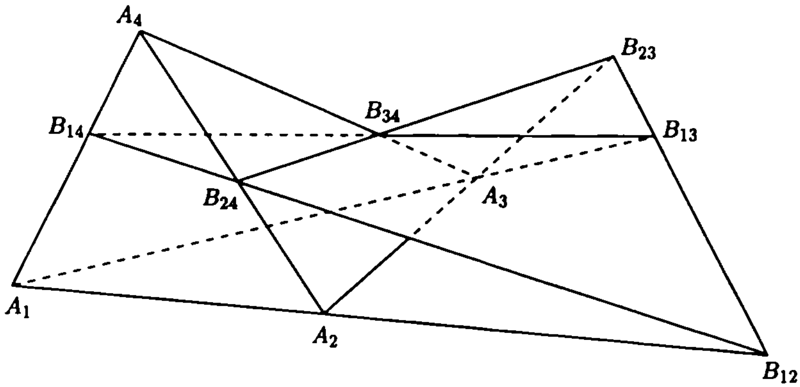
Let A_1, A_2, A_3, A_4 be vertices of a tetrahedron Θ. Let next B_{ij} be a point on the edge A_iA_j but different from points A_i, A_j. We shall formulate a necessary and sufficient condition for points $B_{12}, B_{13}, B_{14}, B_{23}, B_{24}, B_{34}$ to be coplanar (Fig. 1).

PROPOSITION 1. *Points $B_{12}, B_{13}, B_{14}, B_{23}, B_{24}, B_{34}$ are coplanar if and only if the following conditions hold*:

$$
(1)\quad \begin{cases}
(\overline{A_1B_{12}}/\overline{B_{12}A_2}) \cdot (\overline{A_2B_{23}}/\overline{B_{23}A_3}) \cdot (\overline{A_3B_{13}}/\overline{B_{13}A_1}) = -1;\\
(\overline{A_1B_{12}}/\overline{B_{12}A_2}) \cdot (\overline{A_2B_{24}}/\overline{B_{24}A_4}) \cdot (\overline{A_4B_{14}}/\overline{B_{14}A_1}) = -1;\\
(\overline{A_1B_{13}}/\overline{B_{13}A_3}) \cdot (\overline{A_3B_{34}}/\overline{B_{34}A_4}) \cdot (\overline{A_4B_{14}}/\overline{B_{14}A_1}) = -1;\\
(\overline{A_2B_{23}}/\overline{B_{23}A_3}) \cdot (\overline{A_3B_{34}}/\overline{B_{34}A_4}) \cdot (\overline{A_4B_{24}}/\overline{B_{24}A_2}) = -1.
\end{cases}
$$

Proof. Suppose points $B_{12}, B_{13}, B_{14}, B_{23}, B_{24}, B_{34}$ are in a plane α. Let us denote by l_i the common line of planes α and $A_{i+1}A_{i+2}A_{i+3}$, $i = 1,2,3,4$ (we reduce indices modulo 4). Then points B_{12}, B_{13}, B_{23} are on line l_4. By the theorem of Menelaus for triangle $A_1A_2A_3$ we obtain the first of the equalities (1). The remaining ones can be obtained in a similar way.

Assume now that conditions (1) hold. As previously, using Menelaus' theorem, we infer that points B_{12}, B_{13}, B_{23} are collinear. Similarly, points B_{12}, B_{14}, B_{24} as well as points B_{13}, B_{34}, B_{14} and points B_{23}, B_{34}, B_{24} colline. It means that all the points $B_{12}, B_{13}, B_{14}, B_{23}, B_{24}, B_{34}$ are coplanar. ■



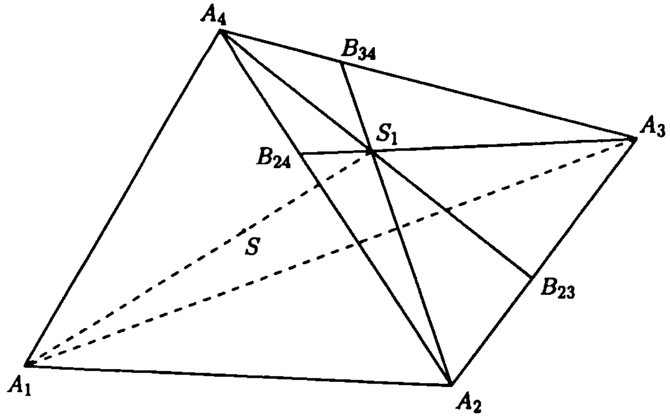
Proposition 1 is, obviously, a spatial version of the theorem of Menelaus. Now we shall state an analogous version of the theorem of Ceva.

PROPOSITION 2. *Planes $A_iA_jB_{kl}$, distinct $i,j,k,l = 1,2,3,4$, have a common point if and only if and only if the following equalities hold*:

$$
(2)\quad \begin{cases}
(\overline{A_1B_{12}}/\overline{B_{12}A_2}) \cdot (\overline{A_2B_{23}}/\overline{B_{23}A_3}) \cdot (\overline{A_3B_{13}}/\overline{B_{13}A_1}) = 1;\\
(\overline{A_1B_{12}}/\overline{B_{12}A_2}) \cdot (\overline{A_2B_{24}}/\overline{B_{24}A_4}) \cdot (\overline{A_4B_{14}}/\overline{B_{14}A_1}) = 1;\\
(\overline{A_1B_{13}}/\overline{B_{13}A_3}) \cdot (\overline{A_3B_{34}}/\overline{B_{34}A_4}) \cdot (\overline{A_4B_{14}}/\overline{B_{14}A_1}) = 1;\\
(\overline{A_2B_{23}}/\overline{B_{23}A_3}) \cdot (\overline{A_3B_{34}}/\overline{B_{34}A_4}) \cdot (\overline{A_4B_{24}}/\overline{B_{24}A_2}) = 1.
\end{cases}
$$

Proof. If planes $A_1A_2B_{34}$, $A_1A_3B_{24}$, $A_1A_4B_{23}$, $A_2A_3B_{14}$, $A_2A_4B_{13}$, $A_3A_4B_{12}$ have a common point S, then lines A_2B_{34}, A_3B_{24}, A_4B_{23} have a common point $S_1 = A_1S \cap A_2A_3A_4$ (fig. 2). Similarly, $A_2S \cap A_1A_3A_4 = S_2$, $A_3S \cap A_1A - 2A_4 = S_3$, $A_4S \cap A_1A_2A_3 = S_4$. Hence, by Ceva's theorem for triangles $A_1A_2A_3$, $A_1A_2A_4$, $A_1A_3A_4$, $A_2A_3A_4$ respectively, the equalities (2) are satisfied.

Suppose now the first of the equalities (2) is fulfilled. Then there exists, by the theorem of Ceva, a point S_4 such that lines A_1B_{23}, A_2B_{13}, A_3B_{12} meet each other in it. Similarly, there exist points S_3, S_2, S_1 for triangles $A_1A_2A_4$, $A_1A_3A_4$, $A_2A_3A_4$, respectively. It is evident that lines A_iS_i, $i = 1, 2, 3, 4$, have a common point S. ■



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