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ON THE EXPONENTIAL MEAN

If $a(x) = \frac{1}{k}\sum_{n=1}^{k} x_n$, $g(x) = (\prod_{n=1}^{k} x_n)^{\frac{1}{k}}$ and $h(x) = k(\sum_{n=1}^{k}\frac{1}{x_n})^{-1}$ are respectively arithmetic, geometric and harmonic means for the sequence $x = (x_1, x_2, \ldots, x_k) \in \mathbf{R}^k$, of positive terms, then, as it is known, for these means the following inequality holds (see [1], [2], [3])

$$h(x) \le g(x) \le a(x) \tag{1}$$

and $h(x) = g(x) = a(x) \Leftrightarrow x_1 = x_2 = \ldots = x_k$.

In this paper we define a new mean. Namely, the function

$$w(x) = \exp\Big\{\Big(\prod_{n=1}^{k} \ln x_n\Big)^{\frac{1}{k}}\Big\} \tag{2}$$

for $x \in A = \{(x_1, x_2, \ldots, x_k) \in \mathbf{R}^k : \ x_1 \ge 1,\ x_2 \ge 1, \ldots,\ x_k \ge 1\}$ will be called the exponential mean. If $x \in A$, then, on account of (1), $(\prod_{n=1}^{k} \ln x_n)^{\frac{1}{k}} \le \frac{1}{k}\sum_{n=1}^{k} \ln x_n$, whence $\ln w(x) \le \ln g(x)$ and finally

$$w(x) \le g(x). \tag{3}$$

It will be shown, that the function $f(x) = w(x) - h(x)$ can change the sign, i.e. that it can be sometimes negative and sometimes positive. Also from equality $f(x) = 0$ it does not result that $x_1 = x_2 = \ldots = x_k$.

In particular the following theorem is true:

THEOREM. *If $k \ge 2$, then there exist points $x^1, x^2 \in A$ such that $f(x^1) \cdot f(x^2) < 0$.*

P r o o f. For $x^1 = (x_1, 1, \ldots, 1) \in \mathbf{R}^k$, $x_1 > 1$ and $k \ge 2$ we have $w(x^1) = \exp\{(\ln x_1 \cdot \ln 1 \ldots \ln 1)^{\frac{1}{k}}\} = \exp 0 = 1$ and $h(x^1) = k(k - 1 + x_1^{-1})^{-1} > 1$, so $f(x^1) < 0$. On the other hand, for $x^3 = (x_1, 2, 2, \ldots, 2) \in \mathbf{R}^k$, $x_1 > 1$ we have

$$f(x^3) = \exp\{((\ln 2)^{k-1} \ln x_1)^{\frac{1}{k}}\} - k\Big(\frac{k-1}{2} + \frac{1}{x_1}\Big)^{-1} = F(x_1).$$

Because $\lim_{x_1\to\infty} F(x_1) = \infty$, there exists a number $c \in \mathbb{R}^+$ and a point $x^2 = (c, 2, 2, \ldots, 2) \in A$ such $F(c) > 0$, whence $f(x^2) > 0$. Finally we have $f(x^1)\cdot f(x^2) < 0$ for some $x^1, x^2 \in A$. If $x_1 = x_2 = \ldots = x_k$, then, of course, $w(x) = h(x)$ i.e. $f(x) = 0$.

Now we define a set

$$B = \Big\{(x_1, x_2 \ldots, x_k) \in A: \ x_1 \le x_2 \le \ldots \le x_k, \ \sum_{n=1}^{k} x_n > kx_1\Big\}.$$

On the base of our theorem, from the continuity and symmetry of the function f, there results that for $k \ge 2$ the set B is not empty. We can show that it is a $(k-1)$-dimensional hyperplane in the space $\mathbb{R}^k$. If $k = 2$, then

$$f(x) = f(x_1, x_2) = \exp\{(\ln x_1 \cdot \ln x_2)^{\frac{1}{2}}\} - \frac{2x_1 \cdot x_2}{x_1 + x_2}$$

and the curve B has the form $B = \{(x_1, x_2) \in A: \ w(x) = h(x), \ x_1 < x_2\}$.

We will prove that the curve B has a common point together with the curve $x_2 = x_1^{t^2}$, $(t > 1)$, which lies in the set A. This common point exists when the equation $P(u) = 0$ has a solution for $u > 1$ and $P(u) = u^{t+1} - 2u^t + 1$. It is easy to see that $P(u) > 0$ for $u \ge 2$, $P(u) < 0$ for $1 < u \le \frac{2t}{1+t}$ and $P(1) = 0$. From the continuity of the function $P(u)$ it follows that there exists a number $u_0 \in (\frac{2t}{1+t}, 2)$ such that $P(u_0) = 0$. This means that the curves $w(x_1, x_2) = h(x_1, x_2)$ and $x_2 = x_1^{t^2}$, $(t > 1)$ have a common point $W(u_0^{\frac{1}{t-1}}, u_0^{\frac{t^2}{t-1}})$, whence $B \ne \emptyset$. Considering the limits of coordinates of the point W when $t \to 1$ and $t \to \infty$, we can see that the straight lines $x_1 = 1$, $x_2 = x_1$ are asymptotes of the curve B. If B has the equation $x_2 = Q(x_1)$, then $f(x_1, x_2) < 0$ for $x_2 < Q(x_1)$ and $f(x_1, x_2) > 0$ for $x_2 > Q(x_1)$. In a similar way we can examine the location of the set B in the k-dimensional space $\mathbb{R}^k$ for $k \ge 3$.

References

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