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REFLECTIONS IN EQUIDISTANT HYPERSURFACES I. ANALYTICAL INVESTIGATIONS

1. Introduction

In this paper we study reflections in some equidistant hypersurfaces of the degenerate hyperbolic space $\mathbf{H}_k^n$ (cf. [3]). Let us remind that the set of points of $\mathbf{H}_k^n$ is the cone $\mathbf{C}_k^n$ contained in projective space $\mathbf{P}_n$. Hyperplanes of $\mathbf{H}_k^n$ are non empty intersections of the hyperplanes of $\mathbf{P}_n$ and the set $\mathbf{C}_k^n$. A hyperplane Q is isotropic iff it corresponds to a projective hyperplane $\overline{Q}$ containing the top $\mathbf{V}$ of the cone $\mathbf{C}_k^n$ (see [4]). We denote by $\Sigma = \Sigma(\mathbf{H}_k^n)$ the class of reflections of $\mathbf{H}_k^n$ in non isotropic hyperplanes, precisely the restrictions to the set $\mathbf{C}_k^n$ of appropriate projective symmetries. Let Ω denote the class of all axial symmetries of $\mathbf{H}_k^n$. In paper [3] we defined, generally, an equidistant hypersurface of $\mathbf{H}_k^n$ to be the orbit of a point α under the centralizer of a symmetry $\sigma = \sigma_Q^q$ in the class Ω. If Q is not isotropic, such an orbit is independent from q ($q \in \mathbf{V}$) and is denoted by $E_Q[\alpha]$. Let S be the class of all sets $E_Q[\alpha]$, where Q is a non isotropic hyperplane of $\mathbf{H}_k^n$ (see [3]). The structure $\overline{\mathbf{H}}_k^n = \langle \mathbf{C}_k^n, S \rangle$ is called an inversive degenerate hyperbolic space. In this structure we shall study the symmetries σ_E^q ($E \in S$ and $q \in \mathbf{V} \setminus \mathbf{V}(E)$), which will be defined below. The set $\mathbf{V}(E) := \{\mathbf{V} \cap \overline{L} : L \subset E\}$ is the top of E.

2. Results

Let $E \in S$ and $q \in \mathbf{V} \setminus \mathbf{V}(E)$. We define the reflection σ_E^q in E with centre q by the condition

DEFINITION 1. Let $\sigma_E^q(x) = y :\Leftrightarrow H(q, E \cap \overline{qx}; x, y)$, where H is a relation of harmonic conjugacy and $\overline{qx}$ denotes the line passing through q and x.

This definition is correct because

PROPOSITION 1. *If K is an isotropic line of $\mathbf{H}_k^n$, $E \in S$, and $K \not\subset E$, then $|K \cap E| = 1$.*

Let $\Lambda = \Lambda(\mathbf{H}_k^n)$ be the class of all symmetries σ_E^q, where $E \in S$. Of course $\Sigma \subset \Lambda$, because S contains the class of non isotropic hyperplanes (see [3]).

First we see that

THEOREM 1. *We have $\Lambda \subset \mathrm{Aut}(\overline{\mathbf{H}_k^n})$, and thus the group $G(\Lambda)$ generated by Λ is a subgroup of $\mathrm{Aut}(\overline{\mathbf{H}_k^n})$.*

Let us remind that T_k^n is the set of maximal generators of $\mathbf{H}_k^n$ i.e. of sets $\langle \alpha, \mathbf{V}\rangle \setminus \mathbf{V}$, where $\langle \alpha, \mathbf{V}\rangle$ is the subspace spanned in $\mathbf{P}_n$ by $\mathbf{V}$ and by point $\alpha \in \mathbb{C}_k^n$. From definition we see that Λ preserves the elements of T_k^n i.e.

THEOREM 2. *If $T \in T_k^n$, $f \in \Lambda$, then $f(T) = T$.*

In [3] we constructed the bijection ϕ of the set $\mathbb{C}_k^n$ onto the halfcylinder $\frac{1}{2}\mathbf{W}_k^n$ contained in $\mathbf{P}_{n+1}$; transformation ϕ is an isomorphism between $\overline{\mathbf{H}_k^n}$ and Laguerre halfspace $\frac{1}{2}\mathbf{L}_k^n$. This transformation correlates the symmetries from Λ and the symmetries of the appropriate Laguerre space. Note that the tops of $\mathbf{L}_k^n$ and $\mathbf{H}_k^n$ are identic.

THEOREM 3. *If $E \in S$ and $\rho \in \mathbf{V}\setminus\mathbf{V}(E)$, then $\phi \circ \sigma_E^\rho \circ \phi^{-1} = \sigma_{\phi(E)}^\rho | \frac{1}{2}\mathbf{W}_k^n$.*

As a consequence we get

THEOREM 4. *If $f : \mathbb{C}_k^n \mapsto \mathbb{C}_k^n$ is a bijection, then $f \in \Lambda$ iff $\phi \circ f \circ \phi^{-1}$ is the restriction of a symmetry of Laguerre space $\mathbf{L}_k^n$.*

Because the reflections in hyperspheres of Laguerre spaces are the restrictions of these projective symmetries of the cylinder $\mathbf{W}_k^n$, which centres belong to the top $\mathbf{V}(\mathbf{W}_k^n)$, we show that

THEOREM 5. *The following groups of symmetries are isomorphic:*

(i) $G(\Sigma(\mathbf{H}_k^n))$

(ii) $G(\Sigma(\mathbf{L}_k^{n-1}))$

(iii) $G(\Sigma(\frac{1}{2}\mathbf{L}_k^{n-1}))$

(iv) $G(\Lambda(\mathbf{H}_k^{n-1}))$.

The symmetries from $\Sigma(\mathbf{H}_k^n)$ preserve the top $\mathbf{V}$ and transform the generators of $\mathbb{C}_k^n$ onto themselves, thus any element of $G(\Sigma(\mathbf{H}_k^n))$ is described in the projective coordinates by matrix

$$(*) \qquad \begin{bmatrix} \Delta_{n-k+1} & 0 \\ B & E \end{bmatrix}, \quad \text{where } \Delta_l = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}_{l \times l}.$$

In general we have

PROPOSITION 2. *$f \in G(\Sigma(\mathbf{H}^n_k))$ if and only if f is described by matrix* (∗), *where* $\det(E) = \pm 1$.

From Theorem 5, Proposition 2, and from the analytical description of the isomorphism ϕ we get an analytical description of the group $G(\Lambda(\mathbf{H}^n_k))$. The set $\mathbb{C}^n_k$ is a subset of the affine space $\mathbf{A}_n$, hence this description will be given in affine coordinates.

Let us recall explicit formulas defining ϕ:

$$\phi((x_2, x_3, \ldots, x_{n+1})) = \left(\sqrt{1 - \sum_{i=2}^{n-k+1} x_i^2}, x_2, x_3, \ldots, x_{n+1}\right).$$

Then we get

THEOREM 6. *Let g be a transformation of $\overline{\mathbf{H}}^n_k$. The following conditions are equivalent:*

(i) $g \in G(\Lambda(\mathbf{H}^n_k))$;

(ii) *there exists a matrix*

$$M = [m_{ij}]_{0 \le i,j \le n+1} = \begin{bmatrix} \Delta_{n-k+2} & O_{n-k+2,k} \\ B & E \end{bmatrix},$$

where $\det(E) = \pm 1$ *such that*

$$g(x)_i = \begin{cases} x_i & \text{for } 1 \le i \le n-k \\ \sum_{j=1}^{n-k} b_{i+1,j+1} x_j + \sum_{j=n-k+1}^{n} e_{i+1,j+1} x_j & \\ +b_{i+1,1}\sqrt{1 - \sum_{j=1}^{n-k} x_j^2} + b_{i+1,0} & \text{for } n-k+1 \le i \le n. \end{cases}$$

3. Proofs and auxiliary lemmas

LEMMA 1. (i) *If $F \in S$, $P : x_n = 0$ is a base of F, and $\rho = [0, \ldots, 0, 1]$, then $|L(\rho, \alpha) \cap F| = 1$ for any $\alpha \in \mathbb{C}^n_k$.*

(ii) *For any $E \in S$, $q \in \mathbf{V}(\mathbb{C}^n_k)$, and $q \notin \mathbf{V}(E)$ there exists an affine transformation ψ such that $\psi \in \mathrm{Aut}(\mathbf{H}^n_k)$, $\psi^*(q) = [0, \ldots, 0, 1]$, and $\psi(E)$ is an equidistant hypersurface with the base P: $x_n = 0$.*

Proof. First we prove (i). Let F satisfy the assumption of (i). By Theorem 2.9 and 2.11 from [3], we get that F is that part of a set with equation: $c^2(-x_0^2 + x_1^2 + \ldots + x_{n-k}^2) + x_n^2 = 0$, which is on the one side of the hyperplane $P : x_n = 0$, or $F = P$. If $F = P$, then the thesis is trivial.

Let $\alpha = [\alpha_0, \alpha_1, \ldots, \alpha_n] \in \mathbb{C}^n_k$, whence $\alpha \neq \rho$ and $L(\rho, \alpha)$ is described by equations:

$$\begin{aligned} x_0 &= \mu\alpha_0 \\ &\vdots \\ x_{n-1} &= \mu\alpha_{n-1} \\ x_n &= \lambda + \mu\alpha_n, \quad \text{where } (\lambda,\mu) \neq (0,0). \end{aligned}$$

Let $\mu = 1$, thus

$$\begin{aligned} c^2(-\alpha_0^2 + \ldots + \alpha_{n-k}^2) + (\lambda + \alpha_n)^2 = 0, \\ \lambda = \pm c\sqrt{\alpha_0^2 - (\alpha_1^2 + \ldots + \alpha_{n-k}^2)} - \alpha_n. \end{aligned}$$

However, F is the part of a set with equation $c^2(-x_0^2+x_1^2+\ldots+x_{n-k}^2)+x_n^2 = 0$, which is on the one side of the hyperplane P. Hence $|L(\rho,\alpha)\cap F| = 1$.

(ii) is a direct consequence of Lemma 2.6 and Theorem 2.7 from [3]. ∎

Proposition 1. is a direct consequence of Lemma 1. ∎

LEMMA 2. *If* $G \in S$ *has the base* $B : \sum_{i=0}^{n} A_i x_i = 0$, $P : x_n = 0$, $\rho = [0,0,\ldots,0,1]$, *and* $F \in S$ *has the base* P, *then* $\sigma_F^\rho(G) \in S$ *and it has the base* $\sigma_P^\rho(B)$.

Proof. Let $F = P$, whence the thesis is trivial because $\sigma_F^\rho = \sigma_P^\rho \in \mathrm{Aut}(\mathbb{H}_k^n)$.

Let $F \in S$ and F be not a hyperplane. Thus, by Theorem 2.9 from [3], F is described by equation: $c^2(-x_0^2 + x_1^2 + \ldots + x_{n-k}^2) + x_n^2 = 0$. Let us see that $\alpha = [1,0,\ldots,0,c] \in F$. By Theorem 2.12 from [3], G is described by the equation:

$$\begin{aligned} &\sum_{j=1}^{n-k}(u^2 + A_j^2)x_j^2 + \sum_{j=n-k+1}^{n} A_j^2 x_j^2 \\ &\qquad + \sum_{\substack{i,j=1 \\ i\neq j}}^{n} A_i A_j x_i x_j + 2\sum_{j=1}^{n} A_0 A_j x_0 x_j = (u^2 - A_0^2)x_0^2. \end{aligned}$$

By Definition 1.2 from [3], $\sigma_P^\rho(B) : \sum_{i=1}^{n-1} A_i x_i - A_n x_n = 0$. Let H be an equidistant hypersurface with the base $\sigma_P^\rho(B)$ such that α is the affine centre of the segment $(L(\rho,\alpha)\cap G)(L(\rho,\alpha)\cap H)$. Whence, in general, H is described by the equation:

$$\sum_{j=1}^{n-k}(u_1^2 + A_j^2)x_j^2 + \sum_{j=n-k+1}^{n} A_j^2 x_j^2 + \sum_{\substack{i,j=1\\ i\neq j}}^{n-1} A_i A_j x_i x_j + 2\sum_{j=1}^{n-1} A_0 A_j x_0 x_j$$

$$-2\sum_{j=1}^{n-1} A_n A_j x_n x_j - 2A_0 A_n x_0 x_n - 2A_0 A_n x_0 x_n = (u_1^2 - A_0^2)x_0^2.$$

Now we can calculate the coordinates of the points $g = L(\rho,\alpha) \cap G$ and $h = L(\rho,\alpha) \cap H$. We get $g = [\lambda', 0, \ldots, \mu']$, $h = [\lambda'', 0, \ldots, \mu'']$ for suitable $\lambda', \mu', \lambda'', \mu''$. Clearly, $\lambda', \lambda'' \neq 0$; thus $g = [1, 0, \ldots, \mu_1]$, $h = [1, 0, \ldots, \mu_2]$. As $g \in G$, hence $A_n^2\mu_1^2 + 2A_0A_n\mu_1 - (u^2 - A_0^2) = 0$, so $\mu_1 = (-A_0 + u)/A_n$, and, analogously, $\mu_2 = (A_0 + u_1)/A_n$. Since α is the affine centre of the segment gh, $u + u_1 = 2A_n c$. Now we prove that $\sigma_F^\rho(G) = H$, i.e. we prove that for every line L of $\mathbf{H}_k^n$ passing through ρ, the affine centre of a segment qr, where $q = L \cap G$, $r = L \cap H$, lies on F. First we calculate the coordinates of the points q and r. L is described by the equations:

$$\begin{aligned}
x_0 &= \lambda \\
x_1 &= \lambda\alpha_1 \\
x_2 &= \lambda\alpha_2 \\
&\;\vdots \\
x_{n-k} &= \lambda\alpha_{n-k} \\
x_{n-k+1} &= 0 \\
&\;\vdots \\
x_{n-1} &= 0 \\
x_n &= \mu, \quad \text{where } (\lambda,\mu) \neq (0,0) \quad \text{and } \sum_{i=1}^{n-k}\alpha_i^2 > 1.
\end{aligned}$$

We can set $\lambda = 1$. As $q \in G$, hence q is characterized by

$$\mu = \Big(- \Big(\sum_{i=1}^{n-k} A_i\alpha_i + A_0\Big) \pm u\sqrt{1 - \textstyle\sum_{i=1}^{n-k}\alpha_i^2}\Big)/A_n, \quad \text{i.e.}$$

$$q = \Big[1, \alpha_1, \ldots \alpha_{n-k}, 0, \ldots, 0, \Big(- \Big(\sum_{i=1}^{n-k} A_i\alpha_i + A_0\Big) + u\sqrt{1 - \textstyle\sum_{i=1}^{n-k}\alpha_i^2}\Big)/A_n\Big].$$

Analogously,

$$r = \Big[1, \alpha_1, \ldots \alpha_{n-k}, 0, \ldots, 0, \Big(\sum_{i=1}^{n-k} A_i\alpha_i + A_0 + u_1\sqrt{1 - \textstyle\sum_{i=1}^{n-k}\alpha_i^0}\Big)/A_n\Big].$$

The affine centre α_1 of the considered segment has the coordinates:

$$\alpha_1 = \left[1, \alpha_1, \ldots \alpha_{n-k}, 0, \ldots, 0, \left(+ (u + u_1)\sqrt{1 - \sum_{i=1}^{n-k} \alpha_i^2}\right)/2A_n\right].$$

However, $u + u_1 = 2A_n c$, hence

$$\alpha_1 = \left[1, \alpha_1, \ldots \alpha_{n-k}, 0, \ldots, 0, \left(+ c\sqrt{1 - \sum_{i=1}^{n-k} \alpha_i^2}\right)/2A_n\right].$$

Now it is easily seen that $\alpha_1 \in F$, because α_1 satisfies the equation of F. Hence $\sigma_F^{\rho}(G) = H$. ∎

LEMMA 3. *If $E \in S$ has a base Q, $q \in \mathbf{V}(\mathbb{C}_k^n)$, and $q \in \mathbf{V}(E)$, $E_1 \in S$ has a base Q_1, then $\sigma_E^{\rho}(E_1) \in S$ and has the base $\sigma_Q^q(Q_1)$.*

Proof. By Lemma 1 (ii), there exists $\psi \in \operatorname{Aut}(\mathbb{H}_k^n)$ such that $\psi^*(q) = \rho = [0, 0, \ldots, 0, 1]$ and $\psi(E) = F$, where F is an equidistant hypersurface with the base $P : x_n = 0$. Whence

$$\sigma_E^q(E_1) = \sigma_{\psi^{-1}(F)}^{\psi^{*-1}(\rho)}(E_1) = \psi^{-1}(\sigma_F^{\rho}(E_1) = \psi^{-1}\sigma_F^{\rho}\psi(E_1).$$

Now, by Lemma 2, $\psi^{-1}\sigma_F^{\rho}\psi(E_1)$ is an equidistant hypersurface belonging to S with the base $\sigma_Q^q(Q_1)$. Hence we get the thesis. ∎

Theorem 1. is a direct consequence of Lemma 3. ∎

Proof of Theorem 5. Note that

$$\Sigma(\mathbb{H}_k^n) = \{\sigma_Q^q|\mathbb{C}_k^n : \sigma_Q^q \in \theta, \mathbf{V}(\mathbb{C}_k^n) \not\subset Q, q \in \mathbf{V}(\mathbb{C}_k^n) \setminus Q\},$$

$$\Sigma(\mathbb{L}_k^{n-1}) = \{\sigma_Q^q|\mathbf{W}_k^{n-1} : \sigma_Q^q \in \theta, \mathbf{V}(\mathbb{C}_k^n) \not\subset Q, q \in \mathbf{V}(\mathbb{C}_k^n) \setminus Q\}, \quad \text{and}$$

$$\Sigma(\tfrac{1}{2}\mathbb{L}_k^{n-1}) = \{\sigma_Q^q|\tfrac{1}{2}\mathbf{W}_k^{n-1} : \sigma_Q^q \in \theta, \mathbf{V}(\mathbb{C}_k^n) \not\subset Q, q \in \mathbf{V}(\mathbb{C}_k^n) \setminus Q\},$$

where θ is the set of symmetries of $\mathbb{P}_n$.

If $k\langle n-1, \mathbf{V}(\mathbb{C}_k^n) \subset Q_i$, and $q_i \in \mathbf{V} \setminus q_i$ for $i = 1, 2$, then it is easily seen that the following conditions are equivalent:

1. $$\sigma_{Q_1}^{q_1} = \sigma_{Q_2}^{q_2};$$
2. $$\sigma_{Q_1}^{q_1}|\mathbb{C}_k^n = \sigma_{Q_2}^{q_2}|\mathbb{C}_k^n;$$
3. $$\sigma_{Q_1}^{q_1}|\mathbf{W}_k^{n-1} = \sigma_{Q_2}^{q_2}|\mathbf{W}_k^{n-1};$$
4. $$\sigma_{Q_1}^{q_1}|\tfrac{1}{2}\mathbf{W}_k^{n-1} = \sigma_{Q_2}^{q_2}|\tfrac{1}{2}\mathbf{W}_k^{n-1}.$$

Thus there exist bijections f_1 and f_2 defined by

$$\sigma_Q^q|\mathbb{C}_k^n \overset{f_1}{\longmapsto} \sigma_Q^q|\mathbf{W}_k^{n-1} \overset{f_2}{\longmapsto} \sigma_Q^q|\tfrac{1}{2}\mathbf{W}_k^{n-1}.$$

Whence f_1 induces, on the generators, an isomorphism between $G(\Sigma(\mathbb{H}_k^n))$ and $G(\Sigma(\mathbb{L}_k^{n-1}))$, and f_2 induces, on the generators, an isomorphism between $G(\Sigma(\mathbb{L}_k^{n-1}))$ and $G(\Sigma(\frac{1}{2}\mathbb{L}_k^{n-1}))$.

From Theorem 2.19 (see [3]) we have $\frac{1}{2}\mathbf{L}_k^{n-1} \cong \overline{\mathbf{H}}_k^{n-1}$. This isomorphism is denoted by ϕ. Thus there exists a bijection f_3 defined by $\sigma \overset{f_3}{\longmapsto} \phi\sigma\phi^{-1}$, where $\sigma \in \Sigma(\overline{\mathbf{H}}_k^{n-1}) = \Lambda(\mathbf{H}_k^{n-1})$ and $\phi\sigma\phi^{-1} \in \Sigma(\frac{1}{2}\mathbf{L}_k^{n-1})$. Whence f_3 induces on the generators an isomorphism between $G(\Lambda(\mathbf{H}_k^{n-1}))$ and $G(\Sigma(\frac{1}{2}\mathbf{L}_k^{n-1}))$. Hence we have the thesis. ∎

The class of all $m \times n$ matrices is denoted by $M_{m,n}$. From among all the matrices we distinguish certain special types of them, i.e. zero-matrices

$$O_{m,n} := \begin{bmatrix} 0 & & 0 \\ & \ddots & \\ 0 & & 0 \end{bmatrix}_{m\times n}, \quad \text{and unit matrices } \Delta_n := \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}_{n\times n}.$$

LEMMA 4. *We have*

$$\det([m_{ij}]_{0<i,j<r+1} - \lambda\Delta_r) = (-1)^r[\lambda^r\lambda^{r-1}M_1 + \lambda^{r-2}M_2 - \ldots + (-1)^r M_r],$$

where M_i is the sum of all minors with i rows and i columns obtained from the matrix $[m_{ij}]_{0<i,j<r+1}$ by cancelling the appropriate quantity of rows and columns possessing equal numbers (*see* [2], p. 104).

LEMMA 5. *Let $l = n - k$. If M is a matrix of the projective collineation f of P_n with the distinguished cylinder $\mathbf{W}_k^{n-1}$, then M has the form*

$$\begin{bmatrix} A & O_{l+1,k} \\ B & E \end{bmatrix} \quad \textit{if and only if } f(\mathbf{V}(\mathbb{C}_k^n)) = \mathbf{V}(\mathbb{C}_k^n).$$

Proof. "⇒" Let $M = \begin{bmatrix} A & O_{l+1,k} \\ B & E \end{bmatrix}$ be a matrix of a collineation f and let $q = [0,\ldots,0,q_{l+1},\ldots,q_n] \in \mathbf{V}(\mathbb{C}_k^n)$. Then $f(q) = [0,\ldots,0,q'_{l+1},\ldots,q'_n]$ and thus $f(\mathbf{V}(\mathbb{C}_k^n)) \subseteq \mathbf{V}(\mathbb{C}_k^n)$. But $\dim(f(\mathbf{V}(\mathbb{C}_k^n))) = \dim(\mathbf{V}(\mathbb{C}_k^n))$ because f is a collineation, hence $f(\mathbf{V}(\mathbb{C}_k^n)) = \mathbf{V}(\mathbb{C}_k^n)$.

"⇐" Let f be a projective collineation of $\mathbf{P}_n$ with the matrix $M = [m_{ij}]_{-1<i,j<n+1}$, such that $f(\mathbf{V}(\mathbb{C}_k^n)) = \mathbf{V}(\mathbb{C}_k^n)$. Since $f(\mathbf{V}(\mathbb{C}_k^n)) = \mathbf{V}(\mathbb{C}_k^n)$, thus $f(q) \in \mathbf{V}$ for any $q \in \mathbf{V}$. If $q \in \mathbf{V}$, then $q = [0,\ldots,0,q_{l+1},\ldots,q_n]$, so $f(q)_i = \sum_{j=0}^n m_{ij}q_j = \sum_{j=l+1}^n m_{ij}q_j$; since $f(q) \in \mathbf{V}(\mathbb{C}_k^n)$, $f(q)_i = 0$ for $i \le l$ and thus $\sum_{j=l+1}^n m_{ij}q_j = 0$ for $i \le l$, $j > l$. Whence $m_{ij} = 0$ for $i \le l$, $j > l$. Hence $M = \begin{bmatrix} A & O_{l+1,k} \\ B & E \end{bmatrix}$. ∎

Whence any projective automorphism of the space $\mathbb{H}_k^n$ has a matrix M such that $M = \begin{bmatrix} A & O_{l+1,k} \\ B & E \end{bmatrix}$.

It is easily seen that the following lemma is true.

LEMMA 6. *If* $M, N \in M_{n+1,n+1}$,

$$M = \begin{bmatrix} A_1 & O_{l+1,k} \\ B_1 & E_1 \end{bmatrix}, \text{ and } N = \begin{bmatrix} A_2 & O_{l+1,k} \\ B_2 & E_2 \end{bmatrix},$$

$$\text{then } MN = \begin{bmatrix} A_1 A_2 & O_{l+1,k} \\ B_3 & E_1 E_2 \end{bmatrix}.$$

Whence we see that the group of matrices which have the form $M = \begin{bmatrix} A & O_{l+1,k} \\ B & E \end{bmatrix}$ contains the subgroup of matrices which have the form $M = \begin{bmatrix} \Delta_{l+1} & O_{l+1,k} \\ B & E \end{bmatrix}$. The appropriate group of transformations with the matrices which have form $M = \begin{bmatrix} \Delta_{l+1} & O_{l+1,k} \\ B & E \end{bmatrix}$ is denoted by I_k^n. Now we see that the following lemma is true.

LEMMA 7. *If* $M, N \in M_{n+1,n+1}$,

$$M = \begin{bmatrix} \lambda_1 \Delta_{l+1} & O_{l+1,k} \\ B & E \end{bmatrix}, \text{ and } N = \begin{bmatrix} \lambda_2 \Delta_{l+1} & O_{l+1,k} \\ C & H \end{bmatrix},$$

$$\text{then } MN = \begin{bmatrix} \lambda_1 \lambda_2 \Delta_{l+1} & O_{l+1,k} \\ D & EH \end{bmatrix},$$

where $D = [d_{ij} = b_{ij}\lambda_2 + \sum_{r=1}^{k} c_{rj} e_{ir}]_{0<i<k+1,-1<j<l+1}$.

From Lemma 7 we have following corollaries.

COROLLARY 1.

(i) $$\begin{bmatrix} \Delta_{l+1} & O_{l+1,k} \\ B_1 & \Delta_k \end{bmatrix} \begin{bmatrix} \Delta_{l+1} & O_{l+1,k} \\ B_2 & \Delta_k \end{bmatrix} = \begin{bmatrix} \Delta_{l+1} & O_{l+1,k} \\ B_1 + B_2 & \Delta_k \end{bmatrix};$$

(ii) $$\begin{bmatrix} \Delta_{l+1} & O_{l+1,k} \\ O_{k,l+1} & E_1 \end{bmatrix} \begin{bmatrix} \Delta_{l+1} & O_{l+1,k} \\ O_{k,l+1} & E_2 \end{bmatrix} = \begin{bmatrix} \Delta_{l+1} & O_{l+1,k} \\ O_{k,l+1} & E_1 E_2 \end{bmatrix};$$

(iii) $$\begin{bmatrix} \Delta_{l+1} & O_{l+1,k} \\ B & \Delta_k \end{bmatrix} \begin{bmatrix} \Delta_{l+1} & O_{l+1,k} \\ O_{k,l+1} & E \end{bmatrix} = \begin{bmatrix} \Delta_{l+1} & O_{l+1,k} \\ B & E \end{bmatrix}.$$

COROLLARY 2.

(i) $$\begin{bmatrix} \Delta_{l+1} & O_{l+1,k} \\ B & \Delta_k \end{bmatrix}^{-1} = \begin{bmatrix} \Delta_{l+1} & O_{l+1,k} \\ -B & \Delta_k \end{bmatrix};$$

(ii) $$\begin{bmatrix} \Delta_{l+1} & O_{l+1,k} \\ O_{k,l+1} & E \end{bmatrix}^{-1} = \begin{bmatrix} \Delta_{l+1} & O_{l+1,k} \\ O_{k,l+1} & E^{-1} \end{bmatrix}.$$

Now we see that the group I_k^n contains three subgroups, the group E_k^n of transformations with the matrices which have the form $\begin{bmatrix} \Delta_{l+1} & O_{l+1,k} \\ O_{k,l+1} & E \end{bmatrix}$, the group T_k^n of transformations with the matrices which have the form

$\begin{bmatrix} \Delta_{l+1} & O_{l+1,k} \\ B & \Delta_k \end{bmatrix}$, and the group $I_k^{n^*}$ of transformations with the matrices which have the form $\begin{bmatrix} \Delta_{l+1} & O_{l+1,k} \\ B & E \end{bmatrix}$, where $\det(E) = \pm 1$.

Of course T_k^n is a subgroup of $I_k^{n^*}$.

As a direct consequence of Corollary 1 we infer

COROLLARY 3. *For any $f \in I_k^n$ there exist g and h such that $h \in E_k^n$, $g \in T_k^n$, and $f = g \circ h$.*

LEMMA 8. *If $\sigma_Q^q \in \Sigma(\mathbf{H}_k^n)$, $q = [q_0, \ldots, q_n]$, and hyperplane Q has the equation $\sum_{i=0}^n = 0$, then the symmetry σ_Q^q has a matrix of the form*

$$M = \begin{bmatrix} \Delta_{l+1} & O_{l+1,k} \\ B & E \end{bmatrix},$$

where $E = \Delta_k - (2/\alpha)[q_i A_j]_{l<i,j<n+1}$, $B = (-2/\alpha)[q_i A_j]_{l<i<n+1,-1<j<l+1}$, $\alpha \neq 0$, and $\det(E) = -1$.

Proof. Since Q is a non isotropic hyperplane, thus $A_s \neq 0$ for some s with $l \leq s \leq n$, and $q \in \mathbf{V}(\mathbb{C}_k^n)$, because $\sigma_Q^q \in \Sigma(\mathbf{H}_k^n)$, whence $q = [0, \ldots, 0, q_{l+1}, \ldots, q_n]$. Now, by Definition 1.2 from [3] we get $\sigma_Q^q(x)_i = \sum_{s=0}^n A_s(q_s x_i - 2q_i x_s) = \sum_{s=0}^n m_{is} x_s$, where

$$m_{is} = \begin{cases} 0 & \text{for } i \neq s,\ 0 \leq i \leq l \\ \sum_{s=l+1}^n A_s Q_s & \text{for } i = s,\ 0 \leq i \leq l \\ -2q_i A_s & \text{for } i \neq s,\ l < i \leq n \\ \sum_{s=l+1}^n A_s q_s - 2q_i A_s & \text{for } i = s,\ l < i \leq n. \end{cases}$$

Let $\alpha = \sum_{s=l+1}^n A_s q_s$. Whence $\alpha \neq 0$, because $q \notin Q$. Matrix $M_0 = [m_{is}]_{-1<i,s<n+1}$ is a matrix of the symmetry σ_Q^q. But $M = (1/\alpha)M_0$, thus M is a matrix of the symmetry σ_Q^q too. The proof will be completed by showing that $\det(E) = -1$ i.e. $\det([m_{is}]_{l<i,s<n+1}) = -\alpha^k$, where $[m_{is}]_{l<i,s<n+1}$ is an appropriate submatrix of M_0. This submatrix has the following form

$$\begin{bmatrix} -2q_{l+1}A_{l+1} & -2q_{l+1}A_{l+2} & \cdots & -2q_{l+1}A_n \\ \vdots & & & \\ -2q_n A_{l+1} & -2q_n A_{l+2} & \cdots & -2q_n A_n \end{bmatrix} + \alpha\Delta_k.$$

By Lemma 4,

$$\det([m_{is}]_{l<i,s<n+1}) = (-1)^k[(-\alpha)^k - (-\alpha)^{k-1}M_1 + (-\alpha)^{k-2}M_2 - \ldots + (-1)^k M_k].$$

It is easily seen that $M_i = 0$ for $1 < i \leq k$. Hence $\det([m_{is}]_{l<i,s<n+1}) = -\alpha^k$. ■

LEMMA 9. *If $g \in I_k^{n*} \cap E_k^n$, then g is a superposition of symmetries of $\mathbb{H}_k^n$ with non isotropic axes.*

Proof. Let f be a skew symmetry of F^k with axis which is a $(k-1)$-dimensional hyperplane A such that $(0,0\ldots,0) \in A$ in F^k. Whence A has the equation $\sum_{j=1}^k A_j x_j = 0$. Let $\rho = [0, \rho_1, \ldots, \rho_k]$ with $\sum_{j=1}^k A_j \rho_j \neq 0$ be a direction of this symmetry. The isotropic hyperplane A' with the equation $\sum_{j=1}^k A_j x_{l+j} = 0$, and the point ρ corresponds to point $q = [0, \ldots, 0, \rho_1, \ldots, \rho_k] \notin A'$, and then $\sigma_{A'}^q \in \Sigma(\mathbb{H}_k^n)$. Now, by Lemma 8, the matrix $M(\sigma_{A'}^q)$ is $\begin{bmatrix} \Delta_{l+1} & O_{l+1,k} \\ O_{k,l+1} & M(f) \end{bmatrix}$, where $M(f)$ is a matrix of the skew symmetry f. Let $g \in I_k^{n*} \cap E_k^n$. By definition of g, $M(g) = \begin{bmatrix} \Delta_{l+1} & O_{l+1,k} \\ O_{k,l+1} & D \end{bmatrix}$, where $\det(D) = \pm 1$. Therefore the transformation $g' : F^k \longmapsto F^k$ with the matrix D is an equiaffine transformation. Thus there exist skew symmetries $f_1, \ldots, f_n$ such that $g' = f_m \circ \ldots \circ f_1$. Each of these symmetries induces in $\mathbb{H}_k^n$ a symmetry $g_i \in \Sigma(\mathbb{H}_k^n)$ such that $M(g_i) = \begin{bmatrix} \Delta_{l+1} & O_{l+1,k} \\ O_{k,l+1} & M(f_i) \end{bmatrix}$. Let us see that $D = M(f_m) \cdot \ldots \cdot M(f_1)$, thus $M(g) = M(g_m) \cdot \ldots \cdot M(g_1)$. Hence $g = g_m \circ \ldots \circ g_1$. ■

LEMMA 10. *If $g \in T_k^n$, then g is a superposition of symmetries of $\mathbb{H}_k^n$ with non isotropic axes.*

Proof. Let $g \in T_k^n$, whence $M(g) = \begin{bmatrix} \Delta_{l+1} & O_{l+1,k} \\ B & \Delta_k \end{bmatrix}$. For $0 \leq j \leq l$ we consider arbitrary $g_j \in T_k^n$ with the matrix $M(g_j) = \begin{bmatrix} \Delta_{l+1} & O_{l+1,k} \\ B_j & \Delta_k \end{bmatrix}$ such that

$$B_j = \begin{bmatrix} & b_{l+1,j} & \\ O_{k,j} & \vdots & O_{k,l-j} \\ & b_{n,j} & \end{bmatrix}$$

and we prove the thesis for such g_j. If $b_{s,j} = 0$ for $l < s \leq n$, then $B_j = O_{k,l+1}$ and, by Lemma 9, g_j is a superposition of symmetries of $\mathbb{H}_k^n$ with non isotropic axes, so we can assume $b_{r,j} \neq 0$. Let us consider two $(n-1)$-dimensional non isotropic hyperplanes of $\mathbb{H}_k^n$ with the equations $A : x_r = 0$ and $A' : x_r + cx_j = 0$, where $c = (-1/2)b_{r,j}$ and let

$q = [0, \ldots, 0, b_{l+1,j}, \ldots, b_{n,j}]$. Then $q \in \mathbf{V}(\mathbb{C}_k^n)$ and $q \notin A, A'$, so $\sigma_A^q, \sigma_{A'}^q \in \Sigma(\mathbb{H}_k^n)$. By Lemma 8, $M(g_j) = M(\sigma_{A'}^q \circ \sigma_A^q)$, hence $g_j = \sigma_{A'}^q \circ \sigma_A^q$. Now clearly, the matrix B is a sum of the matrices B_j $(0 \leq j \leq l)$ as above, so, by Corollary 1(i)

$$\begin{bmatrix} \Delta_{l+1} & O_{l+1,k} \\ B & \Delta_k \end{bmatrix} = M(g_0) \cdot \ldots \cdot M(g_l), \quad \text{thus } g = g_0 \circ \ldots \circ g_l. \ \blacksquare$$

PROPOSITION 2. *We have* $G(\Sigma(\mathbb{H}_k^n)) = I_k^{n^*}$.

Proof. From Lemma 8 and the definition of $I_k^{n^*}$ we have $G(\Sigma(\mathbb{H}_k^n)) \subseteq I_k^{n^*}$. Next, by Corollaries 3, 1(iii), and Lemmas 9, 10, $I_k^{n^*} \subseteq G(\Sigma(\mathbb{H}_k^n))$. ■

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