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SOME REMARKS ON Σ_2^0 SUPPORTED σ-IDEALS

1. Introduction

The paper [KS] contains an interesting characterization of Σ_2^0 supported σ-ideals in a Polish space X. That result was used in [BR] to solve a problem of Mauldin concerning generalized Baire systems. Here some other consequences of the Kechris–Solecki theorem are observed. In Section 2 we establish all possible values of the order $R(I)$ for Σ_2^0 supported σ-ideals in X. In Section 3 we consider Σ_2^0 supported σ-ideals of the form $\mathrm{MGR}(\mathcal{F})$ (it is one of the two kinds of Σ_2^0 supported σ-ideals stated in the Kechris–Solecki theorem). We find a "complementary" σ-ideal to $\mathrm{MGR}(\mathcal{F})$, which generalizes the well known expression of $\mathbb{R}$ as the union of a Lebesgue null set and a set of the first category. In Section 4 we derive from [BR] that the Baire system generated by the family of functions with property K_I (defined in [B1]) either stops on the first stage or requires ω_1 steps.

Let us explain some notation and give necessary definitions.

Denote by X a separable and complete metric space (briefly called *Polish*) which is additionaly uncountable and dense in itself.

Let I be a σ-ideal of subsets of X. In this paper we will assume that all singletons $\{x\}$ belong to I (a σ-ideal which has that property is called *uniform*) and I does not contain nonempty open sets.

Denote by $\mathcal{B}$ the family of all Borel subsets of X, and by Σ_α^0, Π_α^0 (for $0 < \alpha < \omega_1$) - the subclasses of $\mathcal{B}$ defined as in [Mo, 1 F]. In particular, Σ_2^0 is the pointclass of F_σ sets.

We will also need the following definition: a σ-ideal is said to be Σ_2^0 *supported* if for any $A \in I$ there is $B \in \Sigma_2^0 \cap I$ with $A \subset B$.

Let $\mathrm{MGR}(\mathcal{F}) = \{A \subset X : (\forall F \in \mathcal{F}) A \cap F \text{ is meager in } F\}$ for a family $\mathcal{F}$ of closed sets.

Define (cf. [B1]) $R(I) = \min\{\alpha \le \omega_1 : (\forall B \in \mathcal{B})(\exists A \in \Sigma_\alpha^0)(B \triangle A \in I)\}$ where $\Sigma_{\omega_1}^0 = \mathcal{B}$ and $B \triangle A = (B \setminus A) \cup (A \setminus B)$. Observe that Σ_α^0 can be replaced by Π_α^0 in the above definition.

2. $R(I)$ for Σ_2^0 supported σ-ideals

In this section we will establish $R(I)$ for Σ_2^0 supported σ-ideals. Let us start with the above mentioned Kechris-Solecki theorem.

PROPOSITION 1 (see [KS, thm 2]). *Let I be a Σ_2^0 supported σ-ideal. Then precisely one of the following possibilities holds:*

(i) $I = \mathrm{MGR}(\mathcal{F})$ *for a countable family* $\mathcal{F} = \{F_\gamma : \gamma < \alpha\}$, $\alpha < \omega_1$, *of closed subsets of X (moreover it can be assumed that $F_\gamma \subset F_\beta$ for $\beta < \gamma < \alpha$ and $F_{\gamma+1}$ is nowhere dense in F_γ for $\gamma < \alpha$);*

(ii) *there is a homeomorphic embedding* $\varphi : 2^\omega \times \omega^\omega \to X$ *such that* $\varphi[\{\alpha\} \times \omega^\omega] \notin I$ *for any* $\alpha \in 2^\omega$. ■

PROPOSITION 2 (see [BR, proposition 2]). *If a σ-ideal I satisfies condition (ii) of Proposition 1 then I has the following property:*

(M) *there exists a Borel function* $f^{-1}[\{x\}] \notin I$ *for each* $x \in X$. ■

PROPOSITION 3 (see [B3, corollary 2.2]). *If a σ-ideal I has the property (M) then* $R(I) = \omega_1$. ■

THEOREM 1. *If I is a Σ_2^0 supported σ-ideal on X then* $R(I) \in \{1, 2, \omega_1\}$. *Moreover*

(a) $R(I) = 1$ *iff* (i) *holds and* $|\mathcal{F}| = 1$,
(b) $R(I) = 2$ *iff* (i) *holds and* $|\mathcal{F}| > 1$,
(c) $R(I) = \omega_1$ *iff* (ii) *holds* .

Proof. Since the statement

((i) holds and $|\mathcal{F}| = 1$) or ((i) holds and $|\mathcal{F}| > 1$) or (ii) holds

is true and any two of the conditions $R(I) = 1$, $R(I) = 2$, $R(I) = \omega_1$ cannot be true simultaneously, it suffices to prove the implications "$\Leftarrow$" in the statements (a), (b), (c).

(a) Assume that $\mathcal{F} = \{F_0\}$. Consider a Borel set B in X. Of course $B \cap F_0$ is a Borel set in F_0. We can write $B \cap F_0 = G \triangle E$ where $G = U \cap F_0$ for some open set U in X, and E is meager in F_0. Thus

$$(B \triangle U) \cap F_0 = (B \cap F_0) \triangle (U \cap F_0) = (B \cap F_0) \triangle G = E$$

is meager in F_0. So $B \triangle U \in I$ and it follows that $R(I) = 1$.

(b) Assume that $|\mathcal{F}| > 1$ and let $\mathcal{F} = \{F_\gamma : \gamma < \alpha\}$, $\alpha < \omega_1$, fulfil the requirements of (i). To obtain $R(I) > 1$ we will prove that $F_1 \triangle U \notin I$ for each open set $U \in X$.

Consider two cases :

1) $U \cap F_0 = \emptyset$, then $F_1 \triangle U = F_1 \cup U \notin I$ since $F_1 \notin I$.

2) $U \cap F_0 \neq \emptyset$, then $(F_1 \triangle U) \cap F_0 = (F_1 \setminus U) \cup (U \cap F_0 \setminus F_1)$ and since F_1 is nowhere dense in F_0, therefore $U \cap F_0 \setminus F_1$ is of the second category in F_0. Hence $F_1 \triangle U \notin I$.

It remains to prove that $R(I) \le 2$. Let $B \in \mathcal{B}$. For each $\gamma < \alpha$ the set $B \cap F_\gamma$ is Borel in F_γ so there exists an open set $G_\gamma \in X$ such that $(B \cap F_\gamma) \triangle (G_\gamma \cap F_\gamma)$ is meager in F_γ. Let $G = \bigcup_{\gamma<\alpha} (G_\gamma \cap F_\gamma \setminus F_{\gamma+1})$. Clearly $G \in \Sigma_2^0$. Additionally, for each $\gamma < \alpha$, the set $(B \triangle G) \cap F_\gamma = (B \cap F_\gamma) \triangle (G_\gamma \cap F_\gamma \setminus F_{\gamma+1})$ is meager in F_γ. Hence $B \triangle G \in I$ and thus $R(I) \le 2$.

(c) By Propositions 2 and 3, condition (ii) implies $R(I) = \omega_1$. ∎

We say that a σ-ideal I fulfils the *countable chain condition* (the c.c.c.) if any family $\mathcal{A}$ of disjoint Borel sets satisfying $\mathcal{A} \cap I = \emptyset$ is countable.

COROLLARY 1. *A Σ_2^0 supported σ-ideal I of subsets of X satisfies the c.c.c. if and only if $R(I) \le 2$.*

Proof. A Σ_2^0 supported σ-ideal satisfies the c.c.c. iff it is of the form (i) (see [KS, thm 3]). ∎

Remark 1. If we do not assume that I is Σ_2^0 supported, the assertion of Theorem 1 is not valid since, for each α, $1 \le \alpha < \omega_1$, there is a σ-ideal I on the Cantor space 2^ω, satisfying $R(I) = \alpha$ (see [Mi, lemma 3]).

3. A complementary σ-ideal to MGR($\mathcal{F}$)

A σ-ideal MGR($\mathcal{F}$), except for the case $\mathcal{F} = \{X\}$, behaves badly, if one considers its invariance properties (see [KS, corollary]). In particular, for $X = \mathbb{R}$, the only translation invariant σ-ideal MGR($\mathcal{F}$) is obtained if $\mathcal{F} = \{\mathbb{R}\}$, i. e. if MGR($\mathcal{F}$) consists of meager sets in $\mathbb{R}$.

However MGR($\mathcal{F}$), like MGR($\{\mathbb{R}\}$), has a "complementary" measure σ-ideal which will be shown in the following theorem.

THEOREM 2. *If I = MGR($\mathcal{F}$) where $\mathcal{F}$ is a family of the form given in Proposition* 1(i) *then there exist a Borel probability measure μ on X and Borel sets A and B such that $A \in I$, $B \in J$ and $A \cup B = X$ where J is the σ-ideal of null sets with respect to the completion of μ.*

Proof. Let $\mathcal{F} = \{F_\gamma : \gamma < \alpha\}$ (where $\alpha < \omega_1$) satisfy the conditions given in Proposition 1(i). We may assume that all sets F_γ are uncountable. Consider a Borel probability measure μ_γ defined on F_γ as follows.

Let $h_\gamma : F_\gamma \to [0,1]$ be a Borel isomorphism (see [K, §37.II.thm 2]). For any set $A \subset F_\gamma$, $A \in \mathcal{B}$, define

$$\mu_\gamma(A) = \lambda(h_\gamma[A])$$

where λ stands for the Lebesgue measure. Express $\{\gamma : \gamma < \alpha\}$ as $\{\gamma_n : n \in \omega\}$ and put

$$\mu(A) = \sum_{n\in\omega} \frac{1}{2^{n+1}} \mu_{\gamma_n}(A \cap F_\gamma)$$

for $A \subset X$, $A \in \mathcal{B}$. Then μ forms a Borel probability measure on X. Let

$$J = \{A \subset X : (\exists B \in \mathcal{B})(A \subset B \wedge \mu(B) = 0)\},$$

i. e. J is the σ-ideal of null sets with respect to the completion of μ.

It is known (see e.g. [MS, §1(v)]) that for each $\gamma < \alpha$ there are Borel sets A_γ, B_γ such that $A_\gamma \cup B_\gamma = F_\gamma$ and A_γ is of the first category in F_γ and $\mu_\gamma(B_\gamma) = 0$ (it can be assumed that A_γ and B_γ are disjoint, A_γ is of type F_σ and B_γ is of type G_δ).

Define

$$A = \bigcup\nolimits_{\gamma<\alpha} (A_\gamma \setminus F_{\gamma+1}) \cup (X \setminus F_0) \cup \bigcup_{\beta<\alpha, \beta-limit} \Big(\bigcap_{\gamma<\beta} (F_\gamma \setminus F_\beta) \Big)$$

and

$$B = \bigcup_{\gamma<\alpha} (B_\gamma \setminus F_{\gamma+1}).$$

Then A, B are Borel, $A \in I$ and $B \in J$ and $X = A \cup B$. ∎

Remark 2. Since $(X, \mathcal{B}, \mu)$ is isomorphic to $([0,1], \mathcal{B}_{[0,1]}, \lambda)$ (see [P, thm 26.6]) where $\mathcal{B}_{[0,1]}$ denotes the σ-field of Borel sets in $[0,1]$, our result generalizes, in some sense, the classical expression of $[0,1]$ as the union of a meager set and a Lebesgue null set (see [O, thm 1.6]).

4. Property (K_I)

In [BR] generalized Baire systems were studied and, in the main theorem, the Baire order problem of Mauldin was solved. Here we derive one corollary from that theorem. Note that the result of [BR] was proved by the use of Σ^0_2 supported σ-ideals and the Kechris–Solecki theorem.

For any family F of real-valued functions defined on X, let $B_0(F) = F$ and, for each ordinal number $\alpha > 0$, let $B_\alpha(F)$ be the family of all pointwise limits of sequences of functions from $\bigcup_{\gamma<\alpha} B_\gamma(F)$. Now we define $r(F) = \min\{\alpha \le \omega_1 : B_{\alpha+1}(F) = B_\alpha(F)\}$ which is called the *Baire order* of the family F.

For a σ-ideal I of subsets of X, we say that a closed nonempty set $A \subset X$ is *I-perfect* if $U \cap A \neq \emptyset$ implies $U \cap A \notin I$ for all open sets U.

In [B1, def.1] Balcerzak introduced the following definition inspired by the paper of Grande [G]. A function $f : X \to \mathbf{R}$ is said to have the property (K_I) if the set of points of continuity of the function $f|A$ is dense in A for every I-perfect set A.

Denote by K_I the family of all functions with the property (K_I) and denote by C_I the family of all functions $f : X \to \mathbf{R}$ whose sets of points of discontinuity are in I.

PROPOSITION 4 [BR]. *If I is a σ-ideal subsets of X then either $r(C_I) = 1$ or $r(C_I) = \omega_1$.* ∎

THEOREM 3. *If I is a σ-ideal of subsets of X then either $r(K_I) = 1$ or $r(K_I) = \omega_1$.* ∎

We know that $K_I \subset B_1(C_I)$ and $B_1(K_I) = B_2(C_I)$ (see [B1, thm 1, thm 2]). Consider two cases concerning $r(C_I)$ stated in Proposition 4.

1) If $r(C_I) = 1$ then $B_2(C_I) = B_1(C_I)$ and $B_2(C_I)$ is closed under pointwise limits. Hence

$$B_2(K_I) = B_3(C_I) = B_2(C_I) = B_1(K_I)$$

and thus $r(K_I) = 1$.

2) If $r(C_I) = \omega_1$ then $r(K_I) = \omega_1$ since $B_n(K_I) = B_{n+1}(C_I)$ for $1 \leq n < \omega$ and $B_\alpha(K_I) = B_\alpha(C_I)$ for $\omega \leq \alpha < \omega_1$. ∎

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