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# METRIC SPACES OVER ORDERED FIELDS

## 1. Introduction

If $F$ is an ordered field, an $F$-metric space is a set $X$ together with a metric $\rho : X \times X \to F$. This concept appears to be an interesting generalization of the classical real metric space. An example of a nonclassical $F$-metric space is the hyperreal field ${}^*\mathbb{R}$ studied in nonstandard analysis [2,7,10,11,13,14]. Besides having mathematical interest, $F$-metric spaces may have important applications. For example, nonstandard Hilbert spaces, which are a specific class of $F$-metric spaces, have been recently applied to studies in quantum field theory and statistical physics [1,4,6].

As one would expect, an $F$-metric space shares some of the properties of a real metric space and does not share other properties. In this paper we study some of these shared and unshared properties. Some of our proofs are similar to the classical proofs. However, a general $F$-metric space does not have two of the important properties of a real metric space. One of these properties is first countability and the other is the Dedekind completeness of $\mathbb{R}$. Because of this, new techniques must be used for some proofs. It turns out that for an $F$-metric space there exists a cardinal $\alpha$ such that the intersection of a family, with cardinality less than $\alpha$, of open sets is open. In general, $\alpha$ may be an uncountable cardinal so this results in a new type of topological space that we call an $\alpha$-topological space.

We begin with a study of $\alpha$-topological spaces and the notion of convergence in terms of $\alpha$-nets. The dual concept of $\alpha$-compactness is introduced and properties of $\alpha$-compact sets are studied. We next consider the completion of an ordered field. If $F$ is an ordered field and $F^+$ is its set of positive elements, an unbounded subset $I$ of $F^+$ is called cofinal. If $I$ is a cofinal set of smallest ordinality $\beta$, then $I$ is minimal cofinal and $\beta$ is the cofinality of $F$. The $I$-completion $\overline{F}$ of $F$ is constructed and it is shown that $\overline{F}$ is an ordered field. We then study properties of an $F$-metric space where $F$ has uncountable cofinality. In general, it is shown that such spaces are non-metrizable, 0-dimensional and nonseparable. It is also shown that they are compact only if they have finite cardinality.

The next to last section is devoted to $I$-complete $F$-metric spaces. Generalizations of Cantor's characterization of complete metric spaces and the Baire category theorem are proved. A generalization of the uniform boundedness principle and other corollaries of the generalized Baire category theorem are given. The final section discusses some miscellaneous results and presents some open problems.

## 2. $\alpha$-Topologies

This section considers a special type of topology called an $\alpha$-topology. As we shall see in our later work, a metric space over an ordered field results in an $\alpha$-topology.

Let $\mathfrak{J}$ be a topology on a set $X$ and let $\alpha$ be an infinite cardinal. We call $\mathfrak{J}$ an **$\alpha$-topology** if whenever $A_i \in \mathfrak{J}$, $i \in \Delta$, where card$(\Delta) < \alpha$, we have $\bigcap A_i \in \mathfrak{J}$. Thus, in an $\alpha$-topology, the intersection of fewer than $\alpha$ open sets is open. If $\mathfrak{J}$ is an $\alpha$-topology for $X$, we call $(X, \mathfrak{J})$ an **$\alpha$-topological space**. Of course, in an $\alpha$-topological space the union of fewer than $\alpha$ closed sets is closed. Notice that an $\aleph_0$-topological space is just an ordinary topological space, while for $\alpha > \aleph_0$ we have a restrictive condition. In the sequel, we assume that our topological spaces are Hausdorff.

An **$\alpha$-directed set** is a poset $\Delta$ such that $\Delta'$ has an upper bound whenever $\Delta' \subseteq \Delta$ with card$(\Delta') < \alpha$. An **$\alpha$-net** in $X$ is a function $f : \Delta \to X$ where $\Delta$ is an $\alpha$-directed set. Notice that an $\aleph_0$-directed set is an ordinary directed set and an $\aleph_0$-net is an ordinary net. We usually denote an $\alpha$-net in $X$ by $(x_i : i \in \Delta)$ and use the usual net terminology for $\alpha$-nets [3,9]. We denote the set of neighbourhoods of a point $x$ by $\mathfrak{N}(x)$. Many of the proofs in this section are similar to the standard proofs except that extra attention must be given to $\alpha$-conditions.

THEOREM 2.1. *If $A$ is a subset of an $\alpha$-topological space, then $x \in \overline{A}$ if and only if there exists an $\alpha$-net in $A$ converging to $x$.*

Proof. Suppose $x \in \overline{A}$ and let $\{U_i : i \in \Delta\} = \mathfrak{N}(x)$. Direct $\Delta$ by $i \leq j$ if $U_j \subseteq U_i$. If $\Delta' \subseteq \Delta$ with card$(\Delta') < \alpha$, we have $\bigcap_{i \in \Delta'} U_i \in \mathfrak{N}(x)$. Hence there exists a $j \in \Delta$ such that $i \leq j$ for every $i \in \Delta'$ so $\Delta$ is an $\alpha$-directed set. Since $x \in \overline{A}$, $U_i \bigcap A \neq \emptyset$ for every $i \in \Delta$. Choosing $x_i \in U_i \bigcap A$, we have an $\alpha$-net $(x_i : i \in \Delta)$ in $A$ converging to $x$. Conversely, suppose $(x_i : I \in \Delta)$ is an $\alpha$-net in $A$ and $x_i \to x$. Then $x_i$ is eventually in any neighbourhood of $x$. Hence, any neighbourhood of $x$ intersects $A$, so $x \in \overline{A}$. ∎

COROLLARY 2.2. *A subset $A$ of an $\alpha$-topological space is closed if and only if $A$ contains the limits of all converging $\alpha$-nets in $A$.*

COROLLARY 2.3. *A subset $A$ of an $\alpha$-topological space is open if and only if every $\alpha$-net converging to a point in $A$ is eventually in $A$.*

COROLLARY 2.4. *Let $X$ and $Y$ be $\alpha$-topological spaces. Then $f : X \to Y$ is continuous if and only if for every converging $\alpha$-net $x_i \to x$ we have $f(x_i) \to f(x)$.*

Let $\Delta', \Delta$ be $\alpha$-directed sets and let $u : \Delta' \to \Delta$. Then $u$ is **finalizing** if for every $i \in \Delta$, $u(j) \geq i$ eventually. An **$\alpha$-subnet** of an $\alpha$-net $(x_i : i \in \Delta)$ is an $\alpha$-net $(x_{u(j)} : j \in \Delta')$ where $u : \Delta' \to \Delta$ is finalizing. A subset $\Delta'$ of a poset $\Delta$ is **cofinal** if for every $i \in \Delta$ there exists $j \in \Delta'$ such that $j \geq i$. Notice that if $\Delta$ is an $\alpha$-directed set and $\Delta'$ is cofinal in $\Delta$, then $\Delta'$ is an $\alpha$-directed set. If $(x_i : i \in \Delta)$ is an $\alpha$-net and $\Delta' \subseteq \Delta$ is cofinal, then $(x_j : j \in \Delta')$ is an $\alpha$-subnet of $(x_i : i \in \Delta)$. Indeed, define $u : \Delta' \to \Delta$ by $u(j) = j$. Then $u$ is finalizing and $(x_j : j \in \Delta') = (x_{u(j)} : j \in \Delta')$. As usual, $x$ is a **cluster point** of an $\alpha$-net $x_i$ if for every $N \in \mathfrak{N}(x)$, $x_i \in N$ frequently. The proof of the next lemma is the same as the standard proof.

LEMMA 2.5. (a) *If an $\alpha$-net $x_i$ is eventually in a set $A$ then every $\alpha$-subnet of $x_i$ is eventually in $A$.* (b) *Every $\alpha$-subnet of a converging $\alpha$-net converges to the same limit as the $\alpha$-net.* (c) *If an $\alpha$-net is eventually in a set $A$, every cluster point of the $\alpha$-net is in $\overline{A}$.*

THEOREM 2.6. *A point $x$ is a cluster point of an $\alpha$-net $x_i$ in an $\alpha$-topological space if and only if $x_i$ has an $\alpha$-subnet converging to $x$.*

Proof. If $x$ is not a cluster point of $x_i$, then there exists an $N \in \mathfrak{N}(x)$ such that $x_i$ is not frequently in $N$. Hence, $x_i$ is eventually in $N^c$ and by Lemma 2.5(a) so is every $\alpha$-subnet of $x_i$. Hence, no $\alpha$-subnet of $x_i$ can converge to $x$. Conversely, let $x$ be a cluster point of $(x_i : i \in \Delta)$. Let

$$\Delta' = \{(i, N) : i \in \Delta, \ x_i \in N \in \mathfrak{N}(x)\}$$

and define $(i, N) \leq (j, M)$ if $i \leq j$ and $M \subseteq N$. To show that $\Delta'$ is an $\alpha$-directed set, let $\{(j, N_j) : j \in \Gamma\}$ be a subset of $\Delta'$ with $\mathrm{card}(\Gamma) < \alpha$. Let $k' \in \Delta$ be an upper bound for $\{j : j \in \Gamma\}$ and let $M = \bigcap_{j \in \Gamma} N_j$. Then $M \in \mathfrak{N}(x)$ and since $x$ is a cluster point of $x_i$, there exists a $k \in \Delta$ such that $k \geq k'$ and $x_k \in M$. Hence, $(k, M) \in \Delta'$ and $(k, M)$ is an upper bound for $\{(j, N_j) : j \in \Gamma\}$. Define $u : \Delta' \to \Delta$ by $u(i, N) = i$. To show that $u$ is finalizing, let $j \in \Delta$ be given and let $(k, M) \in \Delta'$ satisfy $k \geq j$. Then for $(i, N) \in \Delta'$ with $(i, N) \geq (k, M)$ we have $u(i, N) \geq j$. Hence, $(x_{u(i')} : i' \in \Delta')$ is an $\alpha$-subnet of $(x_i : i \in \Delta)$. To show that $x_{u(i')} \to x$, let

$M \in \mathfrak{N}(x)$. Then there exists an $x_j$, $j \in \Delta$, such that $x_j \in M$. For $i' \in \Delta'$ with

$$i' = (i, N) \geq (j, M)$$

we have $x_{u(i')} \in M$. ∎

Corresponding to the notion of an $\alpha$-topology there is dual notion of $\alpha$-compactness. Let $A \subseteq X$ where $X$ is an $\alpha$-topological space. We say that $A$ is **$\alpha$-compact** if for any open cover $A \subseteq \bigcup_{i \in \Delta} E_i$, there is an open subcover $A \subseteq \bigcup_{i \in \Delta'} E_i$, where $\Delta' \subseteq \Delta$ and $\text{card}(\Delta') < \alpha$. We call such a subcover an **$\alpha$-subcover**. Of course, an $\aleph_0$-compact set is an ordinary compact set. Notice that if $\text{card}(A) < \alpha$, then $A$ is $\alpha$-compact. We now show that $\alpha$-compact sets have some of the usual properties of ordinary compact sets.

THEOREM 2.7. *If $A$ and $B$ are disjoint $\alpha$-compact sets in an $\alpha$-topological space, then there exist open sets $U, V$ such that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$.*

Proof. This is similar to the standard proof. ∎

COROLLARY 2.8. *An $\alpha$-compact subset of an $\alpha$-topological space is closed.*

The usual proof shows that a closed subset of an $\alpha$-compact set is $\alpha$-compact and the continuous image of an $\alpha$-compact set is $\alpha$-compact. A family of sets $\{A_i : i \in \Delta\}$ has the **$\alpha$-intersection property** if $\bigcap_{i \in \Delta'} A_i \neq \emptyset$ for every $\Delta' \subseteq \Delta$ with $\text{card}(\Delta') < \alpha$. The proof of the next result is similar to the standard proof.

THEOREM 2.9. *A subset $A$ of an $\alpha$-topological space is $\alpha$-compact if and only if for any family of closed subsets $\{A_i : i \in \Delta\}$ of $A$ with the $\alpha$-intersection property we have $\cap A_i \neq \emptyset$.*

The next result gives a characterization of $\alpha$-compactness.

THEOREM 2.10. *A subset $A$ of an $\alpha$-topological space is $\alpha$-compact if and only if every $\alpha$-net in $A$ has a subnet converging to a point in $A$.*

Proof. Let $A$ be $\alpha$-compact and let $(x_i : i \in \Delta)$ be an $\alpha$-net in $A$. For each $i \in \Delta$, let $T(i) = \overline{\{x_j : j \geq i\}}$. Then $\{T(i) : i \in \Delta\}$ is a family of closed subsets of $A$ with the $\alpha$-intersection property. Indeed, let $\Delta' \subseteq \Delta$ with $\text{card}(\Delta') < \alpha$ and let $k \in \Delta$ be an upper bound for $\Delta'$. Then $T(k) \subseteq T(i)$ for every $i \in \Delta'$ and $T(k) \neq \emptyset$. By Theorem 2.9, $\bigcap_{i \in \Delta} T(i) \neq \emptyset$. If $x \in \bigcap T(i)$, then $x$ is a cluster point of $x_i$ so by Theorem 2.6 there exists an $\alpha$-subnet of $x_i$ converging to $x$. Conversely, suppose every $\alpha$-net in $A$ has an $\alpha$-subnet converging to a point in $A$. Let $\mathfrak{S}$ be a family of closed subsets of $A$ with the $\alpha$-intersection property. Let $\Delta$ be the family of sets of the form $\bigcap_{j \in \Gamma} B_j$

where $B_j \in \mathfrak{S}$ and $\text{card}(\Gamma) < \alpha$. Since $\alpha$ is an infinite cardinal, $\Delta$ is a family of closed sets with the $\alpha$-intersection property. For $i, j \in \Delta$, define $i \leq j$ if $j \subseteq i$. To show that $\Delta$ is an $\alpha$-directed set, suppose $\Delta' \subseteq \Delta$ with $\text{card}(\Delta') < \alpha$. Then $k = \bigcap_{i \in \Delta'} i \in \Delta$ and $i \leq k$ for all $i \in \Delta'$. For each $i \in \Delta$, choose $x_i \in i$. Then $(x_i : i \in \Delta)$ is a net in $A$ and by Theorem 2.6, $x_i$ has a cluster point $x \in A$. Since $x$ is a cluster point of $x_i$, $x \in T(i)$ for every $i \in \Delta$. But $T(i) \subseteq i$ and therefore $x \in i$ for every $i \in \Delta$. It follows that $x \in \bigcap \mathfrak{S}$. ∎

We conclude this section with some results concerning the cardinality of neighbourhood bases and separability.

THEOREM 2.11. *Let $X$ be an $\alpha$-topological space and let $\beta$ be a cardinal with $\beta < \alpha$. If $x \in X$ has the property that $\{x\}$ is not open, then there does not exist a neighbourhood basis of $x$ with cardinality $\beta$.*

Proof. Suppose $x$ has a neighbourhood basis $\{A_i : i \in \Gamma\}$ where $\text{card}(\Gamma) = \beta < \alpha$. Since $X$ is an $\alpha$-topological space, $A = \bigcap A_i$ is a neighbourhood of $x$. Since $\{x\}$ is not open, there exists a $y \in A \setminus \{x\}$. The Hausdorff postulate implies the existence of an open set $B$ such that $x \in B$ and $y \notin B$. Then $A \cap B$ is a neighbourhood of $x$ that is strictly contained in every $A_i$, $i \in \Gamma$. This is a contradiction so there is no neighbourhood basis of $x$ with cardinality $\beta$. ∎

COROLLARY 2.12. *Let $X$ be an $\alpha$-topological space where $\alpha$ is uncountable. If there exists an $x \in X$ such that $\{x\}$ is not open, then $X$ is not first countable.*

For a cardinal $\beta$, we say that an $\alpha$-topological space $X$ is **$\beta$-separable** if there exists a set $A \subseteq X$ such that $\text{card}(A) \leq \beta$ and $\overline{A} = X$. Of course, $\aleph_0$-separable is the usual notion of separable.

THEOREM 2.13. *Let $X$ be an $\alpha$-topological space and let $\beta$ be a cardinal with $\beta < \alpha$. Then $X$ is $\beta$-separable if and only if* $\text{card}(X) \leq \beta$ *and $X$ is discrete.*

Proof. If $\text{card}(X) \leq \beta$, then clearly $X$ is $\beta$-separable. Conversely, suppose $X$ is $\beta$-separable and let $\{x_i : i \in \Gamma\}$, $\text{card}(\Gamma) \leq \beta$, be dense in $X$. Suppose there exists an $x \in X$ such that $x \neq x_i$ for all $i \in \Gamma$. By the Hausdorff postulate there exist open sets $A_i$, $i \in \Gamma$, such that $x \in A_i$ and $x_i \notin A_i$. Since $\beta < \alpha$, the set $A = \bigcap A_i$ is the neighbourhood of $x$ and $x_i \notin A$ for all $i \in \Gamma$. This contradicts the denseness of $\{x_i : i \in \Gamma\}$. Hence, $X = \{x_i : i \in \Gamma\}$ so $\text{card}(X) \leq \beta$. Now any subset $B \subseteq X$ has the form $B = \bigcup_{i \in \Gamma'} \{x_i\}$ where $\Gamma' \subseteq \Gamma$ so $\text{card}(\Gamma') \leq \beta$. Since singleton sets are closed and $\beta < \alpha$, $B$ is closed. Hence, $X$ is discrete. ∎

COROLLARY 2.24. *Let $X$ be an $\alpha$-topological space where $\alpha$ is uncountable. Then $X$ is separable if and only if* $\mathrm{card}(X) \leq \aleph_0$ *and $X$ is discrete.*

## 3. Completion of ordered fields

Let $F$ be an ordered field and let $F^+$ be its set of positive elements. If $I \subseteq F^+$ is a cofinal set of smallest ordinality $\beta$, then $I$ is **minimal cofinal** and $\beta$ is the **cofinality** of $F$. When we consider $F$ as a topological space, we always assume the topology is the order topology. For $I \subseteq F^+$ cofinal, an **$I$-sequence** in a set $S$ is a map $i \to a_i$ from $I$ into $S$. Notice that if $I$ is minimal cofinal with cardinality $\alpha$, then $I$ is an $\alpha$-directed set and an $I$-sequence is an $\alpha$-net. In the sequel, $I$ will always denote a cofinal subset of $F^+$.

An $I$-sequence $a_i$ in $F$ **converges** to $a \in F$ if for any $\varepsilon \in F^+$ there exists an $n(\varepsilon) \in I$ such that $|a_i - a| < \varepsilon$ for every $i \geq n(\varepsilon)$. An $I$-sequence in $F$ is **null** if it converges to 0. An $I$-sequence $a_i$ in $F$ is **Cauchy** if for every $\varepsilon \in F^+$ there exists an $n(\varepsilon) \in I$ such that $|a_i - a_j| < \varepsilon$ for every $i, j \geq n(\varepsilon)$. We say that $F$ is **$I$-complete** if every Cauchy $I$-sequence in $F$ converges. Denoting the set of null and Cauchy $I$-sequences by $\mathfrak{N}$ and $\mathfrak{C}$, respectively, it is clear that $\mathfrak{N} \subseteq \mathfrak{C}$. For $(a_i), (b_i) \in \mathfrak{C}$, we define $(a_i) + (b_i) = (a_i + b_i)$ and $(a_i)(b_i) = (a_i b_i)$. The proof of the following lemma is classical.

LEMMA 3.1. *Under the above definitions of sum and product, $\mathfrak{C}$ is a commutative ring with unit and $\mathfrak{N}$ is a maximal ideal in $\mathfrak{C}$.*

Let $\overline{F}$ denote the quotient ring $\mathfrak{C}/\mathfrak{N}$. Since $\mathfrak{N}$ is a maximal ideal, it is well known that $\overline{F}$ is a field. We denote the elements of $\overline{F}$ by $\widehat{a} = (a_i) + \mathfrak{N}$. If $a \in F$, we use the notation $\tau(a) = (a, a, \dots) + \mathfrak{N}$. We define the set of **positive** elements $\overline{F}^+ \subseteq \overline{F}$ by

$$\overline{F}^+ = \{\widehat{a} \in \overline{F} : \widehat{a} \neq \tau(0),\ \widehat{a} = (a_i) + \mathfrak{N},\ a_i > 0 \text{ for all } i \in I\}.$$

The proof of the next theorem is similar to the proof of the classical imbedding theorem of the rationals into the reals.

THEOREM 3.2. (a) *The field $\overline{F}$ with positive cone $\overline{F}^+$ is an ordered field.* (b) *The map $\tau : F \to \overline{F}$ is an order-preserving isomorphism of $F$ into $\overline{F}$.* (c) *The range of $\tau$ is dense in the order topology of $\overline{F}$.* (d) *If $\widehat{a} = (a_i) + \mathfrak{N} \in \overline{F}$, then* $\lim \tau(a_i) = \widehat{a}$.

It follows from Theorem 3.2 that we can assume that $F$ is a dense subset of $\overline{F}$. Moreover, $I$ is then a cofinal subset of $\overline{F}$. The proof of the next result is different than the usual proof for $\mathbb{R}$ since the latter uses the well-ordering of $\mathbb{N}$. A longer proof of the next result from a different viewpoint is given in [12].

THEOREM 3.3. *The field $\overline{F}$ is I-complete.*

Proof. Let $\widehat{a}_i$ be a Cauchy $I$-sequence in $\overline{F}$. For any $i \in I$ there exists an $n(i) \in I$ such that $n_i \geq i$ and $p, q \geq n(i)$ implies that $|\widehat{a}_p - \widehat{a}_q| < 1/i$. By the denseness of $F$ in $\overline{F}$, there exist $a_i \in F$ such that $|a_i - \widehat{a}_i| < 1/i$ for every $i \in I$. Let $\varepsilon \in F^+$ and choose a $j \in I$ such that $1/j < \varepsilon/3$. If $p, q \geq n(j)$, then

$$|a_p - a_q| \leq |a_p - \widehat{a}_p| + |\widehat{a}_p - \widehat{a}_q| + |\widehat{a}_q - a_q| < \frac{1}{j} + \frac{1}{j} + \frac{1}{j} < \varepsilon.$$

Hence, $(a_i) \in \mathfrak{C}$ and we define $\widehat{b} = (a_i) + \mathfrak{N}$.

We now prove that $\widehat{a}_i$ converges to $\widehat{b}$. Let $\varepsilon \in \overline{F}^+$. Then there exists an $n \in I$ such that $1/n < \varepsilon/2$. For $i \geq n$ we have

$$|a_i - \widehat{a}_i| < \frac{1}{i} < \frac{1}{n} < \frac{\varepsilon}{2}.$$

Also, from Theorem 3.2 (d), there exists an $m \in I$ such that $|a_i - \widehat{b}| < \varepsilon/2$ for all $i \geq m$. Hence, if $j \geq \max(n, m)$ we have

$$|\widehat{a}_j - \widehat{b}| \leq |\widehat{a}_j - a_j| + |a_j - \widehat{b}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \blacksquare$$

We call $\overline{F}$ the **$I$-completion** of $F$. One can show in the usual way that the $I$-completion of $F$ is unique up to an isomorphism. That is, if $E$ is an ordered field for which there exists an order-preserving isomorphism $\tau : F \to E$ with dense range and if $E$ is $I$-complete, then $E$ and $\overline{F}$ are isomorphic.

Ordered fields with various cofinalities can be constructed using the methods of nonstandard analysis [2,7,10,11,13]. Of course $\mathbb{R}$ is a complete ordered field with minimal cofinal set $\mathbb{N}$. Thus, the cofinality of $\mathbb{R}$ is the first infinite ordinal $\omega_0$. The simplest nonstandard construction of the hyperreal field ${}^*\mathbb{R}$ proceeds as follows. Let $\mathfrak{U}$ be a free ultrafilter on the power set of $\mathbb{N}$. Define ${}^*\mathbb{R} = \mathbb{R}^{\mathbb{N}}/\mathfrak{U}$ where addition, multiplication and order are defined on ${}^*\mathbb{R}$ in the natural way. It is well-known that ${}^*\mathbb{R}$ becomes an ordered field with minimal cofinal set ${}^*\mathbb{N} = \mathbb{N}^{\mathbb{N}}/\mathfrak{U}$. The cofinality of ${}^*\mathbb{R}$, using this construction, is the first uncountable ordinal $\omega_1$. It can be shown that in this model ${}^*\mathbb{R}$ is not ${}^*\mathbb{N}$-complete [8]. Other nonstandard models for ${}^*\mathbb{R}$ can be constructed by replacing $\mathbb{N}$ with large index sets or by using more sophisticated methods of model theory. In this way, models for ${}^*\mathbb{R}$ with arbitrarily large cofinality can be constructed. It can be shown that some of these models for ${}^*\mathbb{R}$ are $I$-complete and others are not $I$-complete [8]. Other kinds of completeness for ${}^*\mathbb{R}$ are studied in [14].

## 4. $F$-metric spaces

Let $F$ be an ordered field and let $X$ be a nonempty set. We call $(X,\rho)$ an $F$-metric space if $\rho : X \times X \to F$ satisfies

(1) $\rho(x,y) \geq 0$,

(2) $\rho(x,y) = 0$ if and only if $x = y$,

(3) $\rho(x,y) = \rho(y,x)$,

(4) $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$.

For $x \in X$, $r \in F^+$, the **ball centered at $x$ with radius $r$** is

$$B(x,r) = \{y \in X :\ \rho(x,y) < r\}.$$

We also define the corresponding **closed ball** by

$$\widehat{B}(x,r) = \{y \in X :\ \rho(x,y) \leq r\}.$$

The family $\{B(x,r) :\ r \in F^+,\ x \in X\}$ is a base for a topology on $X$. Moreover, if $I \subseteq F^+$ is cofinal, then $\{B(x,1/i) : i \in I,\ x \in X\}$ is a base for this topology.

An $I$-sequence $x_i \in X$ **converges** to $x \in X$ if for any $\varepsilon \in F^+$ there exists an $n(\varepsilon) \in I$ such that $i \geq n$ implies $\rho(x_i, x) < \varepsilon$. Convergent $I$-sequeneces determine the topology on $X$ in the sense that $A \subseteq X$ is closed if and only if for every $I$-sequence $x_i \in A$ such that $x_i \to x \in X$ we have $x \in A$. An $I$-sequence $x_i$ is **Cauchy** if for any $\varepsilon \in F^+$ there exists an $n(\varepsilon) \in I$ such that $i,j \geq n$ imply $\rho(x_i, x_j) < \varepsilon$. We say that $X$ is **$I$-complete** if every Cauchy $I$-sequence in $X$ converges. Of course, $F$ itself is an $F$-metric space with metric $\rho(x,y) = |x-y|$ and these concepts of topology, convergent and Cauchy $I$-sequences reduce to the usual concepts on $F$.

THEOREM 4.1. *Let $X$ be an $F$-metric space, let $I$ be a minimal cofinal subset of $F^+$ and let $\alpha = \mathrm{card}(I)$.* (a) *If* $\mathrm{card}(\Delta) < \alpha$ *and $A_i \subseteq X$, $i \in \Delta$, are closed, then $\bigcup A_i$ is closed.* (b) *If* $\mathrm{card}(\Delta) < \alpha$ *and $B_i \subseteq X$, $i \in \Delta$, are open, then $\bigcap B_i$ is open.* (c) *If $A \subseteq X$ and* $\mathrm{card}(A) < \alpha$, *then $A$ is closed.*

Proof. (a) If $\bigcup A_i = X$ we are finished, so suppose $x \notin \bigcup A_i$. Since $A_i$ is closed, there exists an $\varepsilon_i \in F^+$ such that $B(x,\varepsilon_i) \bigcap A_i = \emptyset$, $i \in \Delta$. Since $\mathrm{card}(\Delta) < \alpha$, the set $\{\varepsilon_i^{-1} :\ i \in \Delta\}$ is not cofinal and hence has an upper bound. Thus, there exists an $\varepsilon \in F^+$ such that $\varepsilon < \varepsilon_i$ for all $i \in \Delta$. Then $B(x,\varepsilon) \subseteq B(x,\varepsilon_i)$ for all $i \in \Delta$ so $B(x,\varepsilon) \bigcap A_i = \emptyset$ for all $i \in \Delta$. Hence, $B(x,\varepsilon) \cap (\bigcup A_i) = \emptyset$ so $\bigcup A_i$ is closed. (b) and (c) follow from (a). ∎

Applying Theorem 4.1 we conclude that an $F$-metric space is an $\alpha$-topological space. Hence the work in Section 2 applies to this section as well. Also, when $I \subseteq F^+$ is minimal cofinal, an $I$-sequence is an $\alpha$-net. It follows from the discussion in the last paragraph of Section 3 that there exist non-discrete $\alpha$-topological spaces for a vast set of cardinals $\alpha$.

Since $\mathbb{R}$ has countable cofinality, the interesting new $F$-metric spaces are those in which $F$ has uncountable cofinality. We now study some of the properties of such $F$-metric spaces.

We can consider $\mathbb{N}$ as a subset of $F$. An element $\varepsilon \in F$ is **infinitesimal** if $|\varepsilon| < 1/n$ for every $n \in \mathbb{N}$. We denote the set of the infinitesimals by $\mathfrak{M}(0)$. The proof of the following lemma is straightforward.

LEMMA 4.2. *If $F$ has uncountable cofinality, then $\mathfrak{M}(0) \neq \{0\}$ and $F$ is nonarchimedean.*

Recall that a topological space is **metrizable** if its topology is generated by a real-valued metric. The next result follows from Corollaries 2.12 and 2.14.

THEOREM 4.3. *Let $X$ be an $F$-metric space where $F$ has uncoutable cofinality.* (a) *If there exists an $x \in X$ such that $\{x\}$ is not open, then $X$ is not first countable and hence $X$ is not metrizable.* (b) *$X$ is separable if and only if* $\text{card}(X) \leq \aleph_0$ *and $X$ is discrete.*

For $x \in X$ and $a \in F^+$, we define

$$\mathfrak{M}_a(x) = \{y \in X : \ \rho(y,x) < a\varepsilon, \ \varepsilon \in \mathfrak{M}^+(0)\}$$

where $\mathfrak{M}^+(0) = \mathfrak{M}(0) \cap F^+$. Recall that a topological space $X$ is **0-dimensional** if there exists a neighborhood basis of clopen sets for every $x \in X$.

THEOREM 4.4. *Let $X$ be an $F$-metric space where $F$ has uncountable cofinality.* (a) *$\mathfrak{M}_a(x)$ is clopen for every $x \in X$, $a \in F^+$.* (b) *$X$ is 0-dimensional.*

Proof. (a) Since

$$\mathfrak{M}_a(x) = \bigcup\{B(x, a\varepsilon) : \ \varepsilon \in \mathfrak{M}^+(0)\}$$

we conclude that $\mathfrak{M}_a(x)$ is open. Suppose $y \notin \mathfrak{M}_a(x)$. If $\varepsilon \in \mathfrak{M}^+(0)$, we shall show that $B(y, a\varepsilon) \cap \mathfrak{M}_a(x) = \emptyset$. Indeed if $z \in B(y, a\varepsilon) \cap \mathfrak{M}_a(x)$, then

$$\rho(x,y) \leq \rho(x,z) + \rho(z,y) < a\varepsilon_1 + a\varepsilon = a(\varepsilon_1 + \varepsilon)$$

where $\varepsilon_1 \in \mathfrak{M}^+(0)$. Since $\varepsilon_1 + \varepsilon \in \mathfrak{M}^+(0)$, we have $y \in \mathfrak{M}_a(x)$ which is a contradiction. Hence, $\mathfrak{M}_a(x)$ is closed.

(b) Let $a \in F^+$ and consider $B(x,a)$. Then $\mathfrak{M}_a(x) \subseteq B(x,a)$ since if $y \in \mathfrak{M}_a(x)$ then there exists an $\varepsilon \in \mathfrak{M}^+(0)$ such that $\rho(y,x) < a\varepsilon < a$. Hence, $y \in B(x,a)$. It follows from (a) that $\{\mathfrak{M}_a(x) : \ a \in F^+\}$ is a neighbourhood basis at $x$ of clopen sets. ∎

A subset $A$ of an $F$-metric space is **totally bounded** if for any $\varepsilon \in F^+$ there exists a finite number of balls of radius $\varepsilon$ that cover $A$. The next result shows that compactness is not a useful concept if $F$ has uncountable cofinality.

THEOREM 4.5. *Let $X$ be an $F$-metric space, where $F$ has uncountable cofinality. If $A \subseteq X$, then the following statements are equivalent.* (a) *$A$ is compact.* (b) *$A$ is totally bounded.* (c) $\operatorname{card}(A) < \infty$.

Proof. To show that (a) implies (b), let $A$ be compact and $\varepsilon \in F^+$. Since $\{B(x,\varepsilon) : x \in A\}$ is an open cover of $A$, there exists a finite subcover $B(x_i,\varepsilon)$, $x_i \in A$, $i = 1,\dots,n$. Hence, $A$ is totally bounded. To show that (b) implies (c), let $A$ be totally bounded and suppose $\operatorname{card}(A) = \infty$. Then there exists a subset $B \subseteq A$ with $\operatorname{card}(B) = \aleph_0$. Since $A$ is totally bounded, so is $B$. Let $B = \{y_i : i \in \mathbb{N}\}$. Since $F$ has uncountable cofinality, there exists an $\varepsilon \in F^+$ such that $\varepsilon < \rho(y_i, y_j)$ for every $i,j \in \mathbb{N}$ with $i \neq j$. Since $B$ is totally bounded, there exists an $n \in \mathbb{N}$ such that $B \subseteq \bigcup_{i=1}^n B(x_i,\varepsilon/2)$. Now $B(x_1,\varepsilon/2)$ contains at most one $y_i$. Indeed, if $y_i, y_j \in B(x_1,\varepsilon/2)$, $i \neq j$, then

$$\rho(y_i,y_j) \leq \rho(y_i,x_1) + \rho(x_1,y_j) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

which is a contradiction. Similarly, $B(x_i,\varepsilon/2)$ contains at most one $y_j$. Hence,

$$B \not\subseteq \bigcup_{i=1}^{n} B(x_i,\varepsilon/2)$$

which is a contradiction. We conclude that $\operatorname{card}(A) < \infty$. That (c) implies (a) is trivial. ■

Although compact sets are trivial when $F$ has uncountable cofinality, we still have the notion of an $\alpha$-compact set where $\alpha$ is the cardinality of a minimal cofinal subset $I \subseteq F^+$. In this case, we call the set $I$-**compact**. We say that a set $A \subseteq X$ is $I$-**bounded** if for any $\varepsilon \in F^+$ there exists balls $B_i$, $i \in \Delta$, with $\operatorname{card}(\Delta) < \alpha$ such that $A \subseteq \bigcup_{i\in\Delta} B_i$.

THEOREM 4.6. *Let $A$ be $I$-compact.* (a) *Every $I$-sequence in $A$ has a cluster point in $A$.* (b) *$A$ is $I$-bounded and $I$-complete.*

Proof. (a) This follows from Theorems 2.6 and 2.10. (b) That $A$ is $I$-bounded is similar to the first part of the proof of Theorem 4.4. To show that $A$ is $I$-complete, let $x_i$ be a Cauchy $I$-sequence in $A$. By part (a), $x_i$ has a cluster point $x \in A$. For $\varepsilon \in F^+$ there exists an $n \in I$ such that $\rho(x_i,x_j) < \varepsilon/2$ whenever $i,j \geq n$. Since $x$ is a cluster point of $x_i$, there exists a $k \geq n$ such that $\rho(x_k,x) < \varepsilon/2$. Then $i \geq n$ implies that

$$\rho(x_i,x) \leq \rho(x_i,x_k) + \rho(x_k,x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, $x_i \to x$ so $A$ is $I$-complete. ■

We do not know whether the converses of Theorem 4.6 hold. Following the discussion in the last paragraph of Section 3, $^*\mathbf{R} = \mathbf{R}^{\mathbb{N}}/\mathfrak{U}$ has minimal cofinal set $^*\mathbb{N}$. Assuming the continuum hypothesis, it can be shown that $\text{card}(^*\mathbb{N}) = \aleph_1$. Unfortunately, subsets of $^*\mathbf{R}$ that we would like to be $^*\mathbb{N}$-compact are not. For example,

$$[0,1] = \{x \in {}^*\mathbf{R} : 0 \leq x \leq 1\}$$

is not $^*\mathbb{N}$-compact. Indeed, the family

$$\{\mathfrak{M}_1(x) : x \in [0,1] \cap \mathbf{R}\}$$

forms an open cover of [0,1]. However, $\mathfrak{M}_1(x) \cap \mathfrak{M}_1(y) = \emptyset$ for $x, y \in \mathbf{R}$, $x \neq y$. Hence, there is no subcover of $[0,1]$. In particular, there is no subcover indexed by a set of cardinality $\aleph_0$.

If $I \subseteq F^+$ is cofinal, an **$I$-subsequence** of an $I$-sequence $(x_i : i \in I)$ is an $I$-sequence of the form $(x_{u(i)} : i \in I)$ where $u : I \to I$ satisfies $u(i) \geq i$. If $i \in I$ and $j \geq i$, then $u(j) \geq j \geq i$ so $u$ is finalizing. Thus, if $I$ is minimal cofinal, then an $I$-subsequence is a special case of an $\alpha$-subnet. As usual, we say that $A \subseteq X$ is **bounded** if $A$ is contained in some ball.

LEMMA 4.7. (a) *A point $x \in X$ is a cluster point of an $I$-sequence $x_i$ if and only if $x_i$ has an $I$-subsequence converging to $x$.* (b) *If every $I$-sequence in $A \subseteq X$ has an $I$-subsequence converging to a point in $A$, then $A$ is closed and bounded.*

Proof. (a) Sufficiency is the same as the first part of the proof of Theorem 2.6. For necessity, let $x$ be a cluster point of $x_i$. For $i \in I$, there exists a $u(i) \in I$ such that $u(i) \geq i$ and $x_{u(i)} \in B(x, i^{-1})$. Then $(x_{u(i)} : i \in I)$ is an $I$-subsequence of $x_i$. To show that $x_{u(i)} \to x$, consider a ball $B(x, j^{-1})$, $j \in I$. Then for $i \geq j$, we have $u(i) \geq i \geq j$. Hence,

$$x_{u(i)} \in B(x, i^{-1}) \subseteq B(x, j^{-1}).$$

(b) The proof of this is straightforward. ∎

THEOREM 4.8. *Let $(X, \rho)$ be an $F$-metric space, $(Y, \rho')$ an $F'$-metric space and let $I \subseteq F^+$ be minimal cofinal. If $f : A \to Y$ is continuous, where $A \subseteq X$ is $I$-compact, then $f$ is uniformly continuous.*

Proof. Fix an $\varepsilon \in F'^+$. If $x \in A$, there exists a $\delta(x) \in F^+$ such that $\rho'(f(x), f(y)) < \varepsilon/2$ whenever $y \in A$ with $\rho(x, y) < \delta(x)$. Since $A \subseteq \bigcup_{x \in A} B(x, \delta(x)/2)$, by $I$-compactness there is a subcover $A \subseteq \bigcup_{x_i \in A} B(x_i, \delta(x_i)/2)$ where $i \in \Delta$ with $\text{card}(\Delta) < \text{card}(I)$. Since $I$ is minimal cofinal, there exists a $\delta \in F^+$ such that $\delta < \delta(x_i)/2$ for all $i \in \Delta$. If

$x, y \in A$ with $\rho(x,y) < \delta$, then $x \in B(x_i, \delta(x_i)/2)$ for some $i \in \Delta$. Hence,

$$\rho(x, x_i) < \frac{\delta(x_i)}{2} < \delta(x_i)$$

so $\rho'(f(x), f(x_i)) < \varepsilon/2$. Moreover,

$$\rho(x_i, y) \leq \rho(x_i, x) + \rho(x, y) < \frac{1}{2}\delta(x_i) + \delta$$
$$< \frac{1}{2}\delta(x_i) + \frac{1}{2}\delta(x_i) = \delta(x_i)$$

and hence $\rho'(f(x_i), f(y)) < \varepsilon/2$. We conclude that

$$\rho'(f(x), f(y)) \leq \rho'(f(x), f(x_i)) + \rho'(f(x_i), f(y)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \ \blacksquare$$

It is clear that an $F$-metric space is Hausdorff. Moreover, the classical proof shows that an $F$-metric space is normal [5].

## 5. Complete $F$-metric spaces

We first discuss the completion of an $F$-metric space $X$. Let $I \subseteq F^+$ be cofinal and let $\mathfrak{C}$ be the set of Cauchy $I$-sequences in $X$. For $(x_i), (y_i) \in \mathfrak{C}$ define the equivalence relation $(x_i) \sim (y_i)$ if $\rho(x_i, y_i) \to 0$. Let $\overline{X}$ be the set of equivalence classes relative to $\sim$. For $(x_i), (y_i) \in \mathfrak{C}$, since

$$\rho(x_i, y_i) \leq \rho(x_i, x_j) + \rho(x_j, y_j) + \rho(y_j, y_i)$$

it follows that

$$|\rho(x_i, y_i) - \rho(x_j, y_j)| \leq \rho(x_i, x_j) + \rho(y_j, y_i).$$

Hence, $\rho(x_i, y_i)$ is a Cauchy $I$-sequence in $F$ so $\rho(x_i, y_i)$ can be identified with an element of $\overline{F}$. If $\widehat{x}, \widehat{y} \in \overline{X}$, where $(x_i) \in \widehat{x}$, $(y_i) \in \widehat{y}$, we define $\overline{\rho}(\widehat{x}, \widehat{y}) = (\rho(x_i, y_i)) \in \overline{F}$. Moreover, we define $\sigma : X \to \overline{X}$ by $\sigma(x) = (x, x, \ldots)$. The proof of the following theorem is straightforward.

THEOREM 5.1. *$(\overline{X}, \overline{\rho})$ is an $I$-complete $\overline{F}$-metric space and $\sigma : X \to \overline{X}$ has dense range in $\overline{X}$ and satisfies $\overline{\rho}(\sigma(x), \sigma(y)) = \rho(x, y)$ for all $x, y \in X$.*

We call $(\overline{X}, \overline{\rho})$ the ***I*-completion** of $(X, \rho)$. As usual, $(\overline{X}, \overline{\rho})$ is unique to within an isometry. The proof of the next result is simiar to the classical proof.

THEOREM 5.2. (a) *If $f : X \to X$ is uniformly continuous, then $f$ has a unique uniformly continuous extension $g : \overline{X} \to \overline{X}$.* (b) *If $f : X \to F$ is uniformly continuous, then $f$ has a unique uniformly continuous extension $g : \overline{X} \to \overline{F}$.*

An element $r \in F^+$ is a **bound** for a subset $A$ of an $F$-metric space $X$ if $\rho(x, y) \leq r$ for all $x, y \in A$. If $A_i \subseteq X$ is an $I$-sequence of sets and there exists an $I$-sequence of bounds $r_i$ for the $A_i$, $i \in I$, such that $r_i \to 0$, we write $\lim A_i = 0$. We call $A_i$ a **nested** $I$-sequence if $A_j \subseteq A_i$ for $j \geq i$. The next result generalizes Cantor's characterization of complete metric spaces.

THEOREM 5.3. *Let $X$ be an $F$-metric space and let $I \subseteq F^+$ be minimal cofinal. Then $X$ is $I$-complete if and only if for any nested $I$-sequence of nonempty closed sets $A_i \subseteq X$ with* $\lim A_i = 0$, *we have $\bigcap A_i = \{x\}$ for some $x \in X$.*

Proof. Assume $X$ is $I$-complete and let $A_i \subseteq X$, $i \in I$, satisfy the conditions of the theorem. For each $i \in I$, let $x_i \in A_i$. If $\varepsilon \in F^+$, then there exists an $n \in I$ such that $r_n < \varepsilon$, where $r_n$ is a bound for $A_n$. If $i, j \geq n$, then $x_i \in A_i \subseteq A_n$ and $x_j \in A_j \subseteq A_n$. Hence, $\rho(x_i, x_j) \leq r_n < \varepsilon$ so $x_i$ is Cauchy. Since $X$ is $I$-complete, $x_i \to x$ for some $x \in X$. For each $j \in I$, $x_i \in A_j$ for $i \geq j$ and since $A_j$ is closed, we have $x \in A_j$. Hence, $x \in \bigcap A_j$. If $y \in \bigcap A_j$, then $\rho(x, y) \leq r_j$ for every $j \in I$. Since $r_j \to 0$, we have $\rho(x, y) = 0$ so $y = x$. Therefore, $\bigcap A_j = \{x\}$.

Conversely, assume that $X$ has the stated nested closed sets property. Let $x_i$ be a Cauchy $I$-sequence in $X$. For each $i \in I$, let $A_i$ be the closure of the set $\{x_j : j \geq i\}$. Then $A_i$ is a nested $I$-sequence of nonempty closed sets. Letting $\omega = \operatorname{ord}(I)$ we can write $I = \{i_\alpha : \alpha < \omega\}$ where each $\alpha$ is an ordinal. Define an increasing cofinal sequence $n_\alpha \in I$, $\alpha < \omega$, inductively as follows. Suppose $n_\beta$ is increasing, $n_\beta \geq i_\beta$, and $n_\beta$ has been defined for all ordinals $\beta < \alpha$ where $\alpha < \omega$. Since $I$ is minimal cofinal, there exists an $n \in I$ such that $n > n_\beta$ for all $\beta < \alpha$. Now letting $n_\alpha = \max\{n, i_\alpha\}$ completes the inductive definition. We next define an increasing cofinal sequence $m_\alpha \in I$, $\alpha < \omega$, such that $m_\alpha \geq i_\alpha$ and $\rho(x_p, x_q) < 1/n_\alpha$ whenever $p, q \geq m_\alpha$, inductively as follows. Suppose $m_\beta$ has been defined for all $\beta < \alpha$ where $\alpha < \omega$. Since $I$ is minimal cofinal, there exists an $m \in I$ such that $m > m_\beta$ for all $\beta < \alpha$. Moreover, since $x_i$ is Cauchy, there exists an $r \in I$ such that $\rho(x_p, x_q) < 1/n_\alpha$ whenever $p, q \geq r$. Letting $m_\alpha = \max\{m, r, i_\alpha\}$ completes the inductive definition.

We have thus constructed a nested sequence of nonempty closed sets $A_{m_\alpha} \subseteq X$ with bounds $1/n_\alpha$, $\alpha < \omega$. For $i \in I$, define $B_i = A_{m_\alpha}$ and $n_i = n_\alpha$ where $\alpha$ is the smallest ordinal such that $i \leq m_\alpha$. Then $B_i$ is an $I$-subsequence of $A_i$ with bounds $1/n_i$, $i \in I$. Since $n_i$ is a cofinal $I$-sequence, we have $\lim B_i = 0$. By hypothesis, we conclude that $\bigcap B_i = \{x\}$ for some $x \in X$. Given $\varepsilon \in F^+$, there exists an $i \in I$ such that $1/n_i < \varepsilon$ and there exists a $j \in I$ such that $A_j \subseteq B_i$. Hence, if $p \geq j$, we have $x_p \in A_j \subseteq B_i$. Since $x \in B_i$, we have $\rho(x_p, x) \leq 1/n_i < \varepsilon$. Hence, $x_i \to x$. ∎

For the sufficiency part of Theorem 5.3, we did not need the minimality

COROLLARY 5.4. *If $I \subseteq F^+$ is cofinal and $X$ is $I$-complete, then for any nested $I$-sequence of nonempty closed sets $A_i \subseteq X$ with* $\lim A_i = 0$, *we have* $\bigcap A_i = \{x\}$ *for some $x \in X$.*

We say that $X$ has the **$I$-ball property** if for any set $J \subseteq I$ with $\mathrm{card}(J) < \mathrm{card}(I)$ and any nested $J$-sequence of balls $B_j$, $j \in J$, we have $\bigcap B_j \neq \emptyset$. We now give an example of an $I$-complete $F$-metric space that does not have the $I$-ball property. Let $X = \mathbb{N}$ and define the metric

$$\rho(n,m) = \begin{cases} 0 & \text{if } n = m \\ 1 + 1/\min(m,n) & \text{if } n \neq m. \end{cases}$$

Then $(X,\rho)$ is an $\mathbb{R}$-metric space. Moreover, $(X,\rho)$ is $\mathbb{R}$-complete. Indeed, if $x_i$, $i \in \mathbb{R}$, is a Cauchy $\mathbb{R}$-sequence, then there exists an $n \in \mathbb{R}$ such that $\rho(x_i, x_j) < 1$ if $i,j \geq n$. Hence, $x_i = x_n$ for $i \geq n$ so $x_i \to x_n$. However, $(X,\rho)$ does not have the $\mathbb{R}$-ball property. Indeed, $\mathbb{N} \subseteq \mathbb{R}, \mathrm{card}(\mathbb{N}) < \mathrm{card}(\mathbb{R})$, $B_j = B(j+1, 1+1/j)$, $j \in \mathbb{N}$, is a nested $\mathbb{N}$-sequence of balls and yet $\bigcap B_j = \emptyset$.

We now prove a generalization of the Baire category theorem.

THEOREM 5.5. *Let $X$ be an $F$-metric space, let $I \subseteq F^+$ be minimal cofinal and suppose $X$ is $I$-complete and has the $I$-ball property. If $A \subseteq X$ is a union $\bigcup_{i \in I} A_i$ of nowhere dense sets $A_i \subseteq X$, then $A^c$ is dense in $X$.*

Proof. We can assume that each $A_i$ is closed (otherwise, let $B = \bigcup \overline{A}_i$ and show that $B^c$ is dense in $X$ and since $B^c \subseteq A^c$, then $A^c$ is dense in $X$). Let $U$ be any nonempty open subset of $X$. We shall show that $U \cap A^c \neq \emptyset$ which gives the results. Letting $\omega = \mathrm{ord}(I)$, we can write $A = \bigcup_{\alpha < \omega} A_\alpha$. As in the proof of Theorem 5.3, there exists an increasing cofinal sequence $n_\alpha \in I$. $\alpha < \omega$. Define a nested sequence of closed balls $\widehat{B}_\alpha = \widehat{B}(x_\alpha, r_\alpha)$, where $r_\alpha < 1/n_\alpha$ and $\widehat{B}_\alpha \subseteq A_\alpha^c \cap U$ inductively as follows. Since $A_0$ is nowhere dense, any ball $B \subseteq U$ is not a subset of $A_0$ so $B \cap A_0^c$ is a nonempty open set. Hence, there exists a closed ball $\widehat{B}_0 \subseteq B \cap A_0^c$ with $r_0 < 1/n_0$. Suppose $\widehat{B}_\beta$ is defined for every $\beta < \alpha$ where $\alpha < \omega$. By the $I$-ball property, there exists an $x \in \bigcap_{\beta < \alpha} B_\beta$. Since $I$ is minimal cofinal, there exists a $\delta \in F^+$ such that $\delta < r_\beta - \rho(x, x_\beta)$ for all $\beta < \alpha$. Hence, $B(x,\delta) \subseteq \widehat{B}_\beta$ for every $\beta < \alpha$. Indeed, if $y \in B(x,\delta)$, then

$$\rho(y, x_\beta) \leq \rho(y,x) + \rho(x, x_\beta) < \delta + \rho(x, x_\beta) < r_\beta.$$

Since $A_\alpha$ is nowhere dense, $B(x,\delta)$ is not a subset of $A_\alpha$ so $B(x,\delta) \cap A_\alpha^c$ is a nonempty open set. Letting $x_\alpha = x$, there exists an $r_\alpha < 1/n_\alpha$ such that

$$\widehat{B}(x_\alpha, r_\alpha) \subseteq B(x_\alpha, \delta) \cap A_\alpha^c.$$

This completes the inductive definition.

For $i \in I$, let $\widehat{B}_i = \widehat{B}_\alpha$ where $\alpha$ is the least ordinal such that $i \leq n_\alpha$. We thus obtain a nested $I$-sequence of nonempty closed sets $\widehat{B}_i$ with $\lim \widehat{B}_i = 0$. Since $X$ is $I$-complete, by Theorem 5.3 there exists an $x \in X$ such that $\bigcap \widehat{B}_i = \{x\}$. Then

$$x \in U \cap \left(\bigcap A_i^c\right) = U \cap \left(\bigcup A_i\right)^c = U \cap A^c. \blacksquare$$

The usual Baire category theorem follows from Theorem 5.5. Just let $F = \mathbb{R}$, $I = \mathbb{N}$, and note that the $\mathbb{N}$-ball property holds trivially. The next corollary is a generalization of the uniform boundedness principle.

COROLLARY 5.6. *Suppose $X$ is an $F$-metric space satisfying the hypothese of Theorem 5.5, $Y$ is a topological space and $g : Y \to F^+ \cup \{0\}$ a continuous function. Let $\mathfrak{F}$ be a family of continuous functions from $X$ into $Y$ with the property that for each $x \in X$, there exists an $M_x \in F^+ \cup \{0\}$ such that $(g \circ f)(x) \leq M_x$ for all $f \in \mathfrak{F}$. Then there exists a nonempty open set $A \subseteq X$ and an $M \in F^+ \cup \{0\}$ such that $(g \circ f)(x) \leq M$ for all $f \in \mathfrak{F}$ and all $x \in A$.*

Proof. For $f \in \mathfrak{F}$ and $i \in I$, let $A_{i,f} = (g \circ f)^{-1}([0,i])$. Then $A_{i,f}$ is closed since $[0,i]$ is closed and $g \circ f$ is continuous. Let $A_i$ be the closed set $\bigcap_{f \in \mathfrak{F}} A_{i,f}$. Now $X = \bigcup_{i \in I} A_i$ because if $x \in X$, then $x \in A_i$ for all $i \geq M_x$. By Theorem 5.5, not all $A_i$ are nowhere dense. Hence, some $A_n = \overline{A}_n$, $n \in I$, must contain a nonempty open set $A$. Then for all $x \in A$ and $f \in \mathfrak{F}$, we have $(g \circ f)(x) \in [0,n]$. Taking $M = n$, we have $(g \circ f)(x) \leq M$ for all $f \in \mathfrak{F}$ and all $x \in A$. ■

COROLLARY 5.7. *Let $I \subseteq F^+$ be minimal cofinal. If $X$ is an $F$-metric space that has the $I$-ball property,* $\operatorname{card}(X) = \operatorname{card}(I)$, *and singleton sets in $X$ are not open, then $X$ is not $I$-complete.*

Proof. We can write $X = \{x_i : i \in I\}$. Hence, $X = \bigcup_{i \in I}\{x_i\}$ where $\{x_i\}$ is nowhere dense. If $X$ were $I$-complete, this would contradict Theorem 5.5. ■

COROLLARY 5.8. *Let ${}^*\mathbb{R} = \mathbb{R}^{\mathbb{N}}/\mathfrak{U}$ be the hyperreal field. Assuming the continuum hypothesis, ${}^*\mathbb{R}$ is not ${}^*\mathbb{N}$-complete.*

Proof. We have $\operatorname{card}({}^*\mathbb{R}) = \operatorname{card}({}^*\mathbb{N})$ and ${}^*\mathbb{N} \subseteq {}^*\mathbb{R}^+$ is minimal cofinal. Moreover, singleton sets in ${}^*\mathbb{R}$ are not open. By the continuum hypothesis, if

$$\operatorname{card}(J) < \operatorname{card}({}^*\mathbb{N}) = \aleph_1$$

then $\operatorname{card}(J) \leq \aleph_0$. Since balls in ${}^*\mathbb{R}$ are internal sets, by $\aleph_1$-saturation [2, 3, 10, 11, 13], the intersection of a nested $\mathbb{N}$-sequence of balls is nonempty.

Hence, $^*\mathbb{R}$ has the $^*\mathbb{N}$-ball property. The result now follows from Corollary 5.7. ∎

COROLLARY 5.9. *Let* $^*\mathbb{R} = \mathbb{R}^{\mathbb{N}}/\mathfrak{U}$ *by the hyperreal field, and let* $^*\overline{\mathbb{R}}$ *be the* $^*\mathbb{N}$*-completion of* $^*\mathbb{R}$. *Assuming the continuum hypothesis*, $\text{card}(^*\overline{\mathbb{R}}) = \aleph_2$.

Proof. Suppose $\text{card}(^*\overline{\mathbb{R}}) = \text{card}(^*\mathbb{R}) = \aleph_1$. If $B_i'$, $i \in \mathbb{N}$, is a nested $\mathbb{N}$-sequence of balls in $^*\overline{\mathbb{R}}$, since $^*\mathbb{R}$ is dense in $^*\overline{\mathbb{R}}$, we can construct a nested $\mathbb{N}$-sequence of balls $B_i$ in $^*\mathbb{R}$ such that $B_i \subseteq B_i'$. As in the proof of Corollary 5.8, $\emptyset \neq \bigcap B_i \subseteq \bigcap B_i'$. Hence $^*\overline{\mathbb{R}}$ has the $^*\mathbb{N}$-ball property. Since singleton sets in $^*\overline{\mathbb{R}}$ are not open, it follows from Corollary 5.7 that $^*\overline{\mathbb{R}}$ is not complete. Since this is a contradiction, $\text{card}(^*\overline{\mathbb{R}}) > \aleph_1$. Since

$$\text{card}(^*\overline{\mathbb{R}}) \leq \text{card}(^*\mathbb{R}^{^*\mathbb{N}}) = \aleph_1^{\aleph_1} = \aleph_2$$

the result follows. ∎

## 6. Open problems

This section outlines some miscellaneous results and presents some open problems. In Section 2, we have discussed $\alpha$-topological spaces. Let $X_r$, $r \in \Gamma$, be $\alpha$-topological spaces and let $X = \Pi_r X_r$ be their Cartesian product. The **$\alpha$-product topology** $\mathfrak{J}_\alpha$ on $X$ is the topology with base sets of the form $U = \Pi_r U_r$ where $U_r \subseteq X_r$ are open and $U_r = X_r$ except for $r \in \Gamma' \subseteq \Gamma$ with $\text{card}(\Gamma') < \alpha$. It is easy to show that $(X, \mathfrak{J}_\alpha)$ is an $\alpha$-topological space. Moreover, $\mathfrak{J}_\alpha$ is the weakest $\alpha$-topology on $X$ such that the natural projections $p_r : X \to X_r$, $r \in \Gamma$, are continuous. An open problem is whether Tychonoff's theorem holds in this context. That is, if $X_r$ is $\alpha$-compact for all $r \in \Gamma$, is $X$ $\alpha$-compact?

In connection with Theorem 4.6 we have the following open problem. Are the following statements equivalent? (a) $A$ is $I$-compact, (b) Every $I$-sequence in $A$ has a cluster point in $A$, (c) $A$ is $I$-bounded and $I$-complete, (d) $A$ is $I$-bounded and closed.

We define the **complexification** of an ordered field $F$ to be the set $F_c = F \times F$ with addition and multiplication given by

$$(a,b) + (c,d) = (a+c, b+d),$$
$$(a,b)(c,d) = (ac - bd, ad + bc).$$

It is straightforward to show that $F_c$ is a field. As usual, we write $(a,b) = a + ib$ and define $(a+ib)^* = a - ib$. We call $F$ a **square root field** if for any $a \in F^+$ there exists a unique $b \in F^+$ such that $b^2 = a$. We then write $b = \sqrt{a} = a^{1/2}$. For example, $^*\mathbb{R}$ is a square root field. In the sequel, we assume that $F$ is a square root field. We then define the **modulus** of an element of $F_c$ as $|a + ib| = (a^2 + b^2)^{1/2}$.

An $F_c$-**normed space** is a pair $(X, \|\cdot\|)$ where $X$ is a linear space over $F_c$ and $\|\cdot\| : X \to F^+ \cup \{0\}$ has the usual properties of a norm. Defining $\rho(x,y) = \|x-y\|$, it is clear that an $F_c$-normed space is an $F$-metric space. A linear operator $T : X \to X$ is defined in the usual way, and the proof of the following result is standard.

THEOREM 6.1. *If $T : X \to X$ is a linear operator, then the following statements are equivalent.* (a) *$T$ is continuous.* (b) *$T$ is continuous at* 0. (c) *The set $\{\|Tx\| : \|x\| \leq 1\}$ is bounded.* (d) *There exists an $M \in F^+$ such that $\|Tx\| \leq M\|x\|$ for all $x \in X$.*

If $T$ satisfies condition (d) of Theorem 6.1, we say that $T$ is **bounded** and $M$ is a **bound**. Unlike the standard case, a bounded operator need not have a smallest bound. Moreover, a bounded operator need not have a finite bound. That is, every bound $M$ for $T$ may satisfy $M > n$ for every $n \in \mathbb{N}$. As a consequence of Corollary 5.6, we have the following version of the uniform boundedness principle.

THEOREM 6.2. *Let $I \subseteq F^+$ be minimal cofinal, let $X$ and $Y$ be $F_c$-normed spaces and suppose $X$ is $I$-complete and has the $I$-ball property. For every $a$ in a set $A$, let $T_a : X \to Y$ be a continuous linear operator. If for every $x \in X$, $\{T_a x : a \in A\}$ is bounded, then there exists an $M \in F^+$ such that $\|T_a x\| \leq M\|x\|$ for every $a \in A$, $x \in X$.*

Proof. Applying Corollary 5.6, there exists an $M' \in F^+$ such that $\|T_a z\| \leq M'$ for all $z$ in some ball $B(y,\delta)$ and all $a \in A$. If $\|z\| < \delta$, then $\|(z+y) - y\| < \delta$ so

$$\|T_a z\| \leq \|T_a(z+y)\| + \|T_a y\| \leq 2M'$$

for all $a \in A$. If $x \neq 0$, then $\|\delta x/2\|x\|\| < \delta$. Hence, $\|T_a \delta x/2\|x\|\| < 2M'$ for all $a \in A$. Letting $M = 4M'/\delta$ we have $\|T_a x\| \leq M\|x\|$ for every $a \in A$, $x \in X$. ∎

The usual consequences of the uniform boundedness theorem can now be proved. As an open problem, do other important theorems of functional analysis hold in this context? For example, what about a Hahn-Banach theorem, an open mapping theorem and a closed graph theorem?

An $F_c$-**inner product space** is a pair $(X, \langle\cdot,\cdot\rangle)$ where $X$ is a linear space over $F_c$ and $\langle\cdot,\cdot\rangle : X \times X \to F_c$ has the usual properties of a complex inner product. Defining the **norm** $\|x\| = \langle x, x\rangle^{1/2}$, we see that $X$ is an $F_c$-normed space. It is straightforward to show that Schwarz's inequality $|\langle x,y\rangle| \leq \|x\|\|y\|$ holds. If $\|x-y\| \in \mathfrak{M}(0)$ we write $x \approx y$, and if $a \in F$ satisfies $|a| > n$ for all $n \in \mathbb{N}$ we call $a$ **infinite**. A linear operator $T$ on $X$ is **symmetric** if $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in X$.

Let $T$ be a linear operation on $X$ with domain a subspace $\mathfrak{D}(T) \subseteq X$. As usual, the **point spectrum** $\sigma_p(T)$ of $T$ is the set of $\lambda \in F_c$ such that $\lambda I - T$ is not injective, and if $(\lambda I - T)x = 0$, $x \neq 0$, $\lambda$ is an **eigenvalue** with corresponding **eigenvector** $x$. Moreover, the **continuous spectrum** $\sigma_c(T)$ is the set of $\lambda \in F_c$ such that $\lambda \notin \sigma_p(T)$, the range of $\lambda I - T$ is dense in $X$ but $(\lambda I - T)^{-1}$ is not bounded. The next result generalizes a theorem in [6].

THEOREM 6.3. *Let $T$ be a linear operator on an $F_c$-normed space $X$ where $F$ has uncountable cofinality. If $\lambda \in \sigma_c(T)$, then there exists an $x \in \mathfrak{D}(T)$ with $\|x\| = 1$ such that $Tx \approx \lambda x$.*

The vector $x$ in Theorem 6.3 is called a (unit) **ultraeigenvector** of $T$ corresponding to the **ultraeigenvalue** $\lambda$. The next result also generalizes some theorems in [6].

THEOREM 6.4. *Let $T$ be a symmetric linear operator on an $F_c$-inner product space $X$ where $F$ has uncountable cofinality.* (a) $\sigma_p(T) \cup \sigma_c(T) \subseteq F$. (b) *If $x, x'$ are unit ultraeigenvectors corresponding to distinct ultraeigenvalues $\lambda, \lambda'$ and if there exists an infinite $a \in F^+$ such that*

$$|\lambda - \lambda'| \geq a(\|Tx - \lambda x\| + \|Tx' - \lambda' x'\|)$$

*then $\langle x, x' \rangle \approx 0$.* (c) *If $\lambda, \lambda'$ are distinct ultraeigenvalues of $T$, then there exist unit ultraeigenvectors $x, x'$ corresponding to $\lambda, \lambda'$ such that $\langle x, x' \rangle \approx 0$.*

This last result generalizes a standard Hilbert space theorem. Are there important Hilbert space theorems that carry over to an $I$-complete $F_c$-inner product space? For example, what about the existence of an orthonormal basis, the Riesz representation theorem, the spectral theorem?

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**References**

[1] S. Albeverio, J. Fendstad, R. Høegh-Krohn, and T. Lindstrøm, *Nonstandard Methods in Stochastic Analysis and Mathematical Physics*, Academic Press, Orlando, 1986.

[2] M. Davis, *Applied Nonstandard Analysis*, Wiley, New York, 1977.

[3] N. Dunford and J. Schwartz, *Linear Operators*, Vol. I, Wiley (Interscience), New York, 1958.

[4] C.E. Francis, *Applications of non-standard analysis to relativistic quantum mechanics*, J. Phys. A: Math. Gen. 14 (1981), 2539–2551.

[5] T. Gamelin and R. Greene, *Introduction to Topology*, Saunders, Philadelphia, 1983.

[6] S. Gudder, *Nonstandard Fock spaces*, Intern. J. Theor. Phys. 33 (1994), 865–912.

[7] A.E. Hurd and P.A. Loeb, *An Introduction to Nonstandard Real Analysis*, Academic Press, Orlando, 1985.

[8] J. Keisler and J. Schmerl, *Making the hyperreal line both saturated and complete*, J. Sym. Logic 56 (1991), 1016–1025.

[9] J.L. Kelley, *General Topology*, Van Nostrand, New York, 1955.

[10] T. Lindstrøm, *Invitation to nonstandard analysis*, in *Nonstandard Analysis and its Applications*, N. Cutland, ed. (Cambridge University Press, Cambridge, 1988).

[11] A. Robinson, *Non-Standard Analysis*, North-Holland, Amsterdam, 1966.

[12] D. Scott *On completing ordered fields* in *Applications of Model Theory to Algebras, Analysis, and Probability*, W.A.J. Luxemburg, ed. (Holt, Rinehart and Winston, New York, 1969), pp. 274–278.

[13] K. Stroyan and W.A.J. Luxemburg, *Introduction to The Theory of Infinitesimals*, Academic Press, New York, 1976.

[14] E. Zakon, *Remarks on the nonstandard real axis* in *Applications of Model Theory to Algebras, Analysis, and Probability*, W.A.J. Luxemburg, ed. (Holt, Rinehart and Winston, New York, 1969), pp. 195–227.



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