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FIXED POINT THEOREMS OF MATKOWSKI ON PROBABILISTIC METRIC SPACES[1]

Introduction

Let us mention that probabilistic metric spaces were introduced by Menger [3, 4, 5]. In Menger's theory the concept of distance is considered to be statistical or probabilistic, rather than deterministic: that is to say, given any two points p and q of a metric space, rather than consider a single non-negative real number $d(p,q)$ as a measure of the distance between p and q, a distribution function $F_{pq}(x)$ is introduced which gives the probabilistic interpretation as the distance between p and q is less than $x (x > 0)$. For detailed discussions of probabilstic metric spaces and their applications we refer to Onicescu [6] and Schweizer [7,8].

Let (X,d) denote complete metric space and $T: X \to X$. Matkowski and Benedykt-Matkowski proved the following theorems:

THEOREM 1 [2]. *Let* $G : [0,\infty) \to [0,\infty)$ *fulfil the following conditions: g is nondecreasing in* $[0,\infty)$, $\lim g^n(t) = 0$ *for every* $t > 0$, *and* $d(Tx,Ty) \leq g(d(x,y))$ *for every* $x, y \in X$.

Then T has a unique fixed point x_0 and $\lim d(T^n x, x_0) = 0$ for every $x \in X$.

THEOREM 2 [1]. *Let* $g : [0,\infty) \to [0,\infty)$ *fulfil the following conditions:*

1^0 *g is nondecreasing,*

2^0 $\lim g^n(t) = 0$ *for* $t > 0$.

Suppose that there exists a function $n : X \to N$ *such that for every* $x \in X$ *and* $y \in X$

$$d(T^{n(x)}x, T^{n(y)}y) \leq g(d(x,y)).$$

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Then T has exactly one fixed point $x_0 \in X$ and $\lim T^n(x) = x_0$ *for every $x \in X$.*

(Here g^n as well as T^k denotes the k-th iteration of g and T, respectively). Theorem 1 is an extension the well-known Banach contraction-mapping theorem.

The main purpose of this paper is to prove the counterparts of the above results for probabilistic metric spaces which reads as follows.

THEOREM 3. *Let (E, F, Δ) be a complete PM-Menger space, where Δ is a continuous function satisfying $\Delta(x,x) \geq x$ for each $x \in [0,1]$, and T a mapping of E into itself. Let $g : [0,\infty) \to [0,\infty)$ fulfiles the following conditions*:

1^0 *g is nondecreasing in $[0,\infty)$,*
2^0 $\lim g^n(t) = \infty$ *for every $t > 0$,*
3^0 $F_{TpTq}(x) \geq F_{pq}(g(x))$ *for $x > 0$, and for every $p, q \in E$.*

The T has a unique fixed point p_0 and $\lim F_{T^n p p_0}(x) = 1$ *for every $p \in E$, and $x > 0$.*

THEOREM 4. *Let (E, F, Δ) be a complete PM-Menger space, where Δ is a continuous function satisfying $\Delta(x,x) \geq x$ for each $x \in [0,1]$, and t is mapping of E into itself. Let $g : [0,\infty) \to [0,\infty)$ fulfiles the following conditions*:

1^0 *g is nondecreasing in $[0,\infty)$,*
2^0 $\lim g^n(t) = \infty$ *for every $t > 0$.*

Suppose that there exists a function $n : E \to N$ such that for every $x, y \in E$

$$F_{T^{n(p)}p\,T^{n(q)}q}(x) \geq F_{pq}(g(x)), \quad \text{for every } x > 0.$$

Then T has exactly one fixed point $u \in E$ and $T^n p \to u$ for every $p \in E$.

The proofs will be in section 3.

1. Basic definitions and some auxiliary results

Let R denote the reals and $R^+ = \{x \in R : x \geq 0\}$.

DEFINITION 1. A mapping $F\colon R \to R^+$ is called a distribution function if it is nondecreasing, left-continiuous with $\inf F = 0$ and $\sup F = 1$.

We will denote by L the set of all distribution functions.

DEFINITION 2. A probabilistic metrix space (PM-space) is on ordered pair (E, F), where E is an abstract set of elements and F is a mapping of $E \times E$ into L. We shall denote the distribution function $F(p,q)$ by F_{pq} and

$F_{pq}(x)$ will represent the value of F_{pq} at $x \in R$. The function F_{pq}, $p, q \in E$, are assumed to satisfy the following conditions:

(PM-I) $F_{pq}(x) = 1$ for all $x > 0$, if and only if $p = q$,
(PM-II) $F_{pq}(0) = 0$,
(PM-III) $F_{pq} = F_{qp}$,
(PM-IV) if $F_{pq}(x) = 1$ and $F_{qr}(y) = 1$, then $F_{pr}(x + y) = 1$,

for all $p, q, r \in E$.

Remark. Definition 2 suggests that $F_{pq}(x)$ may be interpreted as probability of the event that the distance between p and q is less than x.

DEFINITION 3. A mapping $\Delta : [0,1] \times [0,1] \to [0,1]$ is a *Δ-norm* if it satisfies

(Δ-I) $\Delta(a,1) = a$, $\Delta(0,0) = 0$,
(Δ-II) $\Delta(a,b) = \Delta(b,a)$,
(Δ-III) $\Delta(c,d) \geq \Delta(a,b)$ for $c \geq a$, $d \geq b$,
(Δ-IV) $\Delta(\Delta(a,b),c) = \Delta(a,\Delta(b,c))$.

Let B denote the set of all Δ-norms, partially ordered by $\Delta_1 \leq \Delta_2$ if and only if $\Delta_1(a,b) \leq \Delta_2(a,b)$ for all $a, b \in [0,1]$ and $\Delta_1, \Delta_2 \in B$.

DEFINITION 4. A *Menger space* is a triplet (E, F, Δ), where (E, F) is PM-space and $\Delta \in B$ satisfies the following triangle inequality:

(PM-IV~) $F_{pr}(x + y) \geq \Delta(F_{pq}(x), F_{qr}(y))$

for all $p, q, r \in E$ and for all $x \geq 0, y \geq 0$.

The concept of a neighborhood in a PM-space was introduced by Schweizer and Sklar [9]. If $p \in E$, and μ, σ are positive reals, then an (μ,σ)-neighborhood of p, denoted by $U_p(\mu,\sigma)$, is defined by

$$U_p(\mu,\sigma) = \{q \in E : F_{pq}(\mu) > 1 - \sigma\}.$$

The following result is due to Schweizer and Sklar [9].

THEOREM 5. *If (E, F, Δ) is a Menger space and Δ is continuous then (E, F, Δ) is a Hausdorff space with the topology induced by the familly $\{U_p(\mu,\sigma) : p \in E, \mu > 0, \sigma > 0\}$ of neighborhoods.*

Note that the above topology satisfies the first axiom of countability. In this topology a sequence $\{p_n\}$ in E converges to a $p \in E$ ($p_n \to p$) if and only if for every $\mu > 0$ and $\sigma > 0$, there exists an integer $M(\mu,\sigma)$ such that $p_n \in U_p(\mu,\sigma)$, i.e., $F_{pp_n}(\mu) > 1 - \sigma$ whenever $n \geq M(\mu,\sigma)$. The sequence $\{p_n\}$ will be called fundamental in E if for each $\mu > 0, \sigma > 0$, there is an integer $M(\mu,\sigma)$ such that $F_{p_n p_m}(\mu) > 1 - \sigma$ whenever $n, m \geq M(\mu,\sigma)$. In analogy with the completion concept of metric spaces, a Menger space E

will be called complete if each fundamental sequence in E converges to an element in E.

The following theorem is easy to prove and it establishes a connection between metric spaces and Menger spaces.

THEOREM 6. *If (E, d) is a metric space then the metric d induces a mapping $\boldsymbol{F}: E \times E \to L$, defined by $F_{pq}(x) = H(x - d(p,q))$, $x \in R$, where $H(x) = 0$ if $x \leq 0$ and $H(x) = 1$ if $x > 0$. Further, the triple $(E, \boldsymbol{F}, \Delta)$ is a Menger space with $\Delta(a, b) = \min\{a, b\}$. This space is complete if (E, d) is complete.*

The space $(E, \boldsymbol{F}, \Delta)$ so obtained will be called *induced Menger space*.

Remark. Metric spaces are special cases of Menger spaces with $\Delta(x, x) \geq x$ for all $x \in [0, 1]$.

2. Banach contraction-mapping theorem on PM-spaces

We first introduce the notion of a contraction mapping on a PM-space.

DEFINITION 5. A mapping T of a PM-space $(E, \boldsymbol{F})$ into itself will be called a *contraction mapping* if and only if there exists a constant k, with $0 < k < 1$, such that for each $p, q \in E$,

$$(*) \qquad F_{TpTq}(kx) \geq F_{pq}(x) \quad \text{for all } x > 0.$$

Expression $(*)$ may be interpreted as follows; the probability that the distance between the image points T_p, T_q is less than kx is at least equal to the probability that the distance between p, q is less than x.

THEOREM 7 [10]. *Let $(E, \boldsymbol{F}, \Delta)$ be a complete Menger space, where Δ is a continuous function satisfying $\Delta(x, x) \geq x$ for each $x \in [0, 1]$. If T is any contraction mapping from E into itself, then there is a unique $p \in E$ such that $Tp = p$. Moreover, $T^n q \to p$ for each $q \in E$.*

Now we state and prove the well-known Banach contraction-mapping theorem; this proof uses the notion of a probabilistic metric.

THEOREM 8. *Let (E, d) be a complete metric space and let $T : E \to E$ satisfy the following condition: there exists a constant $k, 0 < k < 1$, such that $d(Tp, Tq) \leq kd(p, q)$ for all $p, q \in E$. Then T has a unique fixed point $p \in E$ and $T^n q \to p$ for each $q \in E$.*

Proof. If $F : E \times E \to L$ is the mapping induced by the metric d, then from Theorem 6 it follows that $(E, \boldsymbol{F}, \Delta)$ is a complete Menger space, where $\Delta(a, b) = \min\{a, b\}$. Observe that T is a contraction from E into itself. Since for each $x > 0$

$$F_{TpTq}(kx) = H(kx - d(Tp, Tq)) \geq H(kx - kd(p,q)) =$$
$$= H(x - d(p,q)) = F_{pq}(x),$$

The conclusion follows now from Theorem 7.

3. Proofs of the present theorems

Poof of Theorem 3. We first prove the uniqueness. Suppose $p \neq q$ and $Tp = p$, $Tq = q$. Then by $(PM-1)$, there exists an $x > 0$ and an a, with $0 \leq a < 1$, such that $F_{pq}(x) = a$. However, for each positive integer n, we have by 3^0

$$a = F_{pq}(x) = F_{T^n pT^n q}(x) \geq F_{TpTq}(g^n(x)).$$

Since $F_{pq}(g^n(x)) \to 1$ as $n \to \infty$, it follows that $a = 1$. This contradicts the choice of a, and therefore, the fixed point is unique. To prove the existence of the fixed point, consider an arbitrary $q \in E$, and define $p_n = T^n q$, $n = 1, 2, \ldots$ We show that the sequence $\{p_n\}$ is fundamental in E. Let μ, σ be positive reals. Then for $m > n$ and puting $k = m - n$ we have

$$F_{p_n p_m}(\mu) \geq \Delta(F_{p_n p_{n+1}}(\mu k^{-1}), F_{p_{n+1} p_m}(\mu(k-1)k^{-1})) \geq$$
$$\geq \Delta(F_{p_1 q}(d), F_{p_{n+1} p_m}(\mu(k-1)k^{-1})), \quad \text{where } d = g^n(\mu k^{-1}),$$

and

$$F_{p_{n+1} p_m}(\mu(k-1)k^{-1}) \geq \Delta(F_{p_{n+1} p_{n+2}}(\mu k^{-1}), F_{p_{n+2} p_m}(\mu(k-2)k^{-1})) \geq$$
$$\geq \Delta(F_{p_1 q}(g^{n+1}(\mu k^{-1})), F_{p_{n+2} p_m}(\mu(k-2)k^{-1})) \geq$$
$$\geq \Delta(F_{p_1 q}(d), F_{p_{n+2} p_m}(\mu(k-2)k^{-1})).$$

Hence and by the associativity of Δ, and the hypothesis $\Delta(x, x) \geq x$, we have

$$(**) \quad F_{p_n p_m}(\mu) \geq \Delta(F_{p_1 q}(d), \Delta(F_{p_1 q}(d), F_{p_{n+2} p_m}(\mu(k-2)k^{-1}))) =$$
$$= \Delta(\Delta(F_{p_1 q}(d), F_{p_1 q}(d)), F_{p_{n+2} p_m}(\mu(k-2)k^{-1}))) \geq$$
$$\geq \Delta(F_{p_1 q}(d), F_{p_{n+2} p_m}(\mu(k-2)k^{-1})).$$

Using the induction argument we obtain from $(**)$

$$F_{p_n p_m}(\mu) \geq \Delta(F_{p_1 q}(d), \Delta(F_{p_{n+k-2} p_{n+k-1}}(\mu k^{-1}), F_{p_{m-1} p_m}(\mu k^{-1}))) \geq$$
$$\geq \Delta(F_{p_1 q}(d), \Delta(F_{p_1 q}(g^{n+k-2}(\mu k^{-1})), F_{p_1 q}(g^{m-1}(\mu k^{-1})))) \geq$$
$$\geq \Delta(F_{p_1 q}(g^n(\mu k^{-1})).$$

Therefore, if we choose N such that

$$F_{p_n p_m}(g^n(\mu k^{-1})) > 1 - \sigma,$$

$$F_{p_n p_m}(\mu) > 1 - \sigma \quad \text{for all } m > n \geq N.$$

Hence $\{p_n\}$ is a fundamental sequence in E. Since (E, F, Δ) is a complete PM-space, there is a $p \in E$ such that $p_n \to p$, that is $T^n q \to p$. We shall show that $T^n q \to Tp$ also. Let $U_{Tp}(\mu, \sigma)$ be any neighborhood of Tp. Then $p_n \to p$ implies the existence of an integer $N = N(\mu, \sigma)$ such that $p_n \in U_p(\mu, \sigma)$ for all $n \geq N$. However

$$F_{Tp_n Tp}(\mu) \geq F_{p_n p}(g(\mu)) \geq F_{p_n p}(\mu) > 1 - \sigma \quad \text{for all } n \geq N$$

that is $T^n q \to Tp$. Therefore we conclude that $Tp = p$. This proves the existence part of Theorem 3.

Proof of Theorem 4. Let us define: $Sp = T^{n(p)}p$ for $p \in E$. Then

$$F_{SpSq}(x) \geq F_{pq}(g(x)) \quad \text{for any } p, q \in E \quad \text{and} \quad x > 0.$$

According to Th. 3 there is $u \in E$ such that $S(u) = u$. One can easily verify that u is a unique fixed point of T and $T^n q \to u$, for any $q \in E$.

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