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ON-LINE COVERING OF THE UNIT SQUARE BY A SEQUENCE OF CONVEX BODIES

We say that a sequence $Q_1, Q_2, \ldots$ of sets of Euclidean d-space E^d *permits a covering* of a set $S \subset E^d$ if there are rigid motions $\sigma_1, \sigma_2, \ldots$ such that S is contained in the union of sets $\sigma_i Q_i$, where $i = 1, 2, \ldots$ Groemer [2] proved that every sequence of convex bodies of E^d of diameters at most 1 and of total volume at least $(2^d - 1)d^d$ for $d > 3$ (and at least 6 for $d = 2$) permits a covering of the unit cube I^d.

The on-line covering by convex bodies is considered in [3]. The on-line covering restriction means that we learn every Q_i with $i > 1$ only after the motion σ_{i-1} has been provided. This kind of restriction has been introduced for packing by Lassak and Zhang [4] who considered the on-line version of the well known potato-sack problem of Auerbach, Banach, Mazur and Ulam [1]. In this paper we improve the estimate of 28 (see [3]) for the total area of a sequence of planar convex bodies that permits an on-line covering of I^2 to 15.

THEOREM. *Every sequence of planar convex bodies of diameters at most* 1 *whose total area is at least* 15 *permits an on-line covering of the unit square. If the bodies are rectangles, then the total area of* 7.5 *is sufficient for the covering.*

The problem of an on-line covering by planar convex bodies reduces to the problem of an on-line covering by rectangles thanks to the Radziszewski's theorem (see [5]) that every planar convex body P contains a rectangle R whose area $|R|$ is at least $\frac{1}{2}|P|$.

Let $Q_1, Q_2, \ldots$ be an arbitrary sequence of rectangles of sides at most 1. A rectangle S_i of sides s_i, h_i, where $h_i \in \{2^{-1}, 2^{-2}, \ldots\}$ and $0 < s_i \le 2h_i$

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is called *a standard rectangle*. Observe that every rectangle Q_i contains a standard rectangle S_i such that $|S_i| \geq \frac{1}{2}|Q|$. We will cover the unit square by the corresponding sequence of standard rectangles.

If $h_i < \frac{1}{8}$, then S_i is called *a small standard rectangle*. If $h_i \geq \frac{1}{8}$, then S_i is called *a large standard rectangle*. For an on-line covering of I^2 by small standard rectangles a modification of the method from [3] is used. Small rectangles will be used for covering the unit square succesively from the "bottom". Large standard rectangles will be put by another method starting from the "top" of I^2 .

Let $A_k = \{(x,y) : 0 \leq x \leq d_k,\ h_{k-1} \leq y \leq h_k\}$, where $d = d_1 > \ldots > d_n > 0$ and $0 = h_0 < \ldots < h_n = h$. The set $\bigcup_{k=1}^{n} A_k$ is denoted by A_{dh}. Below we describe a method of covering of A_{dh} by arbitrary sequences $B_1, B_2, \ldots$ of small standard rectangles. We call it *the auxiliary method*.

For every $i = 2, 3, \ldots$ we find the greatest number b_i not greater than h such that every point of A_{dh} with the second coordinate smaller than b_i is covered by rectangles $\sigma_1 B_1, \ldots, \sigma_{i-1} B_{i-1}$. We start with $b_1 = 0$.

We find a motion σ_i such that

$$\sigma_i B_i = \{(x,y) : \alpha_i \leq x \leq \alpha_i + s_i \text{ and } \beta_i \leq y \leq \beta_i + h_i\},$$

where

1) β_i is the greatest multiple of $\frac{1}{3}h_i$ not exceeding b_i;

2) α_i is the greatest number such that the set $\{(x,y) \in A_{dh} : 0 \leq x < \alpha_i, y = \beta_i + h_i\}$ is covered by rectangles preceding B_i.

We stop the covering process if $b_i = h$ for an index i.

LEMMA. *A sequence $B_1, B_2, \ldots$ of small standard rectangles permits an on-line covering of A_{dh} if the total area of this sequence is at least* $3|A_{dh}| + \frac{1}{4}h + \frac{3}{16}d$.

Proof. Every rectangle $\sigma_i B_i$ covers for the first time a region containing the rectangle $\{(x,y) : \alpha_i < x \leq \alpha_i + s_i,\ \beta_i + \frac{2}{3}h_i < y \leq \beta_i + h_i\}$. Thus, rectangles $\sigma_1 B_1, \ldots, \sigma_k B_k$ cover a region of area at least $\frac{1}{3}\sum_{i=1}^{k} |B_i|$, for every index k.

Since $s_i \leq \frac{1}{8}$ and $h_i \leq \frac{1}{16}$, every rectangle $\sigma_i B_i$ is contained in the set $A^*_{dh} = \bigcup_{k=1}^{n}(A_k \cup D_k \cup C_k)$, where

$$D_1 = \left\{(x,y) : d_1 < x < d_1 + \tfrac{1}{8},\ 0 \leq y \leq h_k + \tfrac{1}{16}\right\},$$

$$D_k = \left\{(x,y) : d_k < x < d_k + \tfrac{1}{8},\ h_{k-1} + \tfrac{1}{16} \leq y \leq h_k + \tfrac{1}{16}\right\}$$

for $k = 2, \ldots, n$, and

$$C_k = \left\{(x,y) : d_k - d_{k-1} < x < d_k,\ h_k \leq y \leq h_k + \tfrac{1}{16}\right\}$$

for $k = 1, \ldots, n$. Here $d_{n+1} = 0$.

Observe that $|A^*_{dh}| = |A_{dh}| + \frac{1}{8}(h + \frac{1}{16}) + \frac{1}{16}d$. If a point of the set $G = \{(x,y) : d_n \le x < d_n + \frac{1}{8}, h_n \le y \le h_n + \frac{1}{16}\}$ is covered by a rectangle $\sigma_i B_i$, then the set A_{dh} is covered as well. Consequently, every sequence $B_1, B_2, \ldots$ of total area at least $3|A^*_{dh} \backslash G|$ permits a covering of A_{dh}. This number is precisely

$$(1) \qquad 3(|A_{dh}| + \tfrac{1}{8}h + \tfrac{1}{16}d).$$

This estimate can be slightly improved. Denote by Int(S) the interior of S. Observe that if a point of the set

$$D_h = \bigcup_{k=1}^{n} \{(x,y) \in D_k : d_k + \tfrac{1}{16} \le x < d_k + \tfrac{1}{8}\} \backslash G$$

is covered by Int($\sigma_i B_i$), then from the inequality $s_i \le 2h_i$ we conclude that $h_i = \frac{1}{16}$. If some two rectangles from the sequence $\sigma_1 B_1, \sigma_2 B_2, \ldots$ have non-empty intersections with Int(D_h), then they have disjoint interiors. We assume in (1) that D_h is covered by rectangles of total area $3|D_h|$. But the sum of the areas of the rectangles from the sequence $\sigma_1 B_1, \sigma_2 B_2, \ldots$ which have non-empty intersections with Int(D_h) is not greater than $|D_h|$. The estimate (1) remains true if we subtract $2|D_h| = \frac{1}{8}h$ from it.

We conclude that we can cover A_{dh} by every sequence of rectangles $B_1, B_2, \ldots$ of total area at least $3|A_{dh}| + \frac{1}{4}h + \frac{3}{16}d$.

Now we describe an on-line method of covering of I^2 by a sequence $S_1, S_2, \ldots$ of standard rectangles.

If S_i is a small standard rectangle, then let $S_i = B_i$, $T_i = \emptyset$. If S_i is a large standard rectangle, then let $S_i = T_i$, $B_i = \emptyset$. We put every rectangle $B_i \ne \emptyset$ according to the auxiliary method applied to the set $I^2 \backslash \bigcup_{j<i} \sigma_j T_j$. If $B_i = \emptyset$, it is not used for the covering.

We put every rectangle $T_i \ne \emptyset$ in the following way. If $i \ge 2$, then denote by t_i the smallest non-negative number such that all points of I^2 with the second coordinate greater than t_i are covered by rectangles $\sigma_1 T_1, \ldots, \sigma_{i-1} T_{i-1}$. If $i = 1$, then let $t_1 = 1$.

We find a motion σ_i such that

$$\sigma_i T_i = \{(x,y) : \gamma_i - s_i \le x \le \gamma_i,\ \delta_i - h_i \le y \le \delta_i\},$$

where the numbers γ_i, δ_i fulfil the following two conditions :

1) if $h_i = \frac{1}{8}$ or $\frac{1}{4}$, then δ_i is the smallest multiple of h_i not smaller than t_i; if $h_i = \frac{1}{2}$, then δ_i is the smallest number amongst $\frac{1}{2}, \frac{3}{4}, 1$ which is not smaller than t_i;

2) γ_i is the smallest number such that the set $\{(x,y) \in I^2 : \gamma_i < x \le 1,\ m \le y \le \delta_i\}$ is contained in $\bigcup_{i<j} \sigma_j S_j$, where $m = \max(b_i, \delta_i - h_i)$.

We stop the covering process by standard rectangles when the unit square is totally covered.

Proof of Theorem. We cover I^2 by the corresponding sequence of standard rectangles. The reader can easily check that

$$\frac{\sum_{j=1}^{k} |T_j|}{|(\bigcup_{j=1}^{k} \sigma_j T_j) \cap I^2|} < 3$$

for every index k. In the Lemma we suppose that for covering a region of area p by rectangles B_i the total area of rectangles used for the covering is not smaller than $3p$. This means that it is sufficient to consider the following case.

Assume that i is the smallest index such that the set T_i is non-empty and assume that $b_i \geq \frac{7}{8}$. Denote by $\frac{1}{2}s$ the sum of areas of all rectangles $\sigma_j T_j$ used for the covering. If $s \geq 1$, then I^2 is covered. Thus assume that $0 \leq s < 1$. If $A_{dh} = I^2$ in Lemma, then we can assume that the sum of areas of rectangles B_j used for the covering is $3|I^2| + \frac{1}{4} + \frac{3}{16}(1 - s)$.

From this and from the inequality $|Q_i| \leq 2|S_i|$, we conclude that I^2 can be on-line covered by every sequence $Q_1, Q_2, \ldots$ of the total area greater than or equal to

$$2\left(\tfrac{1}{2}s + 3|I^2| + \tfrac{1}{4} + \tfrac{3}{16}(1 - s)\right) \leq 2\left(\tfrac{1}{2} + 3 + \tfrac{1}{4}\right) = 7.5.$$

Consequently, the unit square can be on-line covered by every sequence of planar convex bodies of sides at most 1 whose total area is at least 15.

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