DEMONSTRATIO MATHEMATICA
Vol. XXIX No 1 1996

Nicolae Soare

ON THE EXISTENCE OF THE LINEAR CONNECTIONS COMPATIBLE WITH SOME (f,g)-STRUCTURES

In this note we study the (f,g)-structures determined by a tensor field f of type (1.1) so that $f^{2v+3}-f=0$ and a Riemannian structure g, satisfying a suplimentary condition.

The case of the (f,g)-structures with $f^3+f=0$ was studied by R. Miron and Gh. Atanasiu [5].

Using Obata's operators associated to the considered (f,g)-structures and Wilde's method of characterizing the set of solutions of a system of tensorial equations [10], are found all the linear (f,g)-connections and the groups of transformations of connections of the (f,g)-structures.

1. (f,g)-structures

Let M be a Riemannian manifold with the Riemannian metric g, $C(M)$ the affin modul of the linear connections on M, $\mathcal{T}_s^r(M)$ - the modul of the tensors of type (r,s); for $\mathcal{T}_0^1(M)$ and $\mathcal{T}_1^0(M)$, are used the notations $\mathcal{X}(M)$ and $\mathcal{X}^*(M)$ respectively.

All the objects are of class C^∞.

DEFINITION 1.1. We call f-structure on M, a non-null field of tensors $f\in\mathcal{T}_1^1(M)$, of rank r, where r is a constant everywhere, so that

$$f^{2v+3}-f=0.$$

If M is a $f(2v+3,-1)$-manifold, that is M is an n-dimensional Riemannian manifold, equiped with a f-structure, then for

(1.1) $e=f^{2v+2}$, $\quad m=I-f^{2v+2}$ (I denoting identity operator) we have [7]

$$(1.2)\qquad fe=ef=f,\quad fm=mf=0,\quad f^{2v+2}e=e,\quad f^{2v+2}m=0,$$

$$(1.3)\qquad e+m=I,\quad em=me=0,\quad e^2=e,\quad m^2=m.$$

Thus the operators e and m are complementary projection operators on M.

The Riemannian structure g on M can be considered as $\mathcal{X}^*(M)$-valued differential 1-form and we have $g : \mathcal{X}(M) \to \mathcal{X}^*(M), g(X) = g_X$, where $g_X(Y) = g(X,Y)$ for every $X, Y \in \mathcal{X}(M)$. If $f \in \mathcal{T}_1^1(M)$, then ${}^t f$ is the transpose of f, ${}^t f : \mathcal{X}^*(M) \to \mathcal{X}^*(M)$, ${}^t f(h) = h \circ f$, for every $h \in \mathcal{X}^*(M)$.

DEFINITION 1.2. We call (f,g)-structure on M, a couple made up a f-structure ans a Riemannian structure g so that

$$ {}^t f^{v+1} \circ g \circ f^{v+1} = -g \circ e. \tag{1.4} $$

THEOREM 1.1. *Let M be a paracompact differential manifold with a f-structure. Then, there is a (f,g)-structure.*

Proof. In truth, if γ is a Riemannian metric, fixed on M, then

$$ g = \frac{1}{2}(\gamma - {}^t f^{v+1} \circ \gamma \circ f^{v+1} - \gamma \circ m - {}^t m \circ \gamma + 3 {}^t m \circ \gamma \circ m) \tag{1.5} $$

verifies the condition (1.4).

PROPOSITION 1.1. *For a (f,g)-structure on M and e, m defined by the equations* (1.1) *we have*

$$ \begin{aligned} g \circ f^{v+1} &= -\,{}^t f^{v+1} \circ g, & g^{-1} \circ {}^t f^{v+1} &= -\, f^{v+1} \circ g^{-1} \\ g \circ m &= {}^t m \circ g, & g^{-1} \circ {}^t m &= m \circ g^{-1}. \end{aligned} \tag{1.6} $$

PROPOSITION 1.2. $\omega = g \circ f^{v+1}$ *is a differential 2-form on M.*

DEFINITION 1.3. W call Obata operators associated to f-structure, the applications $A, A^* : \mathcal{T}_1^1(M) \to \mathcal{T}_1^1(M)$ defined by

$$ \begin{aligned} A(w) &= \frac{1}{2}(w - mw - wm + 3wmn - f^{v+1} w f^{v+1}), \\ A^*(w) &= w - A(w). \end{aligned} \tag{1.7} $$

We also consider the Obata operators [6] associated to g:

$$ B(u) = \frac{1}{2}(u - g \circ {}^t u \circ g), \quad B(u) = \frac{1}{2}(u + g^{-1} \circ {}^t u \circ g). \tag{1.8} $$

We can demonstrate

PROPOSITION 1.3. *For a (f,g)-structure on M and for A, A^* and B, B^* defined by* (1.7) *and* (1.8) *we have:*

1) *A and A^* are complementary projections on $\mathcal{T}_1^1(M)$.*
2) *B and B^* commute pairwise with A and A^*.*

3) $A \circ B$ *and* $A^* \circ B^*$ *are projections on* $\mathcal{T}_1^1(M)$.

4) $\operatorname{Ker} A^* \cap \ker B^* = \operatorname{Im}(A \circ B)$.

In truth, by simple calculation, we obtain the result 1).

The affirmation 2) is true, because, taking into account the relations (1.6), we have

$$(A \circ B - B \circ A)(u) = \ldots$$
$$= \frac{1}{4}(m \circ g^{-1} \circ {}^t u \circ g - g^{-1} \circ {}^t m \circ {}^t u \circ g) + (g^{-1} \circ {}^t u \circ g \circ m - g^{-1} \circ {}^t u \circ {}^t m \circ g)$$
$$-3(m \circ g^{-1} \circ {}^t u \circ g \circ m - g^{-1} \circ {}^t m \circ {}^t u \circ {}^t m \circ g)$$
$$+(f^{v+1} \circ g^{-1} \circ {}^t u \circ g \circ f^{v+1} - g^{-1} \circ {}^t f^{v+1} \circ {}^t u \circ {}^t f^{v+1} \circ g) = 0,$$

for every $u \in \mathcal{T}_1^1(M)$, or

$$A \circ B = B \circ A.$$

Thus we have the relations

$$A \circ B^* = B^* \circ A,$$
$$A^* \circ B = B \circ A^*.$$

The above mentioned relations give us the possibility to formulate [10]:

PROPOSITION 1.4. *The system of tensorial equations*

$$A^*(u) = a, \quad B^*(u) = b \tag{1.9}$$

has a solution $u \in \mathcal{T}_1^1(M)$, *if and only if*

$$A(a) = 0, \quad B(b) = 0, \quad A^*(b) = B^*(a). \tag{1.10}$$

If the conditions (1.10) are fulfilled, then the general solution of the system (1.9) is

$$u = a + A(b) + (A \circ B)(w) \quad \forall w \in \mathcal{T}_1^1(M).$$

2. (f, g)-linear connections

In the following $\overset{\circ}{\nabla} \in \mathcal{C}(M)$ will be a linear connection fixed on M. Every tensor field $u \in \mathcal{T}_1^1(M)$ may be considered as a field of $\mathcal{X}(M)$-valued differential 1-forms. So, if ∇ is a linear connection on M, then we note with D and $\widetilde{D}$ the associated connections, acting on the $\mathcal{X}(M)$-valued differential 1-forms and respectively on the differential 1-forms $g : \mathcal{X}(M) \to \mathcal{X}^*(M)$:

$$D_x u = \nabla_X u - u \nabla_X, \quad \forall X \in \mathcal{X}(M), \tag{2.1}$$

$$\widetilde{D}_X g = {}^t\nabla_X \circ g - g \circ \nabla_X, \quad \forall X \in \mathcal{X}(M), \tag{2.2}$$

where

$$({}^t\nabla_X g)(Y,Z) = Xg(Y,Z) - g(\nabla_X Y, Z), \quad \forall X,Y,Z \in \mathcal{X}(M).$$

DEFINITION 2.1. A linear connection ∇ on M is called (f,g)-linear connection if

$$D_X f = o, \quad \widetilde{D}_X g = 0, \quad \forall X \in \mathcal{X}(M). \tag{2.3}$$

Of course, for every (f,g)-linear connection, we have

$$\begin{aligned} &D_X e = \nabla_X e - e\nabla_X = 0, \quad D_X m = \nabla_X m - m\nabla_X = 0,\\ &D_X f^k = \nabla_X f^k - f^k\nabla_X = 0, \quad k \text{ natural number}, \; X \in \mathcal{X}(M). \end{aligned} \tag{2.4}$$

We see that D and $\widetilde{D}$ commute with the operators A, A^*, B, B^*. We take

$$\nabla_X = \overset{\circ}{\nabla}_X + V_X,$$

$X \in \mathcal{X}(M)$, $V \in \mathcal{T}_2^1(M)$, $V_X Y = V(X,Y)$ and find the tensor field V so that ∇ satisfies the conditions (2.3).

∇ will be a (f,g)-linear connection if and only if the field V verifies the system

$$V_X \circ f - f \circ V_X = -\overset{\circ}{D}_X f, \quad {}^tV_X \circ g + gV_X = \overset{\widetilde{\circ}}{D}_X g.$$

This system is equivalent with the system

$$\begin{aligned} A^*(V_X) &= -\frac{1}{2}(f \circ \overset{\circ}{D}_X f + \overset{\circ}{D}_X m - 3m \circ \overset{\circ}{D}_X m)\\ B^*(V_X) &= \frac{1}{2} g^{-1} \circ \overset{\widetilde{\circ}}{D}_X g. \end{aligned} \tag{2.5}$$

Applying Proposition 1.4, it becomes evident that the system (2.5) has the solutions and the general solutions is

$$\begin{aligned} V_X = &-\frac{1}{2}(f \circ \overset{\circ}{D}_X f + \overset{\circ}{D}_X m - 3m \circ \overset{\circ}{D}_X m) + \frac{1}{4}(\overset{\widetilde{\circ}}{D}_X g - {}^tf^{v+1} \circ \overset{\widetilde{\circ}}{D}_X g \circ f^{v+}\\ &- \overset{\widetilde{\circ}}{D}_X g \circ m - {}^tm \circ \overset{\widetilde{\circ}}{D}_X g + 3{}^tm \circ \overset{\widetilde{\circ}}{D}_X g \circ m) + (A \circ B)(W_X), \end{aligned} \tag{2.6}$$

where $W \in \mathcal{T}\frac{1}{2}(M)$. Thus, we have

THEOREM 2.1. *There are (f,g)-linear connections: one of them is*

$$\nabla_X = \overset{\circ}{\nabla}_X + V_X \tag{2.7}$$

where $\overset{\circ}{\nabla}$ is an arbitrary linear connection, fixed on M, and V_X is given by (2.6), W being an arbitrary tensor field.

If $\overset{\circ}{\nabla}$ is the Levi-Civita connection of g, then we have $\tilde{\overset{\circ}{D}}_X g = 0$ and Theorem 2.1 *becomes*

THEOREM 2.2. *For every (f,g)-structure, the following linear connection*

$$\overset{c}{\nabla}_X = \overset{\circ}{\nabla}_X - \frac{1}{2}(f \circ \overset{\circ}{D}_X f + \overset{\circ}{D}_X m - 3m \circ \overset{\circ}{D}_X m), \quad \forall X \in \mathcal{X}(M), \tag{2.8}$$

where $\overset{\circ}{\nabla}$ is the Levi-Civita connection of g, has the following characteristics:

1) *$\overset{c}{\nabla}$ is dependent uniquely of f and g*:
2) *$\overset{c}{\nabla}$ is a (f,g)-linear connection.*

The linear connection $\overset{c}{\nabla}$ will be called the (f,g) canonic connection.

THEOREM 2.3. *The set of all the (f,g)-linear connections is given by*

$$\overline{\nabla}_X = \nabla_X + (A \circ B)(W_X), \quad W \in \mathcal{T}\frac{1}{2}(M), \tag{2.9}$$

where ∇ is a (f,g)-linear connection, in particular $\nabla = \overset{c}{\nabla}$.

Observing that (2.9) can be considered as a transformation of (f,g)-linear connections, we have

THEOREM 2.4. *The set of the transformations of (f,g)-linear connections and the multiplications of the applications is an abelian group, noted with $G(f,g)$, isomorph with the additiv group of the tensors $W \in \mathcal{T}\frac{1}{2}(M)$, which have the characteristic*

$$W_X \in \mathrm{Im}(A \circ B) = \mathrm{Ker}\, A^* \cap \mathrm{Ker}\, B^*, \quad X \in \mathcal{X}(M).$$

References

[1] D. Demetropoulou-Psomopoulou, *On iontegrability conditions of a structure satisfying $f^{2v+3} + f = 0$*, Tensor, N.S. 42 (1985), 252–257.

[2] D. Demetropoulou-Psomopoulou, *Linear connections on manifolds admitting an $f(2v+3,-1)$-structure or an $f(2v+3,1)$-structure*, Tensor, N.S. 47 (1988), 235–238.

[3] S. Goldgerg and K. Yano, *Globally framed f-manifolds*, Ilinois Journal of Math., 15, No. 3(1971), 456–474.

[4] S. Ishihara and K. Yano, *On integrability conditions of a structure satisfying $f^3 + f = 0$*, Quart. J. Math. 15 (1964), 217–222.

[5] R. Miron et Gh. Atanasiu, *Sur les (f,g)-connections linéaires et l'integrabilité des (f,g)-structures*, Bull. Math. Soc. Sci. Math. R.S. Roumanie, (78) 30 (1986), 245–256.

[6] M. Obata, *Affine connections on manifolds with almost complex, quaternionic or Hermitian structure*, Jap. Journ. Math., 26 (1956), 43–77.

[7] A. Petrakis, *Structure f qui satisfait la relation* $f^{2v+3} - f = 0$, Tensor, N.S., 41 (1984), 103–107.

[8] N. Soare, *Sur quelques f-connections linéaire*, Riv. Mat. Univ. Parma (4) 14 (1988), 143–150.

[9] P. Tamia-Dimopoulou, *On direction dependent f-structures satisfying* $f^s + f^t = 0$, Riv. Mat. Univ. Parma (4) 17 (1991), 279–293.

[10] A. C. Wilde, *Commutative Projection Operators*, Atti Sem. Mat. Fis. Univ. Modena, 35 (1987), 167–172.

FACULTATEA DE MATEMATICA
Str. Academiei 14
BUCUREŞTI, ROMANIA

Received May 4, 1994.