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SOME COMMON FIXED POINT THEOREMS IN PARANORMED SPACES

1. Introduction

In [1], [2], [4], [5], [7], the authors established the convergence of some sequences of iterates to a fixed point of a single mapping T under some contractive conditions in a normed space or in a Banach space. On the other hand, L. A. Khan [3] extended some of the above results to the case of a paranormed space using various contractive definitions of the mapping T.

In this paper we extend [3] to the case of two mappings S and T under generalized contractive conditions in a total paranormed space.

In the sequel, we shall assume that X is a paranormed space whose topology is generated by a total paranorm q which has the following properties (see [6], p. 52):

(a) $q(x) \geq 0$, and $q(x) = 0$ iff $x = 0$,

(b) $q(-x) = q(x)$,

(c) $q(x+y) \leq q(x) + q(y)$,

(d) if $\{a_n\}$ is a sequence of real or complex scalars with $a_n \to a$ and $\{x_n\}$ is a sequence in X with $x_n \to x$, then $q(a_n x_n - ax) \to 0$.

Note that a total paranormed space has a paranorm instead of a norm. The paranorm satisfies all the properties of a norm except the homogeneity property, i.e., $q(\lambda x) \neq \lambda q(x)$ for scalar λ and $x \in X$.

2. Main theorems

In this section we prove several common fixed point theorems in a total paranormed space X. We begin with the following theorem.

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THEOREM 2.1. *Let $\{f_n\}$ and $\{g_n\}$ be sequences in a complete paranormed space X. Let u_n and v_n be solutions of the equations $u - Su = f_n$ and $v - Tv = g_n$, respectively, where S and T are mappings on a closed subset C of X into itself satisfying the condition*

$$\text{(I)}\quad \begin{aligned} &q(Sx - Ty) + a_1[q(x - Sx) + q(x - Ty)] + a_2[q(y - Ty) + q(y - Sx)] \\ &\le k \max\{q(x - y),\ q(x - Sx),\ q(y - Ty),\ q(x - Ty),\ q(y - Sx)\} \end{aligned}$$

for all $x, y \in C$ and $0 < k - (a_1 + a_2) < 1$. If $q(f_n) \to 0$ and $q(g_n) \to 0$ as $n \to \infty$, then $\{u_n\}$ and $\{v_n\}$ converge to a unique solution of the equation $x = Sx = Tx$.

Proof. First, we show that at least one of the sequences $\{u_n\}$, $\{v_n\}$ is convergent. For $n, m \ge 1$ by using (I), we have

$$\begin{aligned} &q(u_n - u_m) \le q(u_n - Su_n) + q(Su_n - Tu_m) + q(Tu_m - u_m) \le q(f_n) + q(g_m) \\ &+ k \max\{q(u_n - u_m), q(u_n - Su_n), q(u_m - Tu_m), q(u_n - Tu_m), q(u_m - Su_n)\} \\ &- a_1[q(u_n - Su_n) + q(u_n - Tu_m)] - a_2[q(u_m - Tu_m) + q(u_m - Su_n)] \\ &\le q(f_n) + q(g_m) + k \max\{q(u_n - u_m), q(f_n), q(g_m), q(u_n - u_m) \\ &+ q(g_m), q(u_n - u_m) + q(f_n)\} - a_1[q(f_n) + q(u_n - u_m) + q(g_m)] \\ &- a_2[q(g_m) + q(u_n - u_m) + q(f_n)]. \end{aligned}$$

Or equivalently,

$$q(u_n - u_m) \le \frac{1 + k - a_1 - a_2}{1 - k + a_1 + a_2}[q(f_n) + q(g_m)]$$

which implies $\lim_{n,m\to\infty} q(u_n - u_m) = 0$. Therefore $\{u_n\}$ is a Cauchy sequence in C. But C is a closed subset of X. Then there exists some u in C such that $\lim_{n\to\infty} u_n = u$.

Next, we show that the sequences $\{u_n\}, \{v_n\}$ converge to the same limit u. Using (I), we have

$$\begin{aligned} &q(u_n - v_n) \le q(u_n - Su_n) + q(Su_n - Tv_n) + q(Tv_n - v_n) \\ &\le q(f_n) + q(g_n) \\ &\quad + k \max\{q(u_n - v_n), q(f_n), q(g_n), q(u_n - v_n) + q(g_n), q(u_n - v_n) + q(f_n)\} \\ &\quad - a_1[q(f_n) + q(u_n - v_n) + q(g_n)] - a_2[q(g_n) + q(u_n - v_n) + q(f_n)] \end{aligned}$$

or

$$\text{(1)}\qquad q(u_n - v_n) \le \frac{1 + k - (a_1 + a_2)}{1 - k + a_1 + a_2}[q(f_n) + q(g_n)],$$

which implies $\lim_{n\to\infty} q(u_n - v_n) = 0$, whence $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = u$.

Now, using (I) again, one gets

$$\begin{aligned}(2)\quad q(u-Su) &\le q(u-v_n)+q(v_n-Tv_n)+q(Su-Tv_n) \le q(u-v_n)+q(g_n)\\ &+ k\max\{q(u-v_n), q(u-Su), q(g_n), q(u-v_n)+q(g_n), q(v_n-Su)\}\\ &- a_1[q(u-Su)+q(u-v_n)+q(g_n)] - a_2[q(g_n)+q(v_n-Su)].\end{aligned}$$

Letting $n \to \infty$ in (2), we obtain $q(u-Su) \le [k-(a_1+a_2)]q(u-Su)$, since $0 < k-(a_1+a_2) < 1$, and thus $u = Su$. Similarly $u = Tu$. Hence $u = Su = Tu$. The uniqueness of u follows easily from (I).

If we put $S = T$ and $a_1 = a_2 = 0$ in Theorem 2.1, we obtain the following corollary.

COROLLARY 2.1 ([3], Theorem 1). *Suppose that X is a complete paranormed space and that C is a closed convex subset of X. Let $T : C \to C$ be a mapping satisfying*

$$q(Tx-Ty) \le k\max\{q(x-y), q(x-Tx), q(y-Ty), q(x-Ty), q(y-Tx)\},$$

for all $x, y \in C$, where $0 \le k < 1$. For each $n \ge 1$, let a_n be a solution of equation $Tx - x = A_n$, where $A_n \in X$. If $\lim_{n\to\infty} A_n = 0$, then $\{a_n\}$ converges and its limit point is a unique solution of the equation $Tx = x$.

For any point $x_0 \in X$ and $0 < t < 1$, we shall consider the sequence $\{x_n\}$ associated with T as follows

$$(3)\qquad x_{n+1} = (1-t)x_n + tTx_n, \qquad n \ge 0.$$

THEOREM 2.2 *Let S and T be mappings from a closed subset C of X into itself such that*

(II) $ST = TS$,

$$\begin{aligned}\text{(III)}\quad q(Sx-Ty) &\le r\max\{cq(x-y), q(x-Sx), q(y-Ty), q(x-Ty), q(y-Sx)\}\\ &+ s\max\{q(x-TSx), q(Sx-TSx), q(y-TSx), q(Ty-TSx)\},\end{aligned}$$

for all $x, y \in C$ and $c, r, s \ge 0$ with $r + s < 1$.

If for some x_0 in C the sequence $\{x_n\}$ as in (3) associated with either S or T is convergent, then its limit is a common fixed point of S and T. Moreover, if $rc + s < 1$, then the common fixed point is unique.

Proof. Suppose first that $Tu = u$ for a point u in C. Then, putting $x = y = u$ in (III) and using (II), we easily see that $Su = u$. Similarly $Su = u$ implies $Tu = u$. Now let $\{x_n\}$ be a sequence (3) associated with S and such that $\lim_{n\to\infty} x_n = u$. From (3) we have $x_{n+1} - x_n = t(Sx_n - x_n)$, then $q(Sx_n - x_n) = q(\frac{1}{t}(x_{n+1} - x_n))$. It follows that $\lim_{n\to\infty} Sx_n = u$.

We now show that $\lim_{n\to\infty} TSx_n = u$. Taking $x = x_n$ and $y = Sx_n$ in (III), we have

$$q(Sx_n - TSx_n) \leq r\max\{cq(x_n - Sx_n), q(x_n - Sx_n), q(Sx_n - TSx_n), q(x_n - TSx_n), q(Sx_n - Sx_n)\}$$
$$+s\max\{q(x_n - TSx_n), q(Sx_n - TSx_n), q(Sx_n - TSx_n), q(TSx_n - TSx_n)\},$$

Letting $n \to \infty$, we have

$$q(u - \lim_{n\to\infty} TSx_n) \leq (r+s)q(u - \lim_{n\to\infty} TSx_n).$$

Since $r + s < 1$, then $\lim_{n\to\infty} TSx_n = u$. Using (III) again, one gets

$$(4) \quad q(Sx_n - Tu)$$
$$\leq r\max\{cq(x_n - u), q(x_n - Sx_n), q(u - Tu), q(x_n - Tu), q(u - Sx_n)\}$$
$$+ s\max\{q(x_n - TSx_n), q(Sx_n - TSx_n), q(u - TSx_n), q(Tu - TSx_n)\}.$$

Taking in (4) the limit as $n \to \infty$, we have $q(u - Tu) \leq (r+s)q(u - Tu)$, with $r + s < 0$, whence $Tu = u$, i.e., u is a fixed point of T. In view of our remark at the beginning of the proof, u is a fixed point of S as well. Hence u is a common fixed point of S and T.

If possible let v ($v \neq u$) be another fixed point of S and T. Then using (III), we have $q(u - v) = q(Su - Tv) \leq (rc + s)q(u - v)$. Since $rc + s < 1$, then $u = v$ follows.

If we put $S = T$ in Theorem 2.2, we obtain the following corollary.

COROLLARY 2.2 ([3], Theorem 2). *Let T be a mapping from a closed convex subset C of X into itself satisfying*

$$q(Tx - Ty) \leq r\max\{cq(x - y), q(x - Tx), q(y - Ty), q(x - Ty), q(y - Tx)\}$$
$$+ s\max\{q(x - T^2x), q(Tx - T^2x), q(y - T^2y), q(Ty - T^2y)\}$$

for all $x, y \in C$ and $c, r, s \geq 0$ with $r + s < 1$. If for some $x_0 \in C$ and $0 < t < 1$ the sequence $\{x_n\}$ as in (3) converges to a point u in C, then u is a fixed point of T.

Now, we state the following theorem which can be proved similarly as Theorem 2.2.

THEOREM 2.3. *Let $S, T : C \to C$ be mappings satisfying at least one of the following conditions*:

(IV) $q(Sx-Ty) \leq a\max\{cq(x-y), \frac{1}{2}[q(x-Sx)+q(y-Ty)]\} + b[q(x-Ty) + q(y - Sx)]$,

(V) $q(Sx-Ty) \leq a\max\{cq(x-y), \frac{1}{2}[q(x-Ty)+q(y-Sx)]\} + b[q(x-Sx) + q(y - Ty)]$

for all x, y in C, where $a, b, c \geq 0$ with $a + 2b < 2$. If for some $x_0 \in C$ and $0 < t < 1$, the sequence $\{x_n\}$ defined as above, associated with either S or T, converges to u in C, then $u = Su = Tu$.

If we put $S = T$ in Theorem 2.3, we obtain the following result.

COROLLARY 2.3 ([3], Theorem 3). *Let $T : C \to C$ be a mapping satisfying at least one of the following conditions:*

(IV′) $q(Tx - Ty) \leq a \max\{cq(x - y), \frac{1}{2}[q(x - Tx) + q(y - Ty)]\} + b[q(x - Ty) + q(y - Tx)]$,

(V′) $q(Tx - Ty) \leq a \max\{cq(x - y), \frac{1}{2}[q(x - Ty) + q(y - Tx)]\} + b[q(x - Tx) + q(y - Ty)]$

for all x, y in C, where $a, b, c \geq 0$ with $a + 2b < 2$. If for some $x_0 \in C$ and $0 < t < 1$ the sequence $\{x_n\}$ converges to u in C, then $u = Tu$.

Finally, we give an example of a paranormed space and two mappings which satisfy the contractive conditions of Theorem 2.1 and Theorem 2.2.

EXAMPLE. Let $X = R$, R being the set of real numbers, and q be the total paranorm defined by $q(x) = \frac{|x|}{1+|x|}$ for all $x \in R$. Let $C = [0, 1]$ and define $S, T : C \to C$ by

$$Sx = \begin{cases} 0, & 0 \leq x < 1, \\ \frac{1}{8}, & x = 1, \end{cases} \qquad Tx = \begin{cases} 0, & 0 \leq x < 1, \\ \frac{1}{2}, & x = 1. \end{cases}$$

S and T satisfy the condition (II). Moreover, for $r = \frac{5}{7}$, $s = \frac{1}{4}$ and arbitrary $c \geq 0$, (III) is satisfied as follows.

(i) If $x = y = 1$, then $q(Sx - Ty) = q(\frac{3}{8}) = \frac{3}{11}$,

$$rq(y - Sx) + sq(y - TSx) = \frac{5}{7} q\left(\frac{7}{8}\right) + \frac{1}{4} q(1) = \frac{5}{7} \cdot \frac{7}{15} + \frac{1}{4} \cdot \frac{1}{2} = \frac{11}{24} > \frac{3}{11}.$$

(ii) If $0 \leq x, y < 1$, then (III) is trivially satisfied.

(iii) If $x = 1$, $0 \leq y < 1$, then

$$q(Sx - Ty) = q\left(\frac{1}{8}\right) = \frac{1}{9},$$
$$rq(x - Ty) = \frac{5}{7} q(1) = \frac{5}{7} \cdot \frac{1}{2} = \frac{5}{14} > \frac{1}{9},$$
$$sq(x - TSx) = \frac{1}{4} q(1) = \frac{1}{8} > \frac{1}{9}.$$

(iv) If $0 \leq x < 1$, $y = 1$, then (III) holds (similarly as in (iii)).

Now, taking $k = \frac{5}{7}$, $a_1 = a_2 = 0$, we easily check that (I) holds in cases (i)–(iv). Note that 0 is the unique common fixed point of S and T.

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