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TWO FOURIER–LEGENDRE EXPANSIONS FOR FOX'S H-FUNCTION OF SEVERAL VARIABLES

1. Introduction

Several mathematicians, during the last three decades, have claimed to present various Fourier series and expansions for the G and H-functions of two or more variables. A serious study reveals that almost all of their results may be viewed as the manipulative forms of already known work on Meijer's G-function and Fox's H-function [5]–[7]. It is important to note that the Fourier series and expansions presented there involve only one variable and are expressed in terms of a single series. Therefore, these must be viewed as Fourier series and expansions for a function of one variable. Any Fourier series and expansion for a function of several variables should indeed involve several variables and be presented in terms of a multiple series, as discussed by Carslaw and Jaeger [2] (pp. 180–183) in the case of Fourier series of two variables.

The object of this paper is to establish two Fourier–Legendre expansions for Fox's H-function of several variables with the help of a multiple integral evaluated in this paper.

The following functions

$$1) \qquad f(x_1,\dots,x_r) = (1-x_1^2)^{s_1-1}\dots(1-x_r^2)^{s_r-1} H \begin{bmatrix} z_1(1-x_1^2)^{t_1} \\ \vdots \\ z_r(1-x_r^2)^{t_r} \end{bmatrix}$$

and

$$2) \qquad g(x_1,\dots,x_r) = (1-x_1^2)\dots(1-x_r^2) f(x_1,\dots,x_r),$$

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where H is H-function of r variables given by (1.1) and $-1 < x_i < 1$, $i = 1,\dots,r$, are expanded with respect to the following orthogonal systems

1′) $P_{u_1}^{w_1}(x_1)\dots P_{u_r}^{w_r}(x_r)$, where w_i are fixed non-negative integers and $u_i = w_i, w_{i+1},\dots,\ i = 1,2,\dots,r$,

and

2′) $P_{v_1}^{u_1}(x_1)\dots P_{v_r}^{u_r}(x_r)$, where v_i are fixed positive integers and $u_i = 1,\dots,v_i,\ i = 1,2,\dots,r$,

in the Hilbert spaces

1″) $L^2((-1,1)^r, dx_1,\dots,dx_r)$,

and

2″) $L^2((-1,1)^r,(1-x_1^2)^{-1}dx_1,\dots,(1-x_r^2)^{-1}dx_r)$,

respectively ($P_n^m(x)$ denotes the Legendre function).

The reader is referred to generalizations [4] and definition of Fox's H-function of two and several variables [6] (pp. 22–35), [7] (pp. 82–98, 251–254).

In this paper Fox's H-function of several variables [7] (pp. 251–254) will be represented as follows

$$(1.1)\quad H\begin{bmatrix} z_1\\ \vdots\\ z_r\end{bmatrix} \equiv H^{0,n;m_1,n_1;\dots;m_r,n_r}_{p,q;p_1,q_1;\dots;p_r,q_r}\left[\begin{matrix} z_1\\ \vdots\\ z_r\end{matrix}\left|\begin{matrix} A; C_{p_1};\dots;C_{p_r}\\ B; D_{q_1};\dots;D_{q_r}\end{matrix}\right.\right]$$

$$= H^{0,n;m_1,n_1;\dots;m_r,n_r}_{p,q;p_1,q_1;\dots;p_rq_r}\left[\begin{matrix} z_1\\ \vdots\\ z_r\end{matrix}\left|\begin{matrix}(a_j;\alpha_j',\dots,\alpha_j^{(r)})_{1,p};(c_j';\gamma_j')_{1,p_1};\dots;(c_j^{(r)};\gamma_j^{(r)})_{1,p_r}\\ (b_j;\beta_j',\dots,\beta_j^{(r)})_{1,q};(d_j';\delta_j')_{1,q_1};\dots;(d_j^{(r)};\delta_j^{(r)})_{1,q_r}\end{matrix}\right.\right].$$

The following formulae are required in the proofs: the integral [3] (p. 316, (16))

$$(1.2)\quad \int_{-1}^{1}(1-x^2)^{s-1}P_n^m(x)\,dx$$

$$= \frac{\pi 2^m \Gamma(s+\frac{m}{2})\Gamma(s-\frac{m}{2})}{\Gamma(s+\frac{n}{2}+\frac{1}{2})\Gamma(s-\frac{n}{2})\Gamma(-\frac{m}{2}+\frac{n}{2}+1)\Gamma(-\frac{m}{2}-\frac{n}{2}+\frac{1}{2})}\ \text{for } 2\operatorname{Re}s > |\operatorname{Re}m|$$

and the orthogonality properties of the Legendre functions [3] (p. 279)

$$(1.3)\quad \int_{-1}^{1} P_n^m(x)P_k^m(x)\,dx = \begin{cases} 0 & \text{for } k \neq n, \\ \frac{2(n+m)!}{(2n+1)(n-m)!} & \text{for } k = n,\ m \leq n, \end{cases}$$

$$(1.4)\quad \int_{-1}^{1} (1-x^2)^{-1} P_n^m(x)P_n^k(x)\,dx = \begin{cases} 0 & \text{for } k \neq m, \\ \dfrac{(n+m)!}{m(n-m)!} & \text{for } k = m,\ m \leq n. \end{cases}$$

2. The multiple integral

The multiple integral to be evalued is

$$(2.1)\quad \int_{-1}^{1} \cdots \int_{-1}^{1} \prod_{k=1}^{r} (1-x_k^2)^{s_k-1} P_{u_k}^{w_k}(x_k) H \begin{bmatrix} z_1(1-x_1^2)^{t_1} \\ \vdots \\ z_r(1-x_r^2)^{t_r} \end{bmatrix} dx_1 \ldots dx_r$$

$$= \frac{\pi^r 2^{w_1+\ldots+w_r}}{\prod_{k=1}^{r} \Gamma(\frac{1}{2}(2-w_k+u_k))\Gamma(\frac{1}{2}(2-w_k-u_k))}$$

$$\times H^{\,0,n;m_1,n_1+2;\ldots;m_r n_r+2}_{\,p,q;p_1+2,q_1+2;\ldots;p_r+2,q_r+2} \left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \middle| \begin{matrix} A;(1-s_1-\frac{w_1}{2},t_1), \\ B;D_{q_1},(\frac{1}{2}-s_1-\frac{u_1}{2},t_1), \end{matrix} \right.$$

$$\left. \begin{matrix} (1-s_1+\frac{w_1}{2},t_1),C_{p_1};\ldots;(1-s_r-\frac{w_r}{2},t_r),(1-s_r+\frac{w_r}{2}t_r),C_{p_r} \\ (1-s_1+\frac{u_1}{2}.t_1);\ldots;D_{q_r},(\frac{1}{2}-s_r-\frac{u_r}{2},t_r),(1-s_r+\frac{u_r}{2},t_r) \end{matrix} \right]$$

for $2\operatorname{Re} s_i+2t_i \min_{1\leq j\leq m_i} \left[\operatorname{Re} \frac{d_j}{\delta_j}\right] > \operatorname{Re}|w_i|$, $i=1,\ldots,r$, and the conditions given by (C.4), (C.5), (C.6) in [7] (pp. 252–253).

To establish (2.1), express the H-function in the integrand as (C.1) in [7] (p. 251), change the orders of x-integrals and ζ-integrals, evaluate inner integrals with the help of (1.2) and use (C.1) in [7].

Note 1. The integral (2.1) may be viewed as the several variables analogue of the integral (2.1) in [1] (p. 90) and (2.9.2) in [6] (p. 40).

In the sequel u_i, w_i, v_i, $i=1,\ldots,r$, are non-negative integers.

3. The Fourier–Legendre expansions

The Fourier–Legendre expansions to be established are

$$(3.1)\quad \prod_{i=1}^{r} (1-x_i^2)^{s_i-1} H \begin{bmatrix} z_1(1-x_1^2)^{t_1} \\ \vdots \\ z_r(1-x_r^2)^{t_r} \end{bmatrix}$$

$$= \sum_{u_1=w_1}^{\infty} \cdots \sum_{u_r=w_r}^{\infty} \prod_{i=1}^{r} \frac{\pi^r 2^{w_1+\ldots+w_r-r}(2u_i-1)(u_i-w_i)!P_{u_i}^{w_i}(x_i)}{(u_i+w_i)!\Gamma(\frac{1}{2}(2-w_i+u_i))\Gamma(\frac{1}{2}(1-w_I-u_i))}$$

$$\times H^{0,n;m_1,n_1+2;\ldots;m_r n_r+2}_{p,q;p_1+2,q_1+2;\ldots;p_r+2,q_r+2}\left[\begin{matrix} A;(1-s_1-\frac{w_1}{2},t_1), \\ B;D_{q_1},(\frac{1}{2}-s_1-\frac{u_1}{2},t_1), \end{matrix}\right.$$

$$\left.\begin{matrix}(1-s_1+\frac{w_1}{2},t_1),C_{p_1};\ldots;(1-s_r-\frac{w_r}{2},t_r),(1-s_r+\frac{w_r}{2}t_r),C_{p_r} \\ (1-s_1+\frac{u_1}{2}.t_1);\ldots;D_{q_r},(\frac{1}{2}-s_r-\frac{u_r}{2},t_r),(1-s_r+\frac{u_r}{2},t_r)\end{matrix}\right]$$

valid under the conditions (1.3), (2.1), and

$$(3.2)\quad \prod_{i=1}^{r}(1-x_i^2)^{s_i-1} H\begin{bmatrix} z_1(1-x_1^2)^{t_1} \\ \vdots \\ z_r(1-x_r^2)^{t_r}\end{bmatrix}$$

$$\sim \sum_{u_1=1}^{v_1} \cdots \sum_{u_r=1}^{v_r} \prod_{i=1}^{r} \frac{\pi^r 2^{u_1+\ldots u_r} u_i(v_i-u_i)!P_{v_i}^{u_i}(x_i)}{(v_i+u_i)!\Gamma(\frac{1}{2}(2-u_i+v_i))\Gamma(\frac{1}{2}(1-u_i-v_i))}$$

$$\times H^{0,n;m_1,n_1+2;\ldots;m_r n_r+2}_{p,q;p_1+2,q_1+2;\ldots;p_r+2,q_r+2}\left[\begin{matrix} z_1 \\ \vdots \\ z_r\end{matrix}\middle|\begin{matrix} A;(1-s_1-\frac{u_1}{2},t_1), \\ B;D_{q_1},(\frac{1}{2}-s_1-\frac{v_1}{2},t_1), \end{matrix}\right.$$

$$\left.\begin{matrix}(1-s_1+\frac{u_1}{2},t_1),C_{p_1};\ldots;(1-s_r-\frac{u_r}{2},t_r),(1-s_r+\frac{u_r}{2}t_r),C_{p_r} \\ (1-s_1+\frac{v_1}{2}.t_1);\ldots;D_{q_r},(\frac{1}{2}-s_r-\frac{v_r}{2},t_r),(1-s_r+\frac{v_r}{2},t_r)\end{matrix}\right]$$

valid under the conditions (1.4), (2.1).

To prove (3.1), let

$$(3.3)\quad \prod_{i=1}^{r}(1-x_i^2)^{s_i-1} H\begin{bmatrix} z_1(1-x_1^2)^{t_1} \\ \vdots \\ z_r(1-x_r^2)^{t_r}\end{bmatrix} = \sum_{u_1=0}^{w_1}\cdots\sum_{u_r=0}^{w_r} C_{u_1,\ldots,u_r}\prod_{i=1}^{r}P_{u_i}^{w_i}(x_i).$$

Equation (3.3) is valid, since (cf. [8])

(a) $(P_{u_i}^{w_i}(x_i))_{u_i=w_i}^{\infty}$ is a complete orthogonal system of $L^2[[-1,1],dx_i]$,

$$i=1,\ldots,r, \text{ and hence } \left(\prod_{i=1}^{r}P_{u_i}^{w_i}(x_i)\right)_{(u_1,\ldots,u_r)=(w_1,\ldots,w_r)}^{(\infty,\ldots,\infty)}$$

of $L^2([-1,1]^r, dx_1\ldots,dx_r)$,

(b) the left-hand side of (3.3) is continuous and bounded in $-1 < x_i < 1$, $= 1, \ldots, r$.

Multiplying both sides of (3.3) by $\prod_{i=1}^{r} P_{u_i}^{w_i}(x_i)$, integrating with respect to $x_1, \ldots, x_r$ from -1 to 1 and using (2.1), (1.3), we obtain the value of $C_{u_1,\ldots,u_r}$. Substituting it in (3.3), the Fourier–Legendre expansion (3.1) s obtained.

To establish (3.2), let

$$(3.4)\quad \prod_{i=1}^{r}(1-x_i^2)^{s_i} H \begin{bmatrix} z_1(1-x_1^2)^{t_1} \\ \vdots \\ z_r(1-x_r^2)^{t_r} \end{bmatrix} \sim \sum_{u_1=1}^{v_1} \cdots \sum_{u_r=1}^{v_r} A_{u_1,\ldots,u_r} \prod_{i=1}^{r} P_{v_i}^{u_i}(x_i).$$

Multiplying both sides of (3.4) by $\prod_{i=1}^{r}(1-x_i^2)^{-1} P_{v_i}^{w_i}(x_i)$, integrating with respect to $x_1, \ldots, x_r$ from -1 to 1 and using (2.1), (1.4), we obtain the value of $A_{u_1,\ldots,u_r}$. Substituting it in (3.4), the Fourier–Legendre expansion (3.2) is obtained.

Note 2. The Fourier–Legendre expansion (3.1) may be viewed as the several variables analogue of (3.1) in [1] (p. 91) and (3.6.1) in [2] (p. 70).

On specializing the parameters, Fox's H-function of several variables yields almost all special functions appearing in applied mathematics and physical sciences. Therefore, the results presented in this paper are of a general character and hence may encompass several cases of interest.

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