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HEREDITARY SUBALGEBRAS OF BCK-ALGEBRAS

1. Introduction

The leading motivation for BCK-algebras comes from algebraic logic: a BCK-algebra $\mathbf{A} = \langle A; \cdot, 0\rangle$ is the algebraic counterpart of a purely implicational calculus in which 0 corresponds to truth, $x \cdot y$ is interpreted as $y \to x$ and $x \cdot y = 0$ as $\vdash y \to x$. With $\mathbf{A}$ we associate, as usual, a partially ordered set $\langle A; \leq\rangle$, defining $x \leq y$ iff $x \cdot y = 0$. Hereditary (with respect to $\leq$) subalgebras $\mathbf{B} = \langle B; \cdot, 0\rangle$ of $\mathbf{A}$ correspond to subcalculi of A with an "implicational completeness" property, viz., A contains no consequences of propositions in B that are not themselves in B.

Given a BCK-algebra $\mathbf{A}$ and a variety K of BCK-algebras, there may be various hereditary subalgebras of $\mathbf{A}$ which are members of K. The union $i_K(\mathbf{A})$ of these is also a hereditary subalgebra of $\mathbf{A}$. If it too is a member of K we say that K has the *hereditary subalgebra property for* $\mathbf{A}$. We shall prove, inter alia, that if K is a quasivariety of BCK-algebras then the class of all BCK-algebras for which K has the hereditary subalgebra property is also a quasivariety. We shall also prove that many of the best known varieties of BCK-algebras (but not all varieties of BCK-algebras) have the hereditary subalgebra property for all BCK-algebras $\mathbf{A}$.

2. Conventions

We denote by ω the set of all nonnegative integers. We denote algebras by boldface capitals $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$ and their universes by $A, B, C, \ldots$ By a *quasivariety*, we mean a class of algebras of the same type that is axiomatized by a set of quasi-identities. We assume a familiarity with the theory of BCK-algebras (see survey articles [Cor82] and [IT78]). Henceforth if $\mathbf{A} = \langle A; \cdot, 0\rangle$ is a BCK-algebra, we shall abbreviate the binary operation $\cdot$ on $\mathbf{A}$ by juxtaposition and for $n \in \omega$ and any $a, b, b_1, \ldots, b_n \in A$, we define inductively:

(1) $$ab_1 \dots b_n = (ab_1 \dots b_{n-1})b_n \quad (n > 1),$$
$$ab^0 = a; \quad ab^n = ab^{n-1}b \quad (n > 0).$$

In view of the identity

(2) $$xyz = xzy,$$

which holds in every BCK-algebra [IT78, Theorem1], the order of $b_1, \dots, b_n$ is immaterial in (1). As usual, we associate with $\mathbf{A}$ a poset $\langle A; \leq \rangle$ where for $a, b \in A$, we define $a \leq b$ iff $ab = 0$. This poset has a least element, viz. 0. Recall that every BCK-algebra satisfies the identities

(3) $$xy \leq x,$$

(4) $$x(xy) \leq y,$$

(5) $$(xy)(zy) \leq xz$$

and the quasi-identities

(6) $$x \leq y \Rightarrow xz \leq yz,$$

(7) $$x \leq y \Rightarrow zy \leq zx.$$

For an element a of our BCK-algebra $\mathbf{A}$, we denote by $(a]$ the principal order ideal of the ordered set $\langle A; \leq \rangle$ generated by a, i.e., $(a] = \{x \in A : x \leq a\}$. A set $X \subseteq A$ is said to be a *hereditary subset* of $\mathbf{A}$ if $X \neq \emptyset$ and $(a] \subseteq X$ for all $a \in X$. The set of all hereditary subsets of $\mathbf{A}$ is denoted by $\mathscr{H}(\mathbf{A})$. Clearly $\{0\}$ is the least and A the greatest element of the poset $\langle \mathscr{H}(\mathbf{A}); \subseteq \rangle$. If $\mathscr{S}$ is a nonempty subset of $\mathscr{H}(\mathbf{A})$ then both $\bigcap \mathscr{S}$ and $\bigcup \mathscr{S}$ are elements of $\mathscr{H}(\mathbf{A})$. In particular, $\langle \mathscr{H}(\mathbf{A}); \subseteq \rangle$ is a complete, completely distributive lattice which fails to be a complete sublattice of the power set $\langle \exp A; \subseteq \rangle$ of A only because the supremum of the empty subset of $\exp A$ is $\emptyset$, while the supremum of the empty subset of $\mathscr{H}(\mathbf{A})$ is $\{0\}$. Note also that every hereditary subset of $\mathbf{A}$ is (the universe of) a subalgebra of $\mathbf{A}$. We shall systematically confuse an element B of $\mathscr{H}(\mathbf{A})$ with the subalgebra $\mathbf{B}$ of $\mathbf{A}$ whose universe is B.

3. K-hereditary subsets

For a class K of BCK-algebras, we let $\mathscr{H}_K(\mathbf{A})$ denote the set of all elements B of $\mathscr{H}(\mathbf{A})$ which are the universes of elements of K.

Recall that a sentence φ of a first order language $\mathscr{L}$ is called a *universal existential sentence* (briefly, a $\forall\exists$ *sentence*) if its prenex normal form is $\forall x_1 \dots \forall x_n \exists y_1 \dots \exists y_m \psi$, where ψ is an open formula of $\mathscr{L}$ (i.e., a first order formula of $\mathscr{L}$ not containing any occurrence of a quantifier) and $n, m \in \omega$. (The cases $n = 0$ and $m = 0$ are possible and should be interpreted as the absence of universal and existential quantifiers, respectively, in φ). By a *universal existential class* (of $\mathscr{L}$-structures), we mean a class K of $\mathscr{L}$-structures which is axiomatized by a set of $\forall\exists$ sentences. If $\mathscr{C}$ is a subset of a class K of $\mathscr{L}$-structures and for any $\mathbf{A}, \mathbf{B} \in \mathscr{C}$, either $\mathbf{A}$ is a substructure of $\mathbf{B}$ or $\mathbf{B}$ is a substructure of $\mathbf{A}$, we call $\mathscr{C}$ a *chain of structures* in K. In this case, we may define an $\mathscr{L}$-structure $\mathbf{D} = \bigcup \mathscr{C}$ whose universe is $\bigcup \mathscr{C}$, where $f^{\mathbf{D}}(a_1, \ldots, a_n) = f^{\mathbf{A}}(a_1, \ldots, a_n)$ for any $\mathbf{A} \in \mathscr{C}$, any $a_1, \ldots, a_n \in A$ and any n-ary operation symbol f of $\mathscr{L}$, and where, for any n-ary relation symbol r of $\mathscr{L}$, $r^{\mathbf{D}}(a_1, \ldots, a_n)$ is true if and only if $r^{\mathbf{A}}(a_1, \ldots, a_n)$ is true in some $\mathbf{A} \in \mathscr{C}$ with $a_1, \ldots, a_n \in A$. If $\bigcup \mathscr{C} \in K$ for all chains $\mathscr{C}$ of structures in K, we say that K is closed under the formation of set theoretic unions of chains. It is well known that a class K is closed under the formation of set theoretic unions of chains if and only if K is a universal existential class (e.g., [BS81, Exercise V.1.24, p. 203]).

PROPOSITION 1. *Let K be a nonempty class of BKC-algebras and $\mathbf{A}$ a BCK-algebra.*

(i) *If K is closed under the formation of subalgebras and isomorphic images then $\mathscr{H}_K(\mathbf{A}) \neq \emptyset$ (in particular, $\{0\} \in \mathscr{H}_K(\mathbf{A})$) and $\mathscr{H}(\mathbf{B}) \subseteq \mathscr{H}_K(\mathbf{A})$ for each $\mathbf{B} \in \mathscr{H}_K(\mathbf{A})$.*

(ii) *If K is a universal existential class and $\mathscr{S}$ is a nonempty subchain of the poset $\langle \mathscr{H}_K(\mathbf{A}); \subseteq \rangle$ then $\bigcup \mathscr{S} \in \mathscr{H}_K(\mathbf{A})$. In particular, for every $X \in X_k(\mathbf{A})$, there is a maximal element Y of $\langle \mathscr{H}_K(\mathbf{A}); \subseteq \rangle$ such that $X \subseteq Y$.*

Proof. (i) This follows from the transitivity of subalgebras and hereditary subsets.

(ii) It follows easily from the definitions that $\bigcup \mathscr{S} \in \mathscr{H}(\mathbf{A})$ and since K is a universal existential class, $\bigcup \mathscr{S} \in K$. Consequently $\bigcup \mathscr{S} \in \mathscr{H}_K(\mathbf{A})$. The rest follows from Zorn's Lemma. ∎

We define $i_K(\mathbf{A}) = \bigcup \mathscr{H}_K(\mathbf{A})$. Note that $i_K(\mathbf{A}) \in \mathscr{H}(\mathbf{A})$. If $i_K(\mathbf{A}) \in K$, we say that K has the *hereditary subalgebra property for* $\mathbf{A}$. In this case $i_K(\mathbf{A})$ is, of course, the largest hereditary subalgebra of $\mathbf{A}$ in K. We also define $K^h = \{\mathbf{A} : \mathbf{A}$ is a BCK-algebra for which K has the hereditary subalgebra property$\}$.

PROPOSITION 2. *Let K be a class of BCK-algebras which is closed under the formation of subalgebras and isomorphic images and let $\mathbf{A} \in K^h$. If $\mathscr{S}$ is a nonempty subset of $\mathscr{H}_K(\mathbf{A})$ then the union (resp. intersection) of $\mathscr{S}$ is its supremum (resp. infimum) in $\mathscr{H}_K(\mathbf{A})$, while $\{0\}$ is the supremum of the empty subset of $\mathscr{H}_K(\mathbf{A})$. In particular, $i_K(\mathbf{A})$ is the greatest element of the complete, completely distributive lattice $\langle \mathscr{H}_K(\mathbf{A}); \subseteq \rangle$.*

Proof. For the first assertion it suffices to note that $\bigcup \mathscr{S}$ is a hereditary subuniverse of $i_K(\mathbf{A})$. The rest is trivial. ∎

COROLLARY 3. *Let K be a class of BCK-algebras which is closed under the formation of subalgebras and isomorphic images and let $\mathbf{A} \in K^h$. Then the following conditions are equivalent:*

(i) $\mathbf{A} \in K$;

(ii) $i_K(\mathbf{A}) = A$;

(iii) *For every $x \in A$, we have $(x] \in \mathscr{H}_K(\mathbf{A})$.* ■

COROLLARY 4. *Let K be a class of BCK-algebras which is closed under the formation of subalgebras and isomorphic images and let $\mathbf{A} \in K^h$. Then $X \in \mathscr{H}_K(\mathbf{A})$ iff X is a hereditary subset of $i_K(\mathbf{A})$.*

Proof. This follows from the fact that hereditary subsets of $\mathbf{A}$ are subuniverses of $\mathbf{A}$ and that $i_K(\mathbf{A})$ is the largest of these.

LEMMA 5. *Let K be a class of BCK-algebras which is closed under the formation of subalgebras and isomorphic images and let $\mathbf{A} \in K^h$. For each $x \in A$, we have $x \in i_K(\mathbf{A})$ iff $(x] \in \mathscr{H}_K(\mathbf{A})$.*

Proof. Easy. ■

LEMMA 6. *Suppose that for each $j \in J$ (J a nonempty set), K_j is a class of BCK-algebras which is closed under the formation of subalgebras and isomorphic images and $\mathbf{A} \in (K_j)^h$. Then $\mathbf{A} \in (\bigcap_{j\in J} K_j)^h$ also.*

Proof. Let $K = \bigcap_{j\in J} K_j$. Then $\mathscr{H}_K(\mathbf{A}) = \bigcap_{j\in J} \mathscr{H}_{K_j}(\mathbf{A}) = \{\bigcap_{j\in J} X_j : X_j \in \mathscr{H}_{K_j}(\mathbf{A})$ for each $j \in J\}$, because each K_j is closed under subalgebras. Thus $i_K(\mathbf{A}) = \bigcap_{j\in J} i_{K_j}(\mathbf{A}) \in K_j$ for every $j \in J$, because each K_j has the hereditary subalgebra property for $\mathbf{A}$. Consequently, $i_K(\mathbf{A}) \in K$, as required. ■

PROPOSITION 7. *Let K be a class of BCK-algebras which is closed under the formation of subalgebras and isomorphic images and let $\mathbf{A} \in K^h$. Then any subalgebra of $\mathbf{A}$ is a member of K^h.*

Proof. Let $\mathbf{B}$ be a subalgebra of $\mathbf{A}$. Let $X = \{a \in A : a \leq b$ for some $b \in i_K(\mathbf{B})\}$. Then X is the universe of a hereditary subalgebra $\mathbf{X}$ of $\mathbf{A}$ (by (3)), and $i_K(\mathbf{B})$ is a subalgebra of $\mathbf{X}$, hence of $i_K(\mathbf{A})$. Since $i_K(\mathbf{A}) \in K$, we have $i_K(\mathbf{B}) \in K$, as required. ■

Let $\langle \mathbf{A}_j : j \in J\rangle$ be a nonempty family of BCK-algebras and let Φ be a proper filter on the Boolean algebra $\mathbf{exp}\, J = \langle \exp J; \cup, \cap, ', \emptyset, J\rangle$ of all subsets of J. As usual, we denote the Φ-reduced product of the family $\langle \mathbf{A}_j : j \in J\rangle$ by $\prod_{j\in J} \mathbf{A}_j/\Phi$. For elements f, g of the direct product $\prod_{j\in J} \mathbf{A}_j$ and $X \subseteq \prod_{j\in J} \mathbf{A}_j$, we write

$$f = h \ (\mathrm{mod}\, \Phi) \text{ iff } \{j \in J : f(j) = g(j)\} \in \Phi,$$

$$[f]_\Phi = \{g \in \prod_{j\in J} \mathbf{A}_j : f = g \ (\mathrm{mod}\,\Phi)\},$$
$$[X]_\Phi = \{[f]_\Phi : f \in X\}.$$

The index Φ will frequently be omitted.

PROPOSITION 8. *Let K be a quasivariety of BCK-algebras which has the hereditary subalgebra property for all members of a family $\langle \mathbf{A}_j : j \in J\rangle$ of BCK-algebras. Let Φ be a proper filter on the Boolean algebra* $\mathbf{exp}\, J$. *Then*

$$i_K\Big(\prod_{j\in J} \mathbf{A}_j/\Phi\Big) = \Big[\prod_{j\in J} i_K(\mathbf{A}_j)\Big]_\Phi.$$

Proof. Since the class of all BCK-algebras is a quasivariety, it is closed under the formation of reduced products, so the application of the operator i_K to $\prod_{j\in J} \mathbf{A}_j/\Phi$ is permissible. Let $\Sigma = \{\varphi_k : k \in I\}$ be a set of quasi-identities $\varphi_k(x_1^k, \ldots, x_{n(k)}^k)$ (with free variables $x_1^k, \ldots, x_{n(k)}^k$) axiomatizing K. We shall write $\mathbf{C} = \prod_{j\in J} \mathbf{A}_j/\Phi$ and $D = \prod_{j\in J} \mathbf{A}_j$.

Let $[f] \in [\prod_{j\in J} i_K(\mathbf{A}_j)]_\Phi$, where $f \in D$. Let $k \in I$ and $a_1^k, \ldots, a_{n(k)}^k \in D$ with $[a_1^k], \ldots, [a_{n(k)}^k] \le [f]$. Then

$$S := \{j \in J : f(j) \in i_K(\mathbf{A}_j) \quad \text{and } a_1^k(j), \ldots, a_{n(k)}^k(j) \le f(j)\} \in \Phi.$$

By Lemma 5, for each $j \in S$, we have $(f(j)] \in \mathscr{H}_K(\mathbf{A})$, so $(f(j)] \models \varphi_k(x_1^k, \ldots, x_{n(k)}^k)$. Thus

$$S' := \{j \in J : \varphi_k^{\mathbf{A}_j}(a_j^k(j), \ldots, a_{n(k)}^k(j)) \text{ is true}\} \in \Phi,$$

because $S \subseteq S'$. Since quasi-identities are (equivalent to) Horn formulas, it follows from [BS81, Theorem V.2.7, p. 207] that $\varphi_k^{\mathbf{C}}([a_1^k], \ldots, [a_{n(k)}^k])$ is true. Thus, $\mathbf{C}$ satisfies $\varphi_k(x_1^k, \ldots, x_{n(k)}^k)$. Since quasi-identities are (equivalent to) universal sentences, they are also preserved in the formation of substructures [BS81, Exercise V.1.13. p. 202], so $([f]] \models \varphi_k(x_1^k, \ldots, x_{n(k)}^k)$. Consequently, $([f]] \in \mathscr{H}_K(\mathbf{C})$. By Lemma 5, $[f] \in i_K(\mathbf{C})$.

Conversely, suppose $[f] \in i_K(\mathbf{C})$. Let $T = \{j \in J : f(j) \notin i_K(\mathbf{A}_j)\}$. By Lemma 5, for each $j \in T$, there is an index $k \in I$ and $a_1^{k,j}, \ldots, a_{n(k)}^{k,j} \in A_j$ such that $a_1^{k,j}, \ldots, a_{n(k)}^{k,j} \le f(j)$ and the sentence $\varphi_k^{\mathbf{A}_j}(a_1^{k,j}, \ldots, a_{n(k)}^{k,j})$ is false. Define $b_1^k, \ldots, b_{n(k)}^k \in D$ by: $b_i^k(j) = a_i^{k,j}$ if $j \in T$; $b_i^k(j) = f(j)$ if $j \in J \setminus T$. Then $[b_i^k] \le [f]$ for $i = 1, \ldots, n(k)$ and for all $k \in I$ and

$$T \subseteq \{j \in J : \varphi_k^{\mathbf{A}_j}(b_1^k(j), \ldots, b_{n(k)}^k(j)) \text{ is false for some } k \in I\}.$$

Our assumptions about f imply that $\varphi_k^{\mathbf{C}}([b_1^k], \ldots, [b_{n(k)}^k])$ is true for all $k \in I$, so

$$J \setminus T \supseteq \{j \in J : \varphi_k^{\mathbf{A}_j}(b_1^k(j), \ldots, b_{n(k)}^k(j)) \text{ is true for all } k \in I\} \in \Phi.$$

Thus $J \setminus T \in \Phi$ and so $f(j) \in i_K(\mathbf{A}_j)$ for all j in a member of Φ, whence $[f] \in [\prod_{j \in J} i_K(\mathbf{A}_j)]_\Phi$. ■

THEOREM 9. *Let K be a quasivariety of BCK-algebras. Then K^h is also a quasivariety.*

Proof. Trivially, any one-element BCK-algebra is in K^h. It therefore suffices, by [BS81, Theorem V.2.25, p. 219], to show that K^h is closed under the formation of isomorphic images, subalgebras and reduced products. The first of these closure assertions is trivially true and the second follows from Proposition 7. If K has the hereditary subalgebra property for all members of the family $\langle \mathbf{A}_j : j \in J \rangle$ and Φ is a filter on $\mathbf{exp}\, J$, then by the Third Isomorphism Theorem, $[\prod_{j \in J} i_K(\mathbf{A}_j)]_\Phi \cong \prod_{j \in J} i_K(\mathbf{A}_j)/\Psi$, where Ψ is the restriction of the congruence on $\Pi_{j \in J} \mathbf{A}_j$ associated with Φ to $\Pi_{j \in J} i_K(\mathbf{A}_j)$. Thus $[\prod_{j \in J} i_K(\mathbf{A}_j)]_\Phi$ is isomorphic to the reduced product $\prod_{j \in J} i_K(\mathbf{A}_j)/\Phi$. From the previous result and the fact that the quasivariety is closed under reduced products, we have $i_K(\prod_{j \in J} \mathbf{A}_j/\Phi) \in K$, i.e., $\prod_{j \in J} \mathbf{A}_j/\Phi \in K^h$, as required. ■

A further corollary of Proposition 8 is:

COROLLARY 10. *Let K be a quasivariety of BCK-algebras which has the hereditary subalgebra property for all members of a family $\langle \mathbf{A}_j : j \in J \rangle$ of BCK-algebras. Then*

$$i_K\Big(\prod_{j \in J} \mathbf{A}_j\Big) = \prod_{j \in J} i_K(\mathbf{A}_j).$$

Proof. The result is trivial if $J = \emptyset$, both sides of the above equation being trivial algebras. If $J \neq \emptyset$, apply Proposition 8 with $\Phi = \{J\}$.

4. Varieties having the Hereditary Subalgebra Property for all BCK-algebras

The quasivariety of all BCK-algebras is not a variety [Wro83]. A number of varieties of BCK-algebras have been studied in the literature. In particular, for each $n \in \omega$, the class of all BCK-algebras satisfying the identity

$$(E_n) \qquad xy^n = xy^{n+1}$$

is a variety, denoted by E_n, having

$$(8) \qquad ((xy)(xz))(zy) = 0,$$

$$(9) \qquad 0x = 0,$$

(10) $$x0 = x$$

and (E_n) as an equational base (see [BR95, Proposition 13]). (The identities (8) and (9) are axioms for BCK-algebras and every BCK-algebra satisfies (10): see [IT78].) Clearly E_0 is the trivial variety, while the members of E_1 are known in the literature as *Hilbert algebras* or *positive implicative BCK-algebras*. It is not difficult to see that every finite BCK-algebra belongs to E_n for some $n \in \omega$.

A BCK-algebra $\mathbf{A}$ is called *commutative* if it satisfies the identity

(T) $$x(xy) = y(yx).$$

In this case the associated partially ordered set $\langle A; \leq \rangle$ is a lower semilattice whose infimum operation is definable by

$$x \wedge y = x(xy).$$

The class T of all commutative BCK-algebras is a variety and was studied by Tanaka [Tan75] – also see [Cor82].

A BCK-algebra is called *implicative* or a *Tarski algebra* if it satisfies the identity

(I) $$x(yx) = x.$$

It is well known that the class I of all implicative BCK-algebras is a variety and is the smallest nontrivial quasivariety of BCK-algebras. It is also well known that $I = E_1 \cap T$ (see [Cor82]).

In [Cor81], Cornish considers the class J of all BCK-algebras satisfying the identity

(J) $$x(x(y(yx))) = y(y(x(xy)))$$

and proves that this class is a variety which strictly contains the supremum of the varieties E_1 and T. We shall prove that all of the aforementioned varieties have the hereditary subalgebra property for all BCK-algebras. (In the case of T, this is already known [Stu84].)

LEMMA 11. *For every $k \in \omega$, every BCK-algebra satisfies the identity*:

(11) $$xy^k = x(x(xy))^k.$$

Proof (by induction on k): The lemma is trivially true for $k = 0$, as we have defined $uv^0 = u$. For $k = 1$, we have $x(x(xy)) \leq xy$ as a substitution instance of (4). But from (4) and (7), we conclude that $xy \leq x(x(xy))$ and so $xy = x(x(xy))$. Suppose all BCK-algebras satisfy (11) for some $k \in \omega$. Then

$$\begin{aligned} xy^{k+1} &= (xy^k)y = (x(x(xy))^k)y \quad \text{(by the induction hypothesis)} \\ &= (xy)(x(xy))^k \quad \text{(by (2))} \end{aligned}$$

$$= x(x(xy))(x(xy))^k \quad \text{(by the case } k = 1\text{)}$$
$$= x(x(xy))^{k+1}. \qquad \blacksquare$$

COROLLARY 12. *If* $\mathbf{A}$ *is any BCK-algebra,* $n \in \omega$, $\mathbf{B} \in \mathscr{H}_{E_n}(\mathbf{A})$, $x \in A$ *and* $b \in B$ *then* $bx^n = bx^{n+1}$.

Proof. Since $\mathbf{B}$ is a hereditary subalgebra of $\mathbf{A}$ and $b(bx) \leq b$ (by (3)), we have $b(bx) \in B$. Since $\mathbf{B} \in E_n$, we have $b(b(bx))^n = b(b(bx))^{n+1}$. Applying (11) to this equation, we get $bx^n = bx^{n+1}$. ∎

THEOREM 13. *For each* $n \in \omega$, *the variety* E_n *has the hereditary subalgebra property for all BCK-algebras.*

Proof. Let $\mathbf{A}$ be a BCK-algebra and let $\mathbf{H} = i_{E_n}(\mathbf{A})$. Take $a, b \in H$. There exists $\mathbf{B} \in \mathscr{H}_{E_n}(\mathbf{A})$ such that $b \in B$. By the previous corollary, $ba^n = ba^{n+1}$ hence $\mathbf{H} \in E_n$, as required. ∎

It follows that we cannot replace "quasivariety" by "variety" throughout the statement of Theorem 9: if $K = E_n$, then K^h is the class of all BCK-algebras, which is not a variety.

EXAMPLE 14. Given $n \in \omega$ and a BCK-algebra $\mathbf{A}$, Corollary 12 says that if $a \in i_{E_n}(\mathbf{A})$ then $ax^n = ax^{n+1}$ for each $x \in A$. The converse implication is not true in general, as the following example shows.

Take an element a such that $a \notin \omega$. Set $A = \omega \cup \{a\}$. For $x, y \in \omega$ and $z \in A$, define

$$xy = x \dot{-} y = \max\{0, x - y\}, \qquad ax = a, \qquad za = 0.$$

Then $\mathbf{A} = \langle A; \cdot, 0\rangle$ is a BCK-algebra: in fact, this is a special case of Iséki's adjunction of a unit (our a) to the linearly ordered simple commutative BCK-algebra $\omega = \langle \omega; \dot{-}, 0\rangle$: see [Isé75, Theorem 4]. Note that a is the greatest element of the partially ordered set $\langle A; \leq\rangle$ and the restriction of $\leq$ to ω is the classical linear order on ω. For all $m, p, q \in \omega$, where $q > 0$ and $m > p$, we have

$$m \cdot 1^p = m - p \neq m - (p+1) = m \cdot 1^{p+1} \quad \text{and}$$
$$m \cdot 1^p = m - p \geq m \cdot q^p.$$

Therefore $i_{E_n}(\mathbf{A}) = \{0, \ldots, n\}$ and so $a \notin i_{E_n}(\mathbf{A})$. Nevertheless, $az^n = az^{n+1}$ for all $z \in A$.

THEOREM 15. [Stu84, Theorem 3]. *The variety* T *of all commutative BCK-algebras has the hereditary subalgebra property for all BCK-algebras.* ∎

COROLLARY 16. *The variety* I *of all implicative BCK-algebras has the hereditary subalgebra property for all BCK-algebras.*

Proof. This follows from Lemma 6, Theorems 13 and 15 and the fact that $I = E_1 \cap T$. ■

THEOREM 17. *The variety J has the hereditary subalgebra property for all BCK-algebras.*

Proof. Take a BCK-algebra $\mathbf{A}$ and $a, b \in i_J(\mathbf{A})$, say $a \in C_1 \in \mathscr{H}_J(\mathbf{A})$ and $b \in C_2 \in \mathscr{H}_J(\mathbf{A})$. Then $b(ba) \leq a$ (by (4)) so $b(ba) \in C_1$. Now we may apply (J) to a and $b(ba)$ and we have:

$$\begin{aligned} a(a(b(ba))) &= a(a([b(ba)]0)) \quad \text{(by (10))} \\ &= a(a([b(ba)][(b(ba))a])) \quad \text{(by (4))} \\ &= b(ba)[[b(ba)](a(a[b(ba)]))] \quad \text{(by } (J)) \\ &= b(ba)[[b(a(a[b(ba)]))](ba)] \quad \text{(by (2))} \\ &\leq b(b(a(a[b(ba)]))) \quad \text{(by (5))} \\ &\leq b(b(a(ab))). \end{aligned}$$

The last inequality follows from the fact that $b(ba) \leq b$ (by (3)) and four applications of the quasi-identity (7).

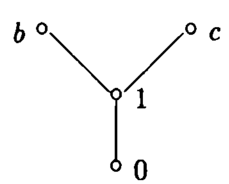
Similarly, from the fact that $a(ab) \in C_2$, we may prove the reverse inequality $b(b(a(ab))) \leq a(a(b(ba)))$; hence $i_J(\mathbf{A})$ satisfies (J), i.e., $i_J(\mathbf{A}) \in J$, as required. ■

EXAMPLE 18. We show that there exists a variety V of BCK-algebras which does not have the hereditary subalgebra property for all BCK-algebras. A BCK-algebra is said to be *directed* if its underlying partially ordered set is upward directed. Let V be the class of all BCK-algebras which are embeddable into directed commutative BCK-algebras. Then V is a variety [Stu82, Theorem 11] and an equational base for V is provided by the identities (8), (9), (10), (T) and

$$xy \wedge yz = 0 \tag{12}$$

[RS87, Corollary 5]. Note that (12) is a $\langle \cdot, 0 \rangle$-identity since $x \wedge y = x(xy)$ holds in every commutative BCK-algebra.

Let $\langle A; \leq \rangle$ be the partially ordered set whose Hasse diagram is depicted on the right. Define a binary operation $\cdot$ on A (abbreviated by juxtaposition) as follows. For $x, y \in A$:



$$b1 = c1 = bc = cb = 1$$

$$xx = 0x = 0; \; x0 = x$$

$$x \leq y \Rightarrow xy = 0.$$

Then $\mathbf{A} = \langle A; \cdot, 0\rangle$ is a BCK-algebra whose associated partial order is $\leq$. Let $\mathbf{B}$ and $\mathbf{C}$ be the subalgebras of $\mathbf{A}$ with $B = \{0, 1, b\}$ and $C = \{0, 1, c\}$. Then clearly $B, C \in \mathscr{H}_V(\mathbf{A})$ and $A = B \cup C$ so $A = i_V(\mathbf{A})$, but $\mathbf{A} \notin V$, since $\mathbf{A}$ violates (12): indeed, $bc \wedge cb = 1 \wedge 1 = 1 \neq 0$.

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