DEMONSTRATIO MATHEMATICA
Vol. XXIX No 1 1996

Marian Hotloś

ON RICCI-PSEUDOSYMMETRIC DOUBLY WARPED PRODUCTS

1. Introduction

Let (M, g) and (M', g') be Riemannian manifolds whose metrics need not be positive definite. If $f : M \longrightarrow (0, \infty)$ and $\phi : M' \longrightarrow (0, \infty)$ are smooth functions then the Cartesian product $\overline{M} = M \times M'$ with the metric $\bar{g} = \phi g \oplus fg'$ (more precisely, $\bar{g} = (\phi \circ \pi')\pi^* g + (f \circ \pi)(\pi')^* g'$, $\pi : \overline{M} \to M$ being the natural projection), is called a doubly warped product [5]. We will use the notation $M_\phi \times_f M'$ for the manifold $(M \times M', \phi g \oplus fg')$. If f or ϕ is constant then we obtain a (singly) warped product [2] (or semi-decomposable space [7]).

It is worth to noticing that a doubly warped product is the special case of a so-called conformal product ($\bar{g} = hg \oplus kg'$, where h and k are functions defined on $M \times M'$) investigated by Yano [11] and Wong [10]. Lorentzian doubly warped products have been studied by Beem and Powell [1]. In this paper we consider only essentially doubly warped products, i.e., such doubly warped products which are not singly warped.

Let (M, g) be an n-dimensional ($n \geq 3$) Riemannian manifold. We denote by $\nabla, \mathcal{R}, R, S$ and K the Levi-Civita connection, the curvature tensor, the Riemannian-Christoffel curvature tensor, the Ricci tensor and the scalar curvature of (M, g), respectively.

A manifold (M, g) is said to be Ricci-pseudosymmetric [4] if at every point of M the following condition is satisfied:

$(*)$ the tensors $R \cdot S$ and $Q(g, S)$ are linearly dependent.

This conditions is trivially satisfied at points at which $S = \frac{K}{n} g$ (it easy to see that the tensor $Q(g, S)$ vanishes at x if and only if $S = \frac{K}{n} g$ at x). Thus the condition (*) is equivalent to the following relation

$$R \cdot S = LQ(g, S) \tag{1}$$

on the set $U = \left\{ x \in M : S \neq \frac{K}{n} g \text{ at } x \right\}$, where L is a function on U.

Obviously, any Ricci-semisymmetric manifold ($R \cdot S = 0$, cf. [8]) is Ricci-pseudosymmetric. Ricci-pseudosymmetric warped products have been studied by Deszcz [3].

The aim of the present paper is to study doubly warped products which are Ricci-pseudosymmetric. Theorem 2.1 contains necessary and sufficient conditions for a doubly warped product to be Ricci-pseudosymmetric.

In section 3 we consider some special cases. In particular, we give necessary and sufficient conditions for a doubly warped product of two Einstein manifolds to be Ricci-pseudosymmetric. Moreover, we prove that if in a Ricci-pseudosymmetric doubly warped product one from the two factors is an Einstein manifold then this product must be Ricci-semisymmetric.

Throughout this paper, by a manifold we mean a connected paracompact manifold of class C^∞ or analytic. By abuse of notation, concerning Riemannian manifolds we often write M instead of (M, g).

2. Preliminaries

Let (M, g) be a Riemannian manifold. For a tensor A of type $(0, p)$, $p \geq 1$, on M we define the tensor fields $R \cdot A$ and $Q(g, A)$ by the formulas

$$\begin{aligned}(R \cdot A)(X_1, \ldots, X_p; X, Y) &= (\mathcal{R}(X, Y) \cdot A)(X_1, \ldots, X_p) = \\ &= -A(\mathcal{R}(X, Y)X_1, X_2, \ldots, X_p) - \cdots - A(X_1, \ldots, X_{p-1}, \mathcal{R}(X, Y)X_p)\end{aligned}$$

and

$$\begin{aligned}Q(g, A)(X_1, \ldots, X_p; X, Y) &= -((X \wedge Y) \cdot A)(X_1, \ldots, X_p) = \\ &= A((X \wedge Y)X_1, X_2, \ldots, X_p) + \cdots + A(X_1, \ldots, X_{p-1}, (X \wedge Y)X_p)\end{aligned}$$

respectively, where $X_i, X, Y \in \Xi(M)$, $\Xi(M)$ being the Lie algebra of vector fields on M, and $\mathcal{R}(X, Y)$ and $X \wedge Y$ are derivations of the algebra of tensor fields on M. These derivations are extensions of endomorphisms $\mathcal{R}(X, Y)$ and $X \wedge Y$ of $\Xi(M)$ defined by

$$\mathcal{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

and

$$(X \wedge Y)Z = g(Z, Y)X - g(Z, X)Y$$

respectively, where $X, Y, Z \in \Xi(M)$. The Riemann-Christoffel curvature tensor R is given by

$$R(X_1, X_2, X_3, X_4) = g(\mathcal{R}(X_1, X_2)X_3, X_4).$$

Let $(\overline{M}, \bar{g})$ be an n-dimensional $(n \geq 2)$ doubly warped product $M_\phi \times_f M'$ $(\dim M = q,\ 1 \leq q < n,\ \dim M' = n - q = s)$.

In a suitable product chart $x^1, \ldots, x^n$ for $\overline{M}$ we have

$$\bar{g}_{ij} dx^i dx^j = \phi g_{ab} dx^a dx^b + f g'_{\alpha\beta} dx_\alpha dx_\beta,$$

where $i, j, \ldots = 1, \ldots, n$, $a, b, \ldots = 1, \ldots, q$, $\alpha, \beta, \ldots = q+1, \ldots, n$, g_{ab} and f are functions of (x^a) only, $g'_{\alpha\beta}$ and ϕ are functions of (x^α) only.

We denote by R_{abcd} and S_{ab} the components of the Riemann-Christoffel curvature tensor R and the Ricci tensor S of (M, g), respectively. Moreover, when Ω is a quantity formed with respect to g, we denote by $\overline{\Omega}$ (resp. Ω') the similar quantity formed with respect to $\bar{g}$ (resp. g'). Analogically, if some formula is indicated by (i), writing (i)' we refer to the similar formula obtaining from (i) by interchanging g and g'.

In the sequel we shall use the following notation

$$(2) \quad \begin{cases} G_{abcd} = g_{ad}g_{bc} - g_{ac}g_{bd}, \\ T_{ab} = -\dfrac{1}{2f}\Big(\nabla_b f_a - \dfrac{1}{2f} f_a f_b\Big), \\ tr(T) = g^{ab} T_{ab}, \\ O = f((s-1)P - tr(T)), \end{cases} \qquad \begin{cases} G'_{\alpha\beta\gamma\delta} = g'_{\alpha\delta}g'_{\beta\gamma} - g'_{\alpha\gamma}g'_{\beta\delta}, \\ T'_{\alpha\beta} = -\dfrac{1}{2\phi}\Big(\nabla'_\beta \phi_\alpha - \dfrac{1}{2\phi}\phi_\alpha\phi_\beta\Big), \\ tr(T') = g'^{\alpha\beta} T'_{\alpha\beta}, \\ O' = \phi((q-1)P' - tr(T')), \end{cases}$$

where $f_b = \partial_b f$, $\phi_\alpha = \partial_\alpha \phi$, $\partial_i = \dfrac{\partial}{\partial x^i}$, $P = g^{ab} f_a f_b / 4f^2 = \dfrac{\Delta_1 f}{4f^2}$, $P' = g'^{\alpha\beta}\phi_\alpha\phi_\beta/4\phi^2 = \dfrac{\Delta'_1\phi}{4\phi^2}$.

We have the following formulas [6]:

$$\bar{R}_{abcd} = \phi R_{abcd} + \frac{\Delta'_1\phi}{4f} G_{abcd},$$

$$\bar{R}_{\alpha\beta\gamma\delta} = f R'_{\alpha\beta\gamma\delta} + \frac{\Delta_1 f}{4\phi} G'_{\alpha\beta\gamma\delta},$$

$$\bar{R}_{a\beta c\delta} = f T_{ac} g'_{\beta\delta} + \phi T'_{\beta\delta} g_{ac},$$

$$\bar{R}_{abc\delta} = \frac{\phi_\delta}{4f}(f_b g_{ac} - f_a g_{bc}),$$

$$\bar{R}_{\alpha\beta\gamma d} = \frac{f_d}{4\phi}(\phi_\beta g'_{\alpha\gamma} - \phi_\alpha g'_{\beta\gamma}),$$

$$\bar{R}_{ab\gamma\delta} = 0,$$

$$\bar{S}_{ab} = S_{ab} - s T_{ab} + \frac{O'}{f} g_{ab},$$

$$\bar{S}_{\alpha\beta} = S'_{\alpha\beta} - qT'_{\alpha\beta} + \frac{O}{\phi} g'_{\alpha\beta},$$
$$\bar{S}_{a\beta} = -\frac{n-2}{4f\phi} f_a \phi_\beta .$$

The last relation implies that $\overline{M}$ cannot be Einsteinian.

Using these formulas, by an elementary but somewhat lenghty calculation, we see that the only non-zero components of $Q(\bar{g}, \bar{S})$ and $\bar{R} \cdot \bar{S}$ are those related to :

(3) $Q(\bar{g}, \bar{S})_{abc\delta} = \frac{n-2}{4f} \phi_\delta (f_b g_{ac} + f_a g_{bc}),$

(3′) $Q(\bar{g}, \bar{S})_{\alpha\beta\gamma d} = \frac{n-2}{4\phi} f_d (\phi_\beta g'_{\alpha\gamma} + \phi_\alpha g'_{\beta\gamma}),$

(4) $Q(\bar{g}, \bar{S})_{ab\gamma d} = \phi g_{bd}(S'_{\alpha\gamma} - qT'_{\alpha\gamma}) - f g'_{\alpha\gamma}(S_{bd} - sT_{bd}) + (O - O') g'_{\alpha\gamma} g_{bd},$

(5) $Q(\bar{g}, \bar{S})_{abcd} = \phi Q(g, S)_{abcd} - \phi s Q(g, T)_{abcd},$

(5′) $Q(\bar{g}, \bar{S})_{\alpha\beta\gamma\delta} = f Q(g', S')_{\alpha\beta\gamma\delta} - f q Q(g', T')_{\alpha\beta\gamma\delta},$

(6)
$$\begin{aligned}(\bar{R} \cdot \bar{S})_{abc\delta} = & -\frac{\phi_\delta}{4f\phi}(f_a S_{bc} + f_b S_{ac} - f^e S_{ea} g_{bc} - f^e S_{eb} g_{ac} \\ & + (q-2)(f_a T_{bc} + f_b T_{ac}) + s(f^e T_{ea} g_{bc} + f^e T_{eb} g_{ac})) \\ & - \frac{n-2}{4f^2} \phi^\omega T'_{\omega\delta}(f_a g_{bc} + f_b g_{ac}),\end{aligned}$$

(6′)
$$\begin{aligned}(\bar{R} \cdot \bar{S})_{\alpha\beta\gamma d} = & -\frac{f_d}{4f\phi}(\phi_\alpha S'_{\beta\gamma} + \phi_\beta S'_{\alpha\gamma} - \phi^\omega S'_{\omega\alpha} g'_{\beta\gamma} - \phi^\omega S'_{\omega\beta} g'_{\alpha\gamma} \\ & + (s-2)(\phi_\alpha T'_{\beta\gamma} + \phi_\beta T'_{\alpha\gamma}) + q(\phi^\omega T'_{\omega\alpha} g'_{\beta\gamma} \\ & + \phi^\omega T'_{\omega\beta} g'_{\alpha\gamma})) - \frac{n-2}{4\phi^2} f^e T_{ed}(\phi_\alpha g'_{\beta\gamma} + \phi_\beta g'\alpha\gamma),\end{aligned}$$

(7)
$$\begin{aligned}(\bar{R} \cdot \bar{S})_{ab\gamma d} = & T'_{\alpha\gamma} S_{bd} - T_{bd} S'_{\alpha\gamma} + (q-s) T'_{\alpha\gamma} T_{bd} + \frac{O' - O}{f} T'_{\alpha\gamma} g_{bd} \\ & + \frac{O' - O}{\phi} T_{bd} g'_{\alpha\gamma} + \frac{f}{\phi}(S_{be} - sT_{be}) T^e_d g'_{\alpha\gamma} \\ & - \frac{\phi}{f}(S'_{\alpha\omega} - qT'_{\alpha\omega}) T'^{\omega}_{\gamma} S_{bd} + \frac{n-2}{(4f\phi)^2}(\Delta_1 f \phi_\alpha \phi_\gamma g_{bd} \\ & - \Delta'_1 \phi f_b f_d g'_{\alpha\gamma}),\end{aligned}$$

(8) $(\bar{R} \cdot \bar{S})_{abcd} = (R \cdot S)_{abcd} - s(R \cdot T)_{abcd} + \frac{\Delta'_1 \phi}{4f\phi} Q(g, B)_{abcd},$

(8′) $(\bar{R} \cdot \bar{S})_{\alpha\beta\gamma\delta} = (R' \cdot S')_{\alpha\beta\gamma\delta} - q(R' \cdot T')_{\alpha\beta\gamma\delta} + \frac{\Delta_1 f}{4f\phi} Q(g', B')_{\alpha\beta\gamma\delta},$

where $B_{ab} = S_{ab} - sT_{ab} + \frac{n-2}{4f^2} f_a f_b$, $B'_{\alpha\beta} = S'_{\alpha\beta} - sT'_{\alpha\beta} + \frac{n-2}{4\phi^2}\phi_\alpha\phi_\beta$.

In the sequel we assume that $\overline{M} = M_\phi \times_f M'$ ($\dim \overline{M} = n \geq 3$) is an essentially doubly warped product (more precisely, we assume that $df \neq 0$ and $d\phi \neq 0$ everywhere). Since $\bar{S} \neq \frac{\bar{K}}{n}\bar{g}$ everywhere, so $\overline{M}$ is Ricci-pseudosymmetric if and only if

$$\bar{R}\cdot\bar{S} = L\, Q(\bar{g}, \bar{S}), \tag{9}$$

where L is a function on $\overline{M}$.

THEOREM 2.1. *A doubly warped product $M_\phi \times_f M'$ is Ricci-pseudosymmetric if and only if the following relations are satisfied:*

(10) $\quad f^e T_{ea} = h f_a$, *where h is a function of (x^a) only,*

(10′) $\quad \phi^\omega T'_{\omega\delta} = \chi\phi_\delta$, *where χ is a function of (x^α) only,*

$$hf + \chi\phi + Lf\phi = 0, \tag{11}$$

$$S_{ab} + (q-2)T_{ab} = \frac{1}{q}(K + (q-2)tr(T))g_{ab}, \tag{12}$$

$$S'_{\alpha\beta} + (s-2)T'_{\alpha\beta} = \frac{1}{s}(K' + (s-2)tr(T'))g'_{\alpha\beta}, \tag{12'}$$

$$(n-2)fT_{be}T^e_d = T_{bd}(V - (n-2)Lf\phi) - hg_{bd}(V + (n-2)\phi\chi) + \frac{n-2}{4f^2}\frac{\Delta'_1\phi}{4\phi}(\Delta_1 f g_{bd} - f_b f_d), \tag{13}$$

$$(n-2)\phi T'_{\alpha\omega}T'^{\omega}_{\gamma} = T'_{\alpha\gamma}(-V - (n-2)Lf\phi) - \chi g'_{\alpha\gamma}(-V - (n-2)hf) + \frac{n-2}{4\phi^2}\frac{\Delta_1 f}{4f}(\Delta'_1\phi g'_{\alpha\gamma} - \phi_\alpha\phi_\gamma), \tag{13'}$$

where $V = \frac{f}{q}(K + (q-2)tr(T)) - O - \frac{\phi}{s}(K' + (s-2)tr(T')) + O'$,

$$f(R\cdot T)_{abcd} + \left(\frac{\Delta'_1\phi}{4\phi} - L\phi f\right)Q(g,T)_{abcd} = \frac{1}{4f^2}\frac{\Delta'_1\phi}{4\phi}Q(g, df\otimes df)_{abcd}, \tag{14}$$

$$\phi(R'\cdot T')_{\alpha\beta\gamma\delta} + \left(\frac{\Delta_1 f}{4f} - Lf\phi\right)Q(g',T')_{\alpha\beta\gamma\delta} = \frac{1}{4\phi^2}\frac{\Delta_1 f}{4f}Q(g', d\phi\otimes d\phi)_{\alpha\beta\gamma\delta}. \tag{14'}$$

P r o o f. Assume that $M_\phi \times_f M'$ is Ricci-pseudosymmetric manifold.

Using (9), (6) and (3), we have

$$(15)\qquad -\frac{\phi_\delta}{\phi}(f_aS_{bc}+f_bS_{ac}-f^eS_{ea}g_{bc}-f^eS_{eb}g_{ac}+(q-2)(f_aT_{bc}+ +f_bT_{ac})+s(f^eT_{ea}g_{bc}+f^eT_{eb}g_{ac}))-\frac{n-2}{f}\phi^\omega T'_{\omega\delta}(f_ag_{bc}+ +f_bg_{ac})= (n-2)L\phi_\delta(f_ag_{bc}+f_bg_{ac}).$$

Multiplying this equation by ϕ_γ and antisymmetrizing the resulting relation, we get

$$(\phi^\omega T'_{\omega\delta}\phi_\gamma-\phi^\omega T'_{\omega\gamma}\phi_\delta)(f_ag_{bc}+f_bg_{ac})=0$$

which implies

$$\phi^\omega T'_{\omega\delta}\phi_\gamma=\phi^\omega T'_{\omega\gamma}\phi_\delta.$$

Thus we have (10′). Analogically, starting from the equation $(\bar R\cdot\bar S)_{\alpha\beta\gamma d}=L\,Q(\bar g,\bar S)_{\alpha\beta\gamma d}$, we obtain (10). Substituting (10) and (10′) into (15), we have

$$(16)\qquad f_aS_{bc}+f_bS_{ac}-f^eS_{ea}g_{bc}-f^eS_{eb}g_{ac}+(q-2)(f_aT_{bc}+f_bT_{ac}) +(sh+(n-2)\phi\Big(\frac{\chi}{f}+L\Big))(f_ag_{bc}+f_bg_{ac})=0.$$

Transvecting now this equation with f^c and using (10), we get (11). Thus $\phi\left(\frac{\chi}{f}+L\right)=-h$ and (16) takes the form

$$(17)\qquad f_aS_{bc}+f_bS_{ac}-f^eS_{ea}g_{bc}-f^eS_{eb}g_{ac}+(q-2)(f_aT_{bc}+f_bT_{ac}) -(q-2)h(f_ag_{bc}+f_bg_{ac})=0.$$

Contracting (17) with g^{bc}, in virtue of (10), we obtain

$$f^eS_{ea}=\frac{1}{q}(K+(q-2)(tr(T)-qh))f_a$$

which turns (17) into

$$f_a\Big(S_{bc}+(q-2)T_{bc}-\frac{1}{q}(K+(q-2)tr(T))g_{bc}\Big) +f_b\Big(S_{ac}+(q-2)T_{ac}-\frac{1}{q}(K+(q-2)tr(T))g_{ac}\Big)=0.$$

This equation immediately leads to (12).

Substituting into the equality $(\bar R\cdot\bar S)_{\alpha b\gamma d}=L\,Q(\bar g,\bar S)_{\alpha b\gamma d}$ the formulas (4), (7), (12), (12′) and then transvecting the resulting relation with f^b (or ϕ^α), by (10), (10′) and (11), we easily obtain (13) (or (13′)).

Finally, taking into account the equality $(\bar R\cdot\bar S)_{abcd}=L\,Q(\bar g,\bar S)_{abcd}$ and using (12), we get (14).

Now assume that relations (10)–(14′) are satisfied. Combining these relations with formulas (3)–(8′) we obtain our assertion. This completes the proof.

Since $\nabla_b(\Delta_1 f) = 2(\nabla_b f_a)f^a$, so using (2) we get

$$f^a T_{ab} = -\frac{1}{4}\nabla_b\left(\frac{\Delta_1 f}{f}\right).$$

Now (10) immediately leads to the following

COROLLARY 2.1. *Let $M_\phi \times_f M'$ be a Ricci-pseudosymmetric manifold. Then $h = 0$ ($\chi = 0$) if and only if $d\left(\frac{\Delta_1 f}{f}\right) = 0$ ($d\left(\frac{\Delta'_1\phi}{\phi}\right) = 0$).*

3. Main results

LEMMA 3.1. *Let $M_\phi \times_f M'$ be a Ricci-pseudosymmetric manifold. If M (resp. M') is an Einstein manifold and* $\dim M > 2$ (*resp.* $\dim M' > 2$), *then $\Delta'_1\phi = 0$ (resp. $\Delta_1 f = 0$).*

Proof. Assume that M is an Einstein manifold and $q > 2$. Substituting $S = \frac{K}{q}g$ into (12), we have $T = \frac{1}{q}tr(T)g$ which implies $R \cdot T = 0$ and $Q(g,T) = 0$. Thus (14) takes the form $\frac{1}{4f^2}\frac{\Delta'_1\phi}{4\phi}Q(g, df \otimes df) = 0$, whence we immediately obtain $\Delta'_1\phi = 0$.

As an immediate consequence of Lemma 3.1 we get

COROLLARY 3.1. *Let $M_\phi \times_f M'$ be a Ricci-pseudosymmetric manifold. If the metric $\phi g \oplus fg'$ is positive definite then neither M nor M' can be an Einstein manifold.*

THEOREM 3.1. *Let M and M'* ($\dim M, \dim M' > 2$) *be Einstein manifolds and f and ϕ be smooth positive functions on M and M', respectively. If the doubly warped product $M_\phi \times_f M'$ is Ricci-pseudosymmetric then M and M' are Ricci-flat, $M_\phi \times_f M'$ is Ricci-semisymmetric and functions f and ϕ satisfy following equations*

$$(18) \qquad \nabla^2 f = \frac{1}{2f}df \otimes df, \qquad \nabla'^2\phi = \frac{1}{2\phi}d\phi \otimes d\phi,$$

$$(19) \qquad \Delta_1 f = 0, \qquad \Delta'_1\phi = 0.$$

Conversely, if M and M' are Ricci-flat manifolds and f and ϕ are positive functions on M and M', respectively, satisfying equations (18) *and* (19), *then $M_\phi \times_f M'$ is Ricci-semisymmetric manifold.*

Proof. Assume that M and M' are Einstein manifolds and $q, s > 2$. Using Lemma 3.1, we have (19), which by Corollary 2.1 implies $h = 0$ and $\chi = 0$. This, in view of (11), leads to $L = 0$. As in the proof of Lemma 3.1, we obtain $T = \frac{1}{q} tr(T)g$. This, by (10) and $h = 0$, gives $T = 0$ which is equivalent to (18). Now it is easy to see that the covector field $d(\sqrt{f})$ is parallel. Thus $R \cdot df = 0$ and $f^a S_{ab} = 0$ which implies $K = 0$. In the same way we get $K' = 0$.

Conversely, taking into account the formulas (6)–(8′), we see that $\bar{R} \cdot \bar{S} = = 0$. This completes the proof.

THEOREM 3.2. *Let $M_\phi \times_f M'$ be Ricci-pseudosymmetric manifold. If M or M' is an Einstein manifold whose dimension is greater than 2, then $M_\phi \times_f M'$ is Ricci-semisymmetric manifold.*

Proof. Assume that M is an Einstein manifold and $q > 2$. Using Lemma 3.1 , we have $\Delta_1'\phi = 0$ which, in view of Corollary 2.1, implies $\chi = 0$. Thus, by (11), $L = -\frac{h}{\phi}$ and our assertion is equivalent to the equality $h = 0$. Suppose that $h \neq 0$. We restrict our considerations to the open subset W on which $h \neq 0$ (and also $d\left(\frac{\Delta_1 f}{f}\right) \neq 0$).

The equation (13′) takes the form

$$(n-2)\phi T'_{\alpha\omega} T'^{\omega}_{\gamma} = T'_{\alpha\gamma}(-V + (n-2)hf) - \frac{\Delta_1 f}{4f}\frac{n-2}{4\phi^2}\phi_\alpha\phi_\gamma, \tag{20}$$

where $V = \frac{f}{q}(K + (q-2)tr(T)) - O - \frac{\phi}{s}(K' + (s-2)tr(T')) + O'$.

Differentiating (20), we have

$$T'_{\alpha\gamma}\partial_a(-V + (n-2)hf) = \partial_a\left(\frac{\Delta_1 f}{4f}\right)\frac{n-2}{4\phi^2}\phi_\alpha\phi_\gamma$$

which, after standard calculation leads to

$$T'_{\alpha\gamma} = c\frac{n-2}{4\phi^2}\phi_\alpha\phi_\gamma, \qquad c = \text{constant}. \tag{21}$$

This turns (20) into

$$c(-V + (n-2)hf) = \frac{\Delta_1 f}{4f}. \tag{22}$$

Substituting (21) into (2) and differentiating the resulting relation covariantly, using Ricci-identity, we have $\phi^\omega R'_{\omega\alpha\beta\gamma} = 0$. Thus $R' \cdot T' = 0$ and (14′)

ta:kes the form

$$\left(\frac{\Delta_1 f}{4f} + hf\right)Q(g',T')_{\alpha\beta\gamma\delta} = \frac{1}{4\phi^2}\frac{\Delta_1 f}{4f}Q(g',d\phi\otimes d\phi)_{\alpha\beta\gamma\delta}.$$

This equation, in view of (21), leads to

$$c(n-2)\left(\frac{\Delta_1 f}{4f} + hf\right) = \frac{\Delta_1 f}{4f}. \tag{23}$$

Comparing (23) with (22), we have

$$-V = (n-2)\frac{\Delta_1 f}{4f}.$$

But this relation, in virtue of definitions of V, O and P, can be written in the form

$$\frac{\phi K'}{s} = \frac{fK}{q} + (q-1)\left(\frac{\Delta_1 f}{4f} + 2hf\right). \tag{24}$$

The relation (23) implies $hf = c_1\frac{\Delta_1 f}{4f}$, c_1 = constant. Substituting this equation into (24), we have

$$\frac{\phi K'}{s} = \frac{fK}{q} + c_2\frac{\Delta_1 f}{4f}, \qquad c_2 = \text{constant}. \tag{25}$$

If $c_2 = 0$, then we immediately obtain f =constant (because K =constant). If $c_2 \neq 0$, then differentiating (25) and using the equality $hf_b = f^a T_{ab} = -\partial_b(\frac{\Delta_1 f}{4f})$, we get

$$\frac{1}{q}f_a K - c_2 h f_a = 0.$$

This implies $h = \frac{K}{c_2 q}$ and consequently, h =constant. Now, differentiating (23), we have $\partial_b\left(\frac{\Delta_1 f}{4f}\right) = 0$ which implies $(\partial_b\left(\frac{\Delta_1 f}{4f}\right) = -hf_b)$ $h = 0$, a contradiction. This completes the proof.

Remar:k. 3.1. Examples of Ricci-flat manifolds on which there exist functions satisfying the conditions (18) and (19) we can obtain ta:king Wal:ker's metrics (see [9], pp. 51, 59).

References

[1] J. K. Beem and T. G. Powell, *Geodesic completness and maximality in Lorentzian warped products*, Tensor, N.S., 39 (1982), 31–36.

[2] R. L. Bishop, B. O'Neill, *Manifolds of negative curvature*, Trans. Amer. Math. Soc., 145 (1969), 1–49.
[3] R. Deszcz, *On Ricci-pseudosymmetric warped products*, Demonstratio Math., 22 (1989), 1053–1065.
[4] R. Deszcz, M. Hotloś, *Remarks on Riemannian manifolds satisfying certain curvature condition imposed on the Ricci tensor*, Prace Nauk. Politech. Szczecin., 11 (1989), 23–34.
[5] P. E. Ehrlich, *Metric deformations of Ricci and sectional curvature on compact Riemannian manifolds*, Ph.D. Dissertation, SUNY, Stony Brook, New York, 1974.
[6] M. Hotloś, *Some theorems on doubly warped products*, Demonstratio Math., 23 (1990), 39–58.
[7] G. I. Kruchkovich, *On semi-decomposable Riemannian spaces*, (in Russian), Dokl. Akad. Nauk SSSR, 115 (1957), 862–865.
[8] J. Mikesh, *On geodesic mappings of Ricci 2-symmetric Riemannian spaces* (in Russian), Mat. Zam., 28 (1980), 313–317.
[9] A. G. Walker, *On Ruse's spaces of recurrent curvature*, Proc. Lond. Math. Soc., 52 (1950), 36–64.
[10] Y. C. Wong, *Some Einstein spaces with conformally separable fundamental tensors*, Trans. Amer. Math. Soc., 53 (1943), 157–194.
[11] K. Yano, *Conformally separable quadratic differential forms*, Proc. Imp. Acad. Tokyo, 16 (1940), 83–86.

INSTITUTE OF MATHEMATICS
WROCŁAW TECHNICAL UNIVERSITY
Wybrzeże Wyspiańskiego 27
50-370 WROCŁAW, POLAND

Received March 4, 1994.