DEMONSTRATIO MATHEMATICA
Vol. XXIX No 1 1996

Bronisław Przybylski

SINGULARITIES OF PRINCIPAL-DIRECTED CURVES IN THE PLANE, I

Principal-directed curves in the plane $\mathbf{R}^2$, called shortly P-directed curves, were introduced in paper [3] in connection with the study of smooth curves with respect to the product final differential structures $\mathcal{S}^1$ and $\mathcal{S}^2$ defined on $\mathbf{R}^2$. These structures determine the differential spaces $\mathbf{R}\times_1\mathbf{R} = (\mathbf{R}^2,\mathcal{S}^1)$ and $\mathbf{R}\times_2\mathbf{R} = (\mathbf{R}^2,\mathcal{S}^2)$ which have many common properties (see [2] for details) and can be considered together as the differential space $\mathbf{R}\times_k\mathbf{R}$ where $k\in\{1,2\}$ is fixed but arbitrary. In particular, the class of all smooth curves in $\mathbf{R}\times_k\mathbf{R}$ does not depend on k and consists of exactly all locally K-subordinate smooth curves in $\mathbf{R}^2$ (see [3], Corollary 2.21), which implies that any such curve must be P-directed. On the other hand, it is easy to construct a P-directed curve in $\mathbf{R}^2$ which is not locally K-subordinate (see [3], Example 2.22), and so, the class of all P-directed curves in $\mathbf{R}^2$ is essentially stronger than that of all smooth ones in $\mathbf{R}\times_k\mathbf{R}$.

In the present paper we treat principal-directed curves in the plane in a way independent of the differential structures $\mathcal{S}^1$ and $\mathcal{S}^2$. For such curves we first prove the so-called statements of singularity (SA) and (SB) of Theorem 1.3. Next, relative to statement (SA) we formulate the special condition of singularity (SCA) as well as the general one (GCA). In particular, this means that any curve satisfying condition (SCA) has to satisfy condition (GCA) for the corresponding data. Our paper is mainly devoted to the study of condition (SCA). More precisely, applying topological and functional analysis methods we obtain appropriate constructions of P-directed curves in $\mathbf{R}^2$ satisfying this condition. Strictly speaking, we replace condition (SCA) by equivalent one (SCA_0) which concerns real smooth functions defined on an interval of $\mathbf{R}$. We prove the basic Theorem 2.8 which implies that there exists a monotonically nondecreasing real smooth function that satisfies condition (SCA_0^+) of Corollary 2.9, being stronger than condition (SCA_0). In turn, this involves that there exists a monotonically nondecreasing smooth curve in $\mathbf{R}^2$ satisfying condition (SCA$^+$) of Corollary 2.10, stronger than condition (SCA). Finally, we ask Question (QSA) and then present a correct answer to it given by statements (A) and (B) of Theorem 3.2. Strictly speaking, statement (A) asserts that any P-directed curve in $\mathbf{R}^2$ satisfying some condition has to be contained in a principal line. Otherwise, by statement (B) one can construct a monotonically nondecreasing P-directed curve in $\mathbf{R}^2$ which is contained in a given principal right angle with a mixed direction but is not in any principal line.

The programme of an investigation of the remaining condition (GCA) is continued in the next papers (see [4], parts II and III). It turns out that the corresponding solutions take account of various kinds of ordinal invariants of smooth geometric curves in $\mathbf{R}^2$. This is fully presented in part III of [4] where such solutions for some realizations of (GCA) are given. However, the preparatory considerations are included in part II of [4] where the concepts of chains and chain fibrations are explained (Section 4) in a form suitable for our further considerations. Give attention that all the conditions discussed above are formulated relative to statement (SA) only. This means that any similar discussion relative to statement (SB) remains completely open.

Our paper has been intended to be part I of series [4] of preprints which have common terminology and notation as well as continuous numeration. Moreover, since they are meant to be continuations of paper [3], we accept for them all the terminology and notation from [3] unless otherwise stated.

1. Statements and conditions of singularity

Let c be a smooth curve in $\mathbf{R}^2$. Following [3], we have defined the sets $\mathrm{dom}(c)$, $\mathrm{dom}_R(c)$, $\mathrm{dom}_S(c)$, $\mathrm{dom}_{CS}(c)$ and $\mathrm{dom}_{NS}(c)$ as well as the sets $\mathrm{dom}_X(c)$ and $\mathrm{loc}_X(c)$ where $X \in \{V, H, P\}$. In addition we adopt the following notations:

$$\mathrm{dom}_{PS}(c) = \mathrm{dom}(c) \setminus \mathrm{loc}_P(c),$$
$$\mathrm{loc}_{RX}(c) = \mathrm{dom}_R(c) \cap \mathrm{loc}_X(c).$$

Obviously, the set $\mathrm{dom}_{PS}(c)$ is closed in dom(c). In turn, the sets $\mathrm{loc}_{RV}(c)$ and $\mathrm{loc}_{RH}(c)$ are disjoint and open in dom(c). Furthermore, $\mathrm{loc}_{RP}(c) = \mathrm{loc}_{RV}(c) \cup \mathrm{loc}_{RH}(c)$ and by definition we get $\mathrm{loc}_{RP}(c) \subseteq \mathrm{dom}_R(c)$ and $\mathrm{loc}_{RP}(c) \subseteq \mathrm{loc}_P(c)$. It is easy to verify the following propositions:

1.1. PROPOSITION. *Let c be a smooth curve in $\mathbf{R}^2$. Then the following conditions are equivalent:*

(a) *c is P-directed, i.e.* $\mathrm{dom}_P(c) = \mathit{dom}(c)$;

(b) $\mathrm{loc}_P(c)$ *is a dense subset of* $\mathrm{dom}(c)$;

(c) $\mathrm{dom}_{PS}(c)$ *is a boundary subset of* $\mathrm{dom}(c)$;

(d) $\mathrm{loc}_{RP}(c) = \mathrm{dom}_R(c)$. ∎

1.2. PROPOSITION. *Let c be a smooth curve in $\mathbf{R}^2$. Then the following conditions are equivalent*:

(a) *c is almost regular, i.e.* $\mathrm{int}\,\mathrm{dom}_S(c) = \emptyset$;

(b) $\mathrm{loc}_{RP}(c) = \mathrm{loc}_P(c)$;

(c) $\mathrm{loc}_{RV}(c) = \mathrm{loc}_V(c)$ *and* $\mathrm{loc}_{RH}(c) = \mathrm{loc}_H(c)$. ∎

The following theorem presents the so-called statements of singularity (SA) and (SB) for a P-directed curve c in $\mathbf{R}^2$. They can be the basis for formulating the corresponding conditions of singularity, which is realized here relative to statement (SA) only.

1.3. THEOREM. *If $X \in \{V, H\}$ and c is a P-directed curve in $\mathbf{R}^2$, then the following statements hold*:

(SA) $\quad \mathrm{dom}_{PS}(c) \subseteq fr\ \mathrm{dom}_{CS}(c) \subseteq \mathrm{fr}\,\mathrm{dom}_S(c)$,

(SB) $\quad \mathrm{dom}_{PS}(c) = \mathrm{cl}\,\mathrm{loc}_{RX}(c) \setminus \mathrm{loc}_X(c) = \mathrm{fr}\,\mathrm{loc}_{RX}(c) \setminus \mathrm{loc}_X(c) = = \mathrm{cl}\,\mathrm{loc}_{RV}(c) \cap \mathrm{cl}\,\mathrm{loc}_{RH}(c) = \mathrm{fr}\,\mathrm{loc}_{RV}(c) \cap \mathrm{fr}\,\mathrm{loc}_{RH}(c)$,

where fr *and* cl *are the boundary and the closure operations in* $\mathrm{dom}(c)$; *respectively.*

Proof. (SA). Since $\mathrm{dom}_{CS}(c)$ and $\mathrm{dom}_S(c)$ are closed subsets of $\mathrm{dom}(c)$, $\mathrm{dom}_{CS}(c) \subseteq \mathrm{dom}_S(c)$ and $\mathrm{int}\,\mathrm{dom}_{CS}(c) = \mathrm{int}\,\mathrm{dom}_S(c)$, we conclude that $\mathrm{fr}\,\mathrm{dom}_{CS}(c) \subseteq \mathrm{fr}\,\mathrm{dom}_S(c)$. Thus, it remains to show the inclusion $\mathrm{dom}_{PS}(c) \subseteq \mathrm{fr}\,\mathrm{dom}_{CS}(c)$ which is equivalent to the following one

(1) $$\mathrm{dom}(c) \setminus \mathrm{fr}\,\mathrm{dom}_{CS}(c) \subseteq \mathrm{loc}_P(c).$$

To prove (1), note first that

(2) $$\mathrm{dom}(c) \setminus \mathrm{fr}\,\mathrm{dom}_{CS}(c) = \mathrm{int}\,\mathrm{dom}_{CS}(c) \cup \mathrm{dom}_{NS}(c).$$

Furthermore, $\mathrm{int}\,\mathrm{dom}_{CS}(c) = \mathrm{loc}_V(c) \cap \mathrm{loc}_H(c) \subseteq \mathrm{loc}_P(c)$ and $\mathrm{dom}_{NS}(c) \subseteq \mathrm{loc}_P(c)$ (see [3], Lemma 1.3 and Proposition 1.13). Hence and from (2) we get (1), and so, the proof of statement (SA) is complete.

(SB). Suppose that $c = (\alpha, \beta)$. First we prove that

(3) $$\mathrm{dom}_{PS}(c) = \mathrm{cl}\,\mathrm{loc}_{RV}(c) \cap \mathrm{cl}\,\mathrm{loc}_{RH}(c).$$

Indeed, let us take a parameter $s \in \mathrm{dom}_{PS}(c)$. If U is an open neighbourhood of s in $\mathrm{dom}(c)$, then there are $v, w \in U$ such that $\dot{\alpha}(v) \neq 0$ and $\dot{\beta}(w) \neq 0$, for otherwise $s \in \mathrm{loc}_P(c)$, a contradiction. Therefore, we can find open neighbourhoods V of v and W of w in $\mathrm{dom}(c)$ such that $V, W \subseteq U$ and $\dot{\alpha}(v') \neq 0$ and $\dot{\beta}(w') \neq 0$ for any $v' \in V$ and $w' \in W$. Since c is P-directed, it follows that $V \subseteq \mathrm{loc}_{RH}(c)$ and $W \subseteq \mathrm{loc}_{RV}(c)$. Consequently, we have $s \in \mathrm{cl}\,\mathrm{loc}_{RV}(c) \cap \mathrm{cl}\,\mathrm{loc}_{RH}(c)$.

Conversely, let us take $s \in \operatorname{cl}\operatorname{loc}_{RV}(c) \cap \operatorname{cl}\operatorname{loc}_{RH}(c)$. In this case for an arbitrary open neighbourhood U of s in $\operatorname{dom}(c)$ we have $U \cap \operatorname{loc}_{RH}(c) \neq \emptyset$ and $U \cap \operatorname{loc}_{RV}(c) \neq \emptyset$. This means that there are $v, w \in U$ such that $\dot{\alpha}(v) \neq 0$ and $\dot{\beta}(w) \neq 0$. Therefore, we conclude that $s \in \operatorname{dom}_{PS}(c)$, for otherwise $s \in \operatorname{loc}_P(c)$, and so, there would exists an open neighbourhood U of s in $\operatorname{dom}(c)$ such that $\dot{\alpha}|U = 0$ or $\dot{\beta}|U = 0$, a contradiction. Summarizing, we have proved the equality (3).

Now, since $\operatorname{loc}_{RP}(c) = \operatorname{loc}_{RV}(c) \cup \operatorname{loc}_{RH}(c)$ and $\operatorname{dom}_{PS}(c)$ are disjoint sets, it follows from (3) that

$$\begin{aligned}\operatorname{dom}_{PS}(c) &= (\operatorname{cl}\operatorname{loc}_{RV}(c) \cap \operatorname{cl}\operatorname{loc}_{RH}(c)) \setminus \operatorname{loc}_{RP}(c)\\ &= (\operatorname{cl}\operatorname{loc}_{RV}(c) \setminus \operatorname{loc}_{RV}(c)) \cap (\operatorname{cl}\operatorname{loc}_{RH}(c) \setminus \operatorname{loc}_{RH}(c))\\ &= \operatorname{fr}\operatorname{loc}_{RV}(c) \cap \operatorname{fr}\operatorname{loc}_{RH}(c).\end{aligned}$$

Analogously, we have

$$\begin{aligned}\operatorname{dom}_{PS}(c) &= (\operatorname{fr}\operatorname{loc}_{RV}(c) \cap \operatorname{fr}\operatorname{loc}_{RH}(c)) \setminus \operatorname{loc}_{P}(c)\\ &= (\operatorname{fr}\operatorname{loc}_{RV}(c) \setminus \operatorname{loc}_{V}(c)) \cap (\operatorname{fr}\operatorname{loc}_{RH}(c) \setminus \operatorname{loc}_{H}(c)),\end{aligned}$$

which implies

$$\operatorname{dom}_{PS}(c) \subseteq \operatorname{fr}\operatorname{loc}_{RX}(c) \setminus \operatorname{loc}_X(c) \subseteq \operatorname{cl}\operatorname{loc}_{RX}(c) \setminus \operatorname{loc}_X(c)$$

for $X \in \{V, H\}$. Thus, to complete the proof, it remains to show that

$$\operatorname{cl}\operatorname{loc}_{RX}(c) \setminus \operatorname{loc}_X(c) \subseteq \operatorname{dom}_{PS}(c). \tag{4}$$

Indeed, since the cases $X = V$ and $X = H$ are analogous, we can further assume that $X = V$. Let us take a parameter $s \in \operatorname{cl}\operatorname{loc}_{RV}(c) \setminus \operatorname{loc}_V(c)$. Consider an open neighbourhood U of s in $\operatorname{dom}(c)$. Obviously, we have $U \cap \operatorname{loc}_{RV}(c) \neq \emptyset$ and note that $U \cap \operatorname{loc}_{RH}(c) \neq \emptyset$, for otherwise we get $\dot{\alpha}|U = 0$, which means that $U \subseteq \operatorname{loc}_V(c)$, a contradiction. Consequently, we conclude that $s \in \operatorname{cl}\operatorname{loc}_{RV}(c) \cap \operatorname{cl}\operatorname{loc}_{RH}(c)$ and from (3) we obtain $s \in \operatorname{dom}_{PS}(c)$, which proves (4) for $X = V$. ∎

Obviously, from this theorem and Proposition 1.2 we get

1.4. COROLLARY. *If c is an almost regular P-directed curve in $\mathbf{R}^2$, then*

$$\operatorname{dom}_{PS}(c) = \operatorname{cl}\operatorname{loc}_V(c) \cap \operatorname{cl}\operatorname{loc}_H(c) = \operatorname{fr}\operatorname{loc}_V(c) = \operatorname{fr}\operatorname{loc}_H(c). \ ∎$$

Statement (SA), also called the *stratification statement* of *singularity* for c, will have a special significance for our further investigations. We shall associate with it the *special condition* (SCA) and the *general condition* (GCA) which will be regarded as those of *singularity* for P-directed curves in $\mathbf{R}^2$. More exactly, the first one will be the main condition under consideration in this paper, but the second is intended to be studied more thoroughly in the next paper (see [4], part III). To formulate these conditions, we first make some necessary observations relative to statement (SA).

In what follows the symbol I stands for an arbitrary but fixed interval of $\mathbf{R}$, that is, a nonsingle-element connected subspace of $\mathbf{R}$, or equivalently, a convex subset of $\mathbf{R}$ with nonempty interior, which will often be regarded as the base topological space. We accept that all topological operations for subsets of I or for functions defined on I are meant relative to I. In particular, this concerns the operations of interior, closure and boundary for subsets of I (see [1]), represented respectively by the symbols int, cl and fr as well as the operation of support for real functions defined on I, represented by the symbol supp. For any subset A of I we denote by A^d the *derived set* of A relative to I, that is,

$$A^d = \{x \in I : x \in \mathrm{cl}(A \setminus \{x\})\}.$$

From the definition of A^d we obviously get

1.5. LEMMA. *For any $A \subseteq I$ the set $A \setminus A^d$ is discrete in I, and so,* $\mathrm{card}(A \setminus A^d) \leq \aleph_0$. ■

We adopt the notation $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. For any $\alpha \in C^\infty(I)$ and $i \in \mathbf{N}_0$ let the symbol $D^i\alpha$ stand for the ith derivative of α. In particular, we accept $D^0\alpha = \alpha$, $D^1\alpha = \alpha' = \dot{\alpha}$ and $D^2\alpha = \alpha''$. Moreover, we adopt the following notations:

$Z(\alpha) = \{x \in I : \alpha(x) = 0\}$;
$Z^k(\alpha) = \bigcap\{Z(D^i\alpha) : 0 \leq i \leq k\}$ for $k \in \mathbf{N}_0$;
$Z^\infty(\alpha) = \bigcap\{Z(D^i\alpha) : i \in \mathbf{N}_0\} = \bigcap\{Z^k(\alpha) : k \in \mathbf{N}_0\}$.

1.6. LEMMA. *If $\alpha \in C^\infty(I)$ and $A = Z(\alpha)$, then $A^d \subseteq Z^\infty(\alpha)$.* ■

Proof. Let us take $a \in A^d$. Of course, there is a strictly monotonic sequence (a_i^0) of elements of the set $Z(D^0\alpha) = A$ such that $\lim_i a_i^0 = a$. Without loss of generality we can further assume that (a_i^0) is strictly increasing, i.e. $a_i^0 < a_{i+1}^0$ for $i \in \mathbf{N}$. Suppose now that for any $k \in \mathbf{N}_0$ we have defined a strictly increasing sequence (a_i^k) of elements of $Z(D^k\alpha)$ such that $\lim_i a_i^k = a$. In particular, we have $(D^k\alpha)(a_i^k) = (D^k\alpha)(a_{i+1}^k) = 0$ for each $i \in \mathbf{N}$, and so, it follows from the Rolle Theorem that there exists a_i^{k+1} in I satisfying $a_i^k < a_i^{k+1} < a_{i+1}^k$ and $(D^{k+1}\alpha)(a_i^{k+1}) = 0$. Therefore, for any $k \in \mathbf{N}_0$ we have defined a strictly increasing sequence (a_i^k) of elements of $Z(D^k\alpha)$ such that $\lim_i a_i^k = a$. Hence we conclude that $a \in Z^k(\alpha)$ for all $k \in \mathbf{N}_0$, which implies that $a \in Z^\infty(\alpha)$. Finally, since a can be an arbitrary element of A^d, it follows that $A^d \subseteq Z^\infty(\alpha)$. ■

To any $\alpha \in C^\infty(I)$ we can assign, for example, the horizontal smooth curve $c_\alpha = (\alpha, 0) : I \to \mathbf{R}^2$. Clearly, the assignment $\alpha \mapsto c_\alpha$ defines a one-to-one correspondence between $C^\infty(I)$ and the family of all smooth curves in $\mathbf{R}^2$ defined on I and contained in the horizontal line $\mathbf{R}\times\{0\}$. This assignment allows us to transfer to $C^\infty(I)$ those necessary notations for smooth curves in $\mathbf{R}^2$ which depend on differential operations. Therefore, we can adopt the following notations: $\mathrm{dom}_S(\alpha) = \mathrm{dom}_S(c_\alpha)$ and $\mathrm{dom}_{CS}(\alpha) = \mathrm{dom}_{CS}(c_\alpha)$. Clearly, we have $\mathrm{dom}_S(\alpha) = Z(\dot\alpha)$ and $\mathrm{dom}_{CS}(\alpha) = \bigcap\{Z(D^k\alpha) : k \in \mathrm{N}\} = Z^\infty(\dot\alpha)$. Finally, note that with the same effect, for any given principal line L in $\mathbf{R}^2$, we can consider in a similar way the appropriate assignment $\alpha \mapsto c_\alpha^L$ from $C^\infty(I)$ to the family of all smooth curves in $\mathbf{R}^2$ which are defined on I and contained in L.

Let us fix $\alpha \in C^\infty(I)$ and set $F = \mathrm{dom}_S(\alpha) = Z(\dot\alpha)$ and $E = \mathrm{dom}_{CS}(\alpha) = Z^\infty(\dot\alpha)$. Obviously, F and E are closed subsets of I such that $E \subseteq F$. By applying Lemma 1.6 for $\dot\alpha$ we get the condition $F^d \subseteq E$. More generally, note that if F and E are closed subsets of I such that $E \subseteq F$, then the condition $F^d \subseteq E$ involves that $\mathrm{int}\, F = \mathrm{int}\, E$. Indeed, we have $\mathrm{int}\, F \subseteq \mathrm{int}\, E$ because $\mathrm{int}\, F \subseteq F^d \subseteq E$ and $\mathrm{int}\, E \subseteq \mathrm{int}\, F$ because $E \subseteq F$. Furthermore, it turns out that the condition $F^d \subseteq E \subseteq F$ is in general essentially stronger than that $\mathrm{int}\, F = \mathrm{int}\, E$ and $E \subseteq F$. For instance, let $I = [0;1]$, $E = \emptyset$ and $F = C$ be the Cantor set lying in I (see [3], Example 2.6). In this case we have $\mathrm{int}\, F = \mathrm{int}\, E = \emptyset$ and $E \subseteq F$, however, the set $F^d = C$ is not contained in E. Clearly, this implies that for such E and F there is no function $\alpha \in C^\infty(I)$ satisfying $\mathrm{dom}_S(\alpha) = F$ and $\mathrm{dom}_{CS}(\alpha) = E$.

Let now $c = (\alpha, \beta) : I \to \mathbf{R}^2$ be a smooth curve. Clearly, we have $F = F_1 \cap F_2$ and $E = E_1 \cap E_2$ where F, F_1, F_2 and E, E_1, E_2 denote respectively the sets $\mathrm{dom}_S(c)$, $\mathrm{dom}_S(\alpha)$, $\mathrm{dom}_S(\beta)$ and the sets $\mathrm{dom}_{CS}(c)$, $\mathrm{dom}_{CS}(\alpha)$, $\mathrm{dom}_{CS}(\beta)$. From the previous considerations it follows that $F_1^d \subseteq E_1 \subseteq F_1$ and $F_2^d \subseteq E_2 \subseteq F_2$, whence we get $F^d \subseteq F_1^d \cap F_2^d \subseteq E \subseteq F$. We have thus for c the condition $F^d \subseteq E \subseteq F$ which is also in general essentially stronger than the condition $\mathrm{int}\, F = \mathrm{int}\, E$ and $E \subseteq F$ (see [3], Lemma 1.3), similarly as for real smooth functions defined on I.

Suppose that E and F are closed subsets of I such that $F^d \subseteq E \subseteq F$. One can ask whether there exists a P-directed curve $c : I \to \mathbf{R}^2$ which is subject to the following condition:

(SCA) $\quad \mathrm{dom}_{PS}(c) = \emptyset, \quad \mathrm{dom}_{CS}(c) = E \quad \text{and} \quad \mathrm{dom}_S(c) = F;$

This condition is consistent with statement (SA) and will be regarded as the purpose for constructions of the corresponding curves. Moreover, we can formulate the most general condition (GCA) as follows. Let D, E and F be closed subsets of I such that D is a boundary set, $F^d \subseteq E \subseteq F$ and $D \subseteq \mathrm{fr}\, E \subseteq \mathrm{fr}\, F$. One can ask whether there exists a P-directed curve $c : I \to \mathbf{R}^2$ for which the following condition holds:

(GCA) $\quad \mathrm{dom}_{PS}(c) = D, \quad \mathrm{dom}_{CS}(c) = E \quad \text{and} \quad \mathrm{dom}_S(c) = F.$

It is seen that any curve satisfying condition (SCA) has to satisfy condition

(GCA) for the corresponding D, E and F. In our paper applying topological and functional analysis methods we present appropriate constructions of P-directed curves in $\mathbf{R}^2$ relative to condition (SCA). In the next paper (see [4], part III) we give similar constructions of such curves relative to condition (GCA).

We first show that condition (SCA) can be replaced by condition (SCA_0) below. Suppose that E and F are closed subsets of I such that $F^d \subseteq E \subseteq F$. One can ask whether there exists a function $\lambda \in C^\infty(I)$ satisfying the following condition:

$$(\mathrm{SCA}_0) \qquad \mathrm{dom}_{CS}(\lambda) = E \quad \text{and} \quad \mathrm{dom}_S(\lambda) = F.$$

Indeed, if $\lambda \in C^\infty(I)$ satisfies condition (SCA_0), then the horizontal curve $c_\lambda = (\lambda, 0)$ clearly satisfies condition (SCA). Conversely, if $c = (\alpha, \beta) : I \to \mathbf{R}^2$ is a P-directed curve satisfying condition (SCA), then observe first that the assumption $\mathrm{dom}_{PS}(c) = \emptyset$ involves that $\mathrm{supp}\,\dot\alpha \cap \mathrm{supp}\,\dot\beta = \emptyset$. This implies that the function $\lambda = \alpha + \beta$ satisfies condition (SCA_0) because $\mathrm{dom}_S(\lambda) = \mathrm{dom}_S(c) = F$ and $\mathrm{dom}_{CS}(\lambda) = \mathrm{dom}_{CS}(c) = E$. Summarizing, we have shown that conditions (SCA) and (SCA_0) are equivalent.

2. Singularities of real smooth functions

Consider the family $C^\infty(I)$ as a real vector space under the pointwise operations. Denote by $\mathfrak{S} = \mathfrak{S}(I)$ the set of all pairs (k, C) where $k \in \mathbf{N}_0$ and C is a compact subset of I. We shall regard $\mathfrak{S}$ as partially ordered set under the ordering $\leq$ defined as follows: $(k', C') \leq (k'', C'')$ if $k' \leq k''$ and $C' \subseteq C''$. For any $\lambda \in C^\infty(I)$ and $\sigma = (k, C) \in \mathfrak{S}$ we set

$$\|\lambda\|_\sigma = \max\{|(D^i\lambda)(t)| : 0 \leq i \leq k,\ t \in C\}.$$

It is seen that every $\|\cdot\|_\sigma$ is a seminorm in $C^\infty(I)$. Moreover, if $\lambda \neq 0$, one can find $\sigma \in \mathfrak{S}$ such that $\|\lambda\|_\sigma \neq 0$. We have thus a topological vector space $C^\infty(I)$ under the weakest topology induced by the family $\{\|\cdot\|_\sigma : \sigma \in \mathfrak{S}\}$ of seminorms. Moreover, this family is monotonically nondecreasing, which means that for any $\sigma, \tau \in \mathfrak{S}$ such that $\sigma \leq \tau$ we have $\|\lambda\|_\sigma \leq \|\lambda\|_\tau$ for each $\lambda \in C^\infty(I)$, written as $\|\cdot\|_\sigma \leq \|\cdot\|_\tau$. Since I is a locally compact space being countable at infinity, we can fix a sequence (C^k) of compact subsets of I such that $C^k \subseteq \mathrm{int}\, C^{k+1}$ for $k \in \mathbf{N}_0$ and $\bigcup\{C^k : k \in \mathbf{N}_0\} = I$. Let us set $\|\cdot\|_k = \|\cdot\|_{(k, C^k)}$ for $k \in \mathbf{N}_0$. One can see that for each $\sigma \in \mathfrak{S}$ there is $k \in \mathbf{N}_0$ such that $\|\cdot\|_\sigma \leq \|\cdot\|_k$. It follows that the topology of $C^\infty(I)$ is induced by the family $\{\|\cdot\|_k : k \in \mathbf{N}_0\}$ of seminorms too. This means that $C^\infty(I)$ is a metrizable topological vector space. In fact, $C^\infty(I)$ is a separable Frechet space (compare [5], the definitions of a Frechet space in 1.8 (f) and of $C^\infty(\Omega)$ in 1.46). In this topology the convergence $\lambda_n \to \lambda$ in $C^\infty(I)$ means that for each $i \in \mathbf{N}_0$, $D^i\lambda_n$ is convergent to $D^i\lambda$ uniformly on every compact subset of I. In the sequel we shall always regard that $C^\infty(I)$ is a Frechet space with respect to the above-described family $\|\cdot\|_k : k \in \mathbf{N}_0$ of seminorms.

Let $\lambda_1, \lambda_2, \ldots$ be an infinite sequence of functions of the Frechet space $C^\infty(I)$. The series $\sum_{n=1}^\infty \lambda_n$ is called *absolutely convergent* in $C^\infty(I)$ if for each $\sigma \in \mathfrak{S}$ the series $\sum_{n=1}^\infty \|\lambda_n\|_\sigma$ is convergent, or equivalently, if for each $k \in \mathbf{N}_0$ the series $\sum_{n=1}^\infty \|\lambda_n\|_k$ is convergent. Clearly, every series $\sum_{n=1}^\infty \lambda_n$ absolutely convergent in $C^\infty(I)$ is convergent in $C^\infty(I)$ but not conversely in general. We need the following lemma which can be proved, more generally, for an arbitrary Frechet space $C^\infty(\Omega)$ (see [5], 1.46).

2.1. LEMMA. *If the series $\sum_{n=1}^\infty \alpha_n$ is absolutely convergent in $C^\infty(I)$, then for each $i \in \mathbf{N}$ so is the series $\sum_{n=1}^\infty D^i\alpha_n$. Furthermore, if $\alpha = \sum_{n=1}^\infty \alpha_n$, then $D^i\alpha = \sum_{n=1}^\infty D^i\alpha_n$.* ∎

2.2. LEMMA. *If $\alpha_1, \alpha_2, \ldots$ is an infinite sequence of functions of $C^\infty(I)$, there exists an infinite sequence $\varepsilon_1, \varepsilon_2, \ldots$ of positive real numbers such that the series $\sum_{n=1}^\infty \varepsilon_n\alpha_n$ is absolutely convergent in $C^\infty(I)$.*

Proof. It is clear that for any $n \in \mathbf{N}$ one can find $\varepsilon_n > 0$ such that $\varepsilon_n\|\alpha_n\|_n < 2^{-n}$. This implies that for each $k \in \mathbf{N}_0$ the series $\sum_{n=1}^\infty \|\varepsilon_n\alpha_n\|_k$ is convergent because the family $(\|\cdot\|_k)$ is monotonically nondecreasing, and so, we have $\|\varepsilon_n\alpha_n\|_k = \varepsilon_n\|\alpha_n\|_k \le \varepsilon_n\|\alpha_n\|_n < 2^{-n}$ for $n \ge k$. Thus the series $\sum_{n=1}^\infty \varepsilon_n\alpha_n$ is absolutely convergent in $C^\infty(I)$. ∎

2.3. LEMMA. *For any closed subset A of I, there exists a nonnegative function $\alpha \in C^\infty(I)$ such that $Z(\alpha) = Z^\infty(\alpha) = A$.*

Proof. The assertion is trivial if $A = I$. Therefore we can further assume that $A \ne I$. Since A is a G_δ-subset of the normal topological space I, it follows that there is an infinite sequence $U_1, U_2, \ldots$ of open neighbourhoods of A in I such that $\operatorname{cl} U_{n+1} \subseteq U_n$ for $n \in \mathbf{N}$ and $\bigcap\{U_n : n \in \mathbf{N}\} = A$. In turn, it is known that for each n there exists a nonnegative function $\alpha_n \in C^\infty(I)$ such that $Z(\alpha_n) = \operatorname{cl} U_{n+1}$. By Lemma 2.2 there exists an infinite sequence (ε_n) of positive real numbers such that the series $\alpha = \sum_{n=1}^\infty \varepsilon_n\alpha_n$ is absolutely convergent in $C^\infty(I)$. Clearly, we have $Z(\alpha) = A$. Moreover, from Lemma 2.1 it follows that $Z^\infty(\alpha) = A$. ∎

It is clear that I equipped with the differential structure $C^\infty(I)$ is a paracompact connected one-dimensional differentiable manifold (with boundary in case I is not open in $\mathbf{R}$) being a normal topological space. Applying a smooth partition of unity it is easy to prove the following lemma which can be generalized for paracompact finite-dimensional manifolds too.

2.4. LEMMA. *Let A and B be closed disjoint subsets of I. If U and V are open neighbourhoods of A and B in I respectively such that $\operatorname{cl} U \cap \operatorname{cl} V = \emptyset$,* *then there exists a function $\lambda \in C^\infty(I)$ such that $\lambda|U = 1$, $\lambda|V = 0$ and $0 \le \lambda(x) \le 1$ for $x \in I$.* ∎

2.5. LEMMA. *For any discrete subset A of I there exists a nonnegative function $\alpha \in C^\infty(I)$ such that $A \subseteq Z^1(\alpha)$ and $\alpha''(a) > 0$ ($\alpha''(a) \ne 0$) for each $a \in A$.*

Proof. The assertion is obviously trivial if $A = \emptyset$. Therefore we can further assume that $A \ne \emptyset$. Since A is a discrete subset of I, it follows that $1 \le \operatorname{card} A \le \aleph_0$. We can thus suppose that $A = \{a_1, \ldots\}$ where (a_n) for $n < \min\{\operatorname{card} A+1, \aleph_0\}$ is a finite or infinite sequence of all distinct elements of A. Next, since A is a discrete subset of the normal topological space I, it follows that there exists a sequence (W_n) of disjoint open subsets of I such that $W_n \cap A = \{a_n\}$ for each n. Furthermore, by applying induction on n we conclude that for $n < \min\{\operatorname{card} A + 1, \aleph_0\}$ there exist disjoint open sets U_n and V_n in I such that

(1) $\quad a_n \in U_n \subseteq W_n \cap V(n), \quad I \setminus W_n \subseteq V_n \quad \text{and} \quad \operatorname{cl} U_n \cap \operatorname{cl} V_n = \emptyset$

where $V(1) = I$ and $V(n) = \bigcap\{V_i : 1 \le i \le n-1\}$ for $n > 1$. Moreover, by Lemma 2.4 we conclude that for each n there exists a function $\lambda_n \in C^\infty(I)$ such that

(2) $\quad \lambda_n|U_n = 1, \quad \lambda_n|V_n = 0 \quad \text{and} \quad 0 \le \lambda_n(x) \le 1 \quad \text{for } x \in I.$

Let us set

$$\alpha_n(x) = \lambda_n(x) \cdot (x - a_n)^2 \quad \text{for } x \in I$$

and note that α_n is a nonnegative function of $C^\infty(I)$ satisfying the conditions:

(3) $\quad \operatorname{supp} \alpha_n \subseteq W_n, \quad \alpha_n(a_n) = \alpha'(a_n) = 0 \quad \text{and} \quad \alpha''(a_n) = 2.$

In the case when $m = \operatorname{card} A < \aleph_0$ we set $\alpha = \alpha_1 + \ldots + \alpha_m$. Note that α is a nonnegative function of $C^\infty(I)$ such that $\alpha|U_n = \alpha_n|U_n$ for each $1 \le n \le m$, which follows from (1) and (2) because for $i \ne n$ we have $\lambda_n|U_i = 0$, and so, $\alpha_n|U_i = 0$. Therefore, from (3) we infer that in this case α satisfies the assertion of our lemma. Otherwise, if A is infinite, then from Lemma 2.2 it follows that there exists an infinite sequence (ε_n) of positive real numbers such that the series $\sum_{n=1}^\infty \varepsilon_n \alpha_n$ is absolutely convergent in $C^\infty(I)$. Clearly, in that case $\alpha = \sum_{n=1}^\infty \varepsilon_n \alpha_n$ is a nonnegative function of $C^\infty(I)$. Moreover, similarly as in the previous case, we conclude that $\alpha|U_n = \varepsilon_n \alpha_n|U_n$ for each $n \in \mathbb{N}$. Consequently, it follows from (3) that α satisfies the assertion of our lemma too. ∎

2.6. Remark. Observe that if α is a nonnegative function of $C^\infty(I)$ and $a \in Z^1(\alpha)$, then the condition $\alpha''(a) \ne 0$ involves that $\alpha''(a) > 0$. Indeed, suppose to the contrary that $\alpha''(a) < 0$. Then there exists $\varepsilon > 0$

such that $\alpha''(x) < 0$ for each $x \in U_\varepsilon = I \cap (a - \varepsilon; a + \varepsilon)$. It follows from the Taylor formula that for any $x \in U_\varepsilon$ there exists $\xi_x \in (0;1)$ such that

$$\alpha(x) = \frac{1}{2} \cdot \alpha''(a + \xi_x(x-a)) \cdot (x-a)^2,$$

which implies immediately that $\alpha(x) < 0$ for each $x \in U_\varepsilon \setminus \{a\}$, a contradiction. ∎

If necessary, we shall regard $C^\infty(I)$ as a real algebra under the pointwise operations. For any $A \subseteq I$ and $k \in \mathbb{N}_0 \cup \{\infty\}$ let us set

$$J^k(A) = \{\alpha \in C^\infty(I) : A \subseteq Z^k(\alpha)\}.$$

Clearly, we have $J^k(A) = J^k(\operatorname{cl} A)$. Moreover, note that $J^k(A)$ is an ideal of the algebra $C^\infty(I)$.

2.7. Lemma. *Let A and B be disjoint subsets of I. Suppose that A is discrete, B is closed and $A^d \subseteq B$. Then there exists a nonnegative function φ of $C^\infty(I)$ such that $A \subseteq Z^1(\varphi)$, $\varphi''(a) > 0$ for each $a \in A$, $Z^\infty(\varphi) = B$ and $Z(\varphi) = A \cup B$.*

Proof. By Lemma 2.5 there exists a nonnegative function α of $C^\infty(I)$ such that $A \subseteq Z^1(\alpha)$ and $\alpha''(a) > 0$ for each $a \in A$. In turn, by Lemma 2.3 there exists a nonnegative function β of $C^\infty(I)$ such that $Z(\beta) = Z^\infty(\beta) = B$. Let us consider the nonnegative function $\vartheta = \alpha\beta$ of $C^\infty(I)$. Since $\alpha \in J^1(A)$, $\beta \in J^\infty(B)$ and both $J^1(A)$ and $J^\infty(B)$ are ideals of $C^\infty(I)$, it follows that $A \subseteq Z^1(\vartheta)$ and $B \subseteq Z^\infty(\vartheta)$. Moreover, by applying the Leibniz rule we conclude that $\vartheta''(a) = \alpha''(a)\beta(a) > 0$ for each $a \in A$ because $\alpha(a) = \alpha'(a) = 0$, $\alpha''(a) > 0$ and $\beta(a) > 0$. Next, since $A \cup B$ is a closed subset of I, it follows from Lemma 2.3 that there exists a nonnegative function $\omega \in C^\infty(I)$ such that $Z(\omega) = Z^\infty(\omega) = A \cup B$. Finally, note that the function $\varphi = \vartheta + \omega$ satisfies the assertion of our lemma. ∎

2.8. Theorem. *Let E and F be closed subsets of I such that $F^d \subseteq E \subseteq F$. Then there exists a nonnegative function $\varphi \in C^\infty(I)$ such that $Z(\varphi) = F$, $Z^\infty(\varphi) = E$, $F \setminus E \subseteq Z(\varphi')$ and $\varphi''(x) > 0$ for each $x \in F \setminus E$.*

Proof. This theorem is a consequence of Lemma 2.7 for $A = F \setminus E$ and $B = E$, where $A \subseteq F \setminus F^d$ is a discrete subset of I by Lemma 1.5. ∎

2.9. Corollary. *Let E and F be closed subsets of I such that $F^d \subseteq E \subseteq F$. Then there exists a monotonically nondecreasing function $\vartheta \in C^\infty(I)$ satisfying the following condition*:

$$(\mathrm{SCA}_0^+) \qquad \begin{aligned} &\operatorname{dom}_{CS}(\vartheta) = E, \quad \operatorname{dom}_S(\vartheta) = F; \\ &F \setminus E \subseteq Z(\vartheta''), \quad (D^3\vartheta)(x) > 0 \ \forall x \in F \setminus E. \end{aligned}$$

In particular, ϑ satisfies condition (SCA_0).

Let φ be a function satisfying Theorem 2.8. We define the function $\vartheta \in C^\infty(I)$ to be an antiderivative of φ, that is, satisfying $\vartheta' = \varphi$. It is seen that ϑ satisfies the assertion of our corollary. ∎

A curve $c = (\alpha, \beta) : I \to \mathbf{R}^2$ is called monotonically nondecreasing if so are both the functions α and β. If c is smooth, then for any $i \in \mathbf{N}$ we set

$$D^i c = [D^i\alpha, D^i\beta] \quad \text{and} \quad Z(D^i c) = Z(D^i\alpha) \cap Z(D^i\beta).$$

2.10. Corollary. *Let E and F be closed subsets of I such that $F^d \subseteq E \subseteq F$. Let L be a given principal line in $\mathbf{R}^2$. Then there exists a monotonically nondecreasing P-directed curve $c : I \to \mathbf{R}^2$ contained in L and satisfying the following condition*:

$$(\mathrm{SCA}^+) \qquad \begin{array}{l} \mathrm{dom}_{PS}(c) = \emptyset, \quad \mathrm{dom}_{CS}(c) = E, \quad \mathrm{dom}_S(c) = F; \\ F \setminus E \subseteq Z(D^2 c), \quad (D^3 c)(x) \neq 0 \ \forall x \in F \setminus E; \end{array}$$

which is stronger than condition (SCA).

Proof. By definition, L has to be of the form $\mathbf{R} \times \{a\}$ or $\{a\} \times \mathbf{R}$ for some $a \in \mathbf{R}$. Therefore if we take a function ϑ satisfying Corollary 2.9, then the smooth curve $c = (\vartheta, a)$ (resp. $c = (a, \vartheta)$) in $\mathbf{R}^2$ satisfies clearly the assertion of our corollary. ∎

3. Singularity of locally P-subordinate smooth curves

Following [3], a smooth curve c in $\mathbf{R}^2$ is said to be *almost regular* if $\mathrm{int}\,\mathrm{dom}_S(c) = \emptyset$. Recall that c is *locally P-subordinate* if $\mathrm{loc}_P(c) = \mathrm{dom}(c)$, or equivalently, $\mathrm{dom}_{PS}(c) = \emptyset$. Clearly, every locally P-subordinate smooth curve in $\mathbf{R}^2$ is P-directed. We need the following (see [3], Proposition 1.9).

3.1. Lemma. *Every almost regular locally P-subordinate smooth curve in $\mathbf{R}^2$ is contained in a principal line.* ∎

Of course, if c is a P-directed curve in $\mathbf{R}^2$ and $\mathrm{dom}_{PS}(c) \neq \emptyset$, then c cannot be contained in any principal line. It is of interest to know whether there exists a locally P-subordinate smooth curve in $\mathbf{R}^2$ or, more generally, satisfying condition (SCA) which is not contained in any principal line. From Lemma 3.1 it follows that if c is a locally P-subordinate smooth curve in $\mathbf{R}^2$ not contained in any principal line, then it cannot be almost regular. On the other hand, a totally stationary smooth curve c in $\mathbf{R}^2$, that is, satisfying $\mathrm{dom}_S(c) = \mathrm{dom}(c)$, is clearly not almost regular however is contained in a principal line as a constant map. More generally, note that if c is a P-directed curve in $\mathbf{R}^2$ such that the set $\mathrm{dom}_R(c)$ is connected, then c is contained in a principal line (Corollary 3.3).

One can ask the following question formulated relative to condition (SCA):

(QSA) *For what closed subsets E and F of I satisfying $F^d \subseteq E \subseteq F$, there exists a P-directed curve c in $\mathbf{R}^2$ such that*

$$\mathrm{dom}_{PS}(c) = \emptyset, \quad \mathrm{dom}_{CS}(c) = E, \quad \mathrm{dom}_S(c) = F$$

and c is not contained in any principal line.

To obtain an affirmative answer to this question (Theorem 3.2), we first consider the operation of C^∞-gluing for P-directed curves in $\mathbf{R}^2$ which corresponds to that for real smooth functions. This operation can be regarded as a particular case of the procedure of gluing for differential or differentiable spaces (see [6] and [7]).

For any nonempty $S \subseteq I$ we adopt the following notations:

$I^-(S) = \{x \in I : x \leq s \ \forall s \in S\}$,
$I^+(S) = \{x \in I : x \geq s \ \forall s \in S\}$.

Clearly, $I^-(S)$ and $I^+(S)$ are convex subsets of I. In particular, for $a \in I$ we accept $I^-(a) = I^-(\{a\})$ and $I^+(a) = I^+(\{a\})$.

Let us take $a \in \mathrm{int}\, I$ and observe that $I^-(a)$ and $I^+(a)$ are subintervals of I satisfying the identities $I = I^-(a) \cup I^+(a)$ and $I^-(a) \cap I^+(a) = \{a\}$. The pair (I_1, I_2) where $I_1 = I^-(a)$ and $I_2 = I^+(a)$ is called the *cut* of I *by* a, and a is referred to as the *contact parameter* of I_1 and I_2. In this case we also say that I_1 and I_2 are *contact* intervals of $\mathbf{R}$ at a. Conversely, if I_1 and I_2 are intervals of $\mathbf{R}$ such that $\sup I_1 = \max I_1 = \inf I_2 = \min I_2$, then I_1 and I_2 are contact intervals of $\mathbf{R}$ at the parameter $a = \max I_1 = \min I_2$. Moreover, $I = I_1 \cup I_2$ is an interval of $\mathbf{R}$ and the pair (I_1, I_2) is the cut of I by a.

Suppose further that I_1 and I_2 are contact intervals of $\mathbf{R}$ at a. We say that functions $\alpha_1 \in C^\infty(I_1)$ and $\alpha_2 \in C^\infty(I_2)$ are C^∞-*contact* (*at* a) in the case when $(D^i\alpha_1)(c) = (D^i\alpha_2)(c)$ for each $i \in \mathbf{N}_0$, which is equivalent to the fact that there is a unique function $\alpha \in C^\infty(I)$ such that $\alpha_1 = \alpha|I_1$ and $\alpha_2 = \alpha|I_2$, called the (*smooth*) *gluing* of α_1 and α_2 (*at* a) and denoted by $\alpha_1 \amalg_a \alpha_2$, or shortly, $\alpha_1 \amalg \alpha_2$. Let now $c_1 = (\alpha_1, \beta_1) : I_1 \to \mathbf{R}^2$ and $c_2 = (\alpha_2, \beta_2) : I_2 \to \mathbf{R}^2$ be smooth curves. We say that c_1 and c_2 are C^∞-*contact* if the functions α_1 and α_2 as well as β_1 and β_2 are C^∞-contact at a. Clearly, in this case there exists a unique smooth curve $c : I \to \mathbf{R}^2$ where $I = I_1 \cup I_2$ such that $c_1 = c|I_1$ and $c_2 = c|I_2$, called the (*smooth*) *gluing* of c_1 and c_2 (*at* a) and denoted by $c_1 \amalg_a c_2$ or $c_1 \amalg c_2$ for short. It is seen that if in addition c_1 and c_2 are P-directed, then so is the gluing $c_1 \amalg c_2$.

For any $(a, b) \in \mathbf{R}^2$ and $\sigma, \tau \in \{-, +\}$ we define the set

$$A(a^\sigma, b^\tau) = \mathbf{R}^\sigma(a) \times \{b\} \cup \{a\} \times \mathbf{R}^\tau(b)$$

and call it the *principal right angle* at (a, b) with (σ, τ)-*direction*. In particular, we have the sets $A(a^-, b^+)$ and $A(a^+, b^-)$ called the *principal right angles* at (a, b) with *mixed directions*. It is seen that the principal cross $\mathbb{K}_p$

for $p = (a, b)$ (see [3]) is a union of four distinct principal right angles at p with all possible directions which correspond to the four quarters of the plane determined by $\mathbf{K}_p$.

For any subset S of I denote by conv S the *convex hull* of S in I, i.e. $\operatorname{conv} S = \{tx + (1-t)y : x, y \in S \text{ and } 0 \le t \le 1\}$. It is clear that conv S is a subinterval of I provided that S has at least two distinct elements.

3.2. THEOREM. *Let E and F be closed subsets of I such that $F^d \subseteq E \subseteq F$. Then the following statements hold:*

(A) *If* $\operatorname{int} F \cap \operatorname{conv}(I \setminus F) = \emptyset$, *then any P-directed curve c in $\mathbf{R}^2$ such that* $\operatorname{dom}_{PS}(c) = \emptyset$, $\operatorname{dom}_{CS}(c) = E$ *and* $\operatorname{dom}_S(c) = F$ *is contained in a principal line.*

(B) *If* $\operatorname{int} F \cap \operatorname{conv}(I \setminus F) \neq \emptyset$, *then for any principal right angle A at a point $p \in \mathbf{R}^2$ with a mixed direction, there exists a monotonically non-decreasing smooth curve $c : I \to \mathbf{R}^2$ satisfying condition* (SCA$^+$) *of Corollary* 2.10 *and such that c is contained in A but is not in any principal line.*

Proof. (A). In the case when $\operatorname{int} F = \emptyset$, c is almost regular, so it is contained in a principal line by Lemma 3.1. In turn, if $I \setminus F = \emptyset$, then c is totally stationary, which means that it is contained in a principal line as a constant map. Therefore we can further assume that $\operatorname{int} F \neq \emptyset$ and $I \setminus F \neq \emptyset$. In the sequel we put $R = I \setminus F$. Since $\operatorname{int} F \cap \operatorname{conv} R = \emptyset$, it follows that $\operatorname{int} F \subseteq I^-(R) \cup I^+(R)$. Without loss of generality one can also assume that $\operatorname{int} F \cap I^-(R) \neq \emptyset$. Thus, we can set $a = \max I^-(R)$ and note that $x < a < y$ for any $x \in \operatorname{int} F \cap I^-(R)$ and $y \in R$, so $a \in \operatorname{int} I$. Let (I_1, I_2) be the cut of I by a. We set $c_1 = c_1|I_1$ and $c_2 = c_2|I_2$, which means that $c = c_1 \amalg_a c_2$. It is seen that c_1 is totally stationary, i.e. $c(t) = c(a)$ for each $t \in I_1$. In turn, observe that c_2 satisfies the condition:

$$\operatorname{dom}_R(c_2) = R \quad \text{and} \quad \operatorname{int} F_2 \subseteq I^+(R)$$

where $F_2 = F \cap I_2 = \operatorname{dom}_S(c_2)$.

Consider first the case when $\operatorname{int} F_2 = \emptyset$, which means that c_2 is almost regular. Then since c_2 is locally P-subordinate, it follows from Lemma 3.1 that c_2 is contained in some principal line in $\mathbf{R}^2$, say L. In this case c is also contained in L because $c = c_1 \amalg_a c_2$ and c_1 is totally stationary.

Similarly, in the case when $\operatorname{int} F_2 \neq \emptyset$, it suffices also to show that c_2 is contained in a principal line. Indeed, in that case we can set $b = \min I_2^+(R) = \min I^+(R) \in \operatorname{int} I_2$. Consider the cut (I_{21}, I_{22}) of I_2 by b, i.e. $I_{21} = I_2^-(b)$ and $I_{22} = I_2^+(b)$. Let us set $c_{21} = c_2|I_{21}$ and $c_{22} = c_2|I_{22}$, which means that $c_2 = c_{21} \amalg_b c_{22}$. It is seen that c_{21} is almost regular but c_{22} is totally stationary. Therefore, analogously as for c in the previous case, we conclude that c_2 is contained in a principal line in $\mathbf{R}^2$, which completes the proof in the case under consideration.

(B). Since int $F \cap \operatorname{conv} R \neq \emptyset$, we infer that there exists a connected component G of int F such that $I^-(G) \cap R \neq \emptyset$ and $I^+(G) \cap R \neq \emptyset$. Let us take $g \in G \subseteq \operatorname{int} I$ and consider the cut (I_1, I_2) where $I_1 = I^-(g)$ and $I_2 = I^+(g)$. Moreover, note that for $k = 1,2$ the sets $E_k = E \cap I_k$ and $F_k = F \cap I_k$ are closed subsets of I_k such that $F_k^d \subseteq E_k \subseteq F_k$. By Corollary 2.10 there exists a monotonically nondecreasing smooth curve $c_k : I_k \to \mathbf{R}^2$ such that

$$\text{(1)} \qquad \begin{aligned} &\operatorname{dom}_{PS}(c_k) = \emptyset, \quad \operatorname{dom}_{CS}(c_k) = E_k, \quad \operatorname{dom}_S(c_k) = F_k, \\ &F_k \setminus E_k \subseteq Z(D^2 c), \quad (D^3 c)(x) \neq 0 \ \forall x \in F_k \setminus E_k, \end{aligned}$$

and furthermore, we can require that the following inclusions hold:

$$\text{(2)} \qquad \begin{aligned} &c_1(I_1) \subseteq \mathbf{R} \times \{0\} \quad \text{and} \quad c_2(I_2) \subseteq \{0\} \times \mathbf{R} \\ &(c_1(I_1) \subseteq \{0\} \times \mathbf{R} \quad \text{and} \quad c_2(I_2) \subseteq \mathbf{R} \times \{0\}). \end{aligned}$$

From (2) we obviously have $c_1 = (\lambda_1, 0)$ and $c_2 = (0, \lambda_2)$ ($c_1 = (0, \lambda_1)$ and $c_2 = (\lambda_2, 0)$) where $\lambda_k \in C^\infty(I_k)$. One can further assume that $\lambda_1(g) = \lambda_2(g) = 0$, for otherwise we can replace λ_1 by $\lambda_1 - \lambda_1(g)$ and λ_2 by $\lambda_2 - \lambda_2(g)$ with preserving all above-mentioned properties for c_1 and c_2. Obviously, this additional assumption means that $c_1(g) = c_2(g) = (0,0)$. Furthermore, observe that c_1 and c_2 are C^∞-contact at g because $g \in G \cap I_k \subseteq E \cap I_k = E_k$ for $k = 1,2$, and so, $(D^i c_1)(g) = (D^i c_2)(g) = [0,0]$ for each $i \in \mathbf{N}$. We have thus a smooth curve $c = c_1 \amalg_g c_2$ which is monotonically nondecreasing and P-directed because so are c_1 and c_2. From (1) we infer that c satisfies the conditions: $\operatorname{dom}_{CS}(c) = E$, $\operatorname{dom}_S(c) = F$, $F \setminus E \subseteq Z(D^2 c)$ and $(D^3 c)(x) \neq 0$ for each $x \in F \setminus E$. Since $g \in \operatorname{int} F$, it follows that c is locally stationary at g, which implies that $g \in \operatorname{loc}_P(c)$. This involves that $\operatorname{dom}_{PS}(c) = \emptyset$ because $I \setminus \{g\} = (I_1 \setminus \{g\}) \cup (I_2 \setminus \{g\}) \subseteq \operatorname{loc}_P(c_1) \cup \operatorname{loc}_P(c_2) \subseteq \operatorname{loc}_P(c)$. Consequently, we conclude that c satisfies condition (SCA$^+$) of Corollary 2.10.

Finally, consider the principal right angles $A(0^-, 0^+)$ and $A(0^+, 0^-)$. Obviously, from properties of c_1 and c_2 and from the definition of c we conclude that c is contained in $A(0^-, 0^+)$ ($A(0^+, 0^-)$). Moreover, observe that c cannot be contained in a principal line. Indeed, since $I_1 \cap R \neq \emptyset$ and $I_2 \cap R \neq \emptyset$, there exist $t_1 \in I_1 \cap R = \operatorname{dom}_R(c_1)$ and $t_2 \in I_2 \cap R = \operatorname{dom}_R(c_2)$ where $t_1 < g < t_2$. Clearly, these sets are open and locally convex subsets of I, which implies that there exist $s_1 \in \operatorname{dom}_R(c_1)$ and $s_2 \in \operatorname{dom}_R(c_2)$ such that $t_1 < s_1 < g < s_2 < t_2$,

$$[t_1; s_1] \subseteq \operatorname{dom}_R(c_1) \quad \text{and} \quad [s_2; t_2] \subseteq \operatorname{dom}_R(c_2).$$

Since $c_1 = (\lambda_1, 0)$ and $c_2 = (0, \lambda_2)$ ($c_1 = (0, \lambda_1)$ and $c_2 = (\lambda_2, 0)$) are monotonically nondecreasing smooth curves, so are the real smooth functions λ_1 and λ_2. Furthermore, we get $\lambda_1'(\xi_1) > 0$ for $\xi_1 \in (t_1; s_1)$ and $\lambda_2'(\xi_2) > 0$

for $\xi_2 \in (s_2; t_2)$ because $(t_1; s_1) \subsetneq \mathrm{dom}_R(c_1)$ and $(s_2; t_2) \subsetneq \mathrm{dom}_R(c_2)$. Next, from the Lagrange Mean-value Theorem it follows that there exist $\bar{\xi}_1 \in (t_1; s_1)$ and $\bar{\xi}_2 \in (s_2; t_2)$ such that $\lambda_k(t_k) = \lambda_k(s_k) + \lambda'_k(\bar{\xi}_k) \cdot (t_k - s_k)$ where $k = 1, 2$. For these reasons we conclude that

$$\lambda_1(t_1) = \lambda_1(s_1) + \lambda'_1(\bar{\xi}_1) \cdot (t_1 - s_1) < \lambda_1(s_1) \leq \lambda_1(g) = 0,$$
$$\lambda_2(t_2) = \lambda_2(s_2) + \lambda'_2(\bar{\xi}_2) \cdot (t_2 - s_2) > \lambda_2(s_2) \geq \lambda_2(g) = 0.$$

This means that

$$c(t_1) = c_1(t_1) \in H_0 \setminus \{o\}, \quad c(t_2) = c_2(t_2) \in V_0 \setminus \{o\}$$
$$(c(t_1) = c_1(t_1) \in V_0 \setminus \{o\}, \quad c(t_2) = c_2(t_2) \in H_0 \setminus \{o\})$$

where $H_0 = \mathbf{R} \times \{0\}$, $V_0 = \{0\} \times \mathbf{R}$ and $o = (0,0)$. Hence we infer that there is no principal line in $\mathbf{R}^2$ containing points $c(t_1)$ and $c(t_2)$, so c is not contained in any principal line. This completes the proof of statement (B). ∎

From statement (A) of this theorem we obviously obtain the following corollaries:

3.3. COROLLARY. *If c is a P-directed curve in $\mathbf{R}^2$ such that the set* $\mathrm{dom}_R(c)$ *is connected, then c is contained in a principal line.* ∎

3.4. COROLLARY. *If c is a locally P-subordinate smooth curve in $\mathbf{R}^2$ not contained in any principal line, then*

$$\mathrm{int}\, \mathrm{dom}_S(c) \cap \mathrm{conv}\, \mathrm{dom}_R(c) \neq \emptyset.$$

The following example shows that there exist monotonically nondecreasing locally P-subordinate smooth curves in $\mathbf{R}^2$ with images much more complicated than that of such a curve considered in the proof of statement (B) of Theorem 3.2.

3.5. EXAMPLE. Let $0 < \varepsilon < 1/2$. By Lemma 2.3 we infer that there exist nonnegative functions $\alpha', \beta' \in C^\infty(\mathbf{R})$ satisfying the following conditions ($n \in \mathbf{Z}$):

$\alpha'(t) > 0$, $\beta'(t) = 0$ if $n + \varepsilon < t < n + 2\varepsilon$ where n is even;
$\alpha'(t) = 0$, $\beta'(t) > 0$ if $n + \varepsilon < t < n + 2\varepsilon$ where n is odd;
$\alpha'(t) = \beta'(t) = 0$ otherwise.

In addition, one can require that

$$\int_n^{n+1} \alpha'(t)\, dt = 1 \text{ if } n \text{ is even}, \qquad \int_n^{n+1} \beta'(t)\, dt = 1 \text{ if } n \text{ is odd}.$$

Define the functions $\alpha, \beta \in C^\infty(\mathbf{R})$ by

$$\alpha(s) = \int_0^s \alpha'(t)\, dt \quad \text{and} \quad \beta(s) = \int_0^s \beta'(t)\, dt$$

and observe that $c = (\alpha, \beta) : \mathbf{R} \to \mathbf{R}^2$ is a monotonically nondecreasing P-subordinate smooth curve. Let us set

$$I_n = \{t \in \mathbf{R} : n \leq t \leq n+1\}$$

for $n \in \mathbf{Z}$ and note that $c(\mathbf{R}) = \bigcup\{c(I_n) : n \in \mathbf{Z}\}$. Of course, for each $k \in \mathbf{Z}$ we have $c(I_{2k}) = [k; k+1] \times \{k\}$ and $c(I_{2k-1}) = \{k\} \times [k-1; k]$. Therefore the image $c(\mathbf{R})$ is a *principal* broken line (edge path) consisting of countable many distinct principal closed segments (its edges). Furthermore, observe that c cannot be contained in any finite union of principal angles (crosses). ∎

A chain (i.e. linearly ordered set) S is called *integral* in case it is order-isomorphic to a subchain of $\mathbf{Z}$. This means that for each $s \in S$ there exist in S at most one predecessor and one successor of s, which are usually denoted by $'s$ and s' respectively. We say that s_1 and s_2 are *successive elements* of S provided that $s_1' = s_2$, or equivalently, $'s_2 = s_1$. Clearly, every integral chain is order-isomorphic to an interval (convex subset) of $\mathbf{Z}$ (compare [4], part II, Proposition 4.2). If S is an integral chain, then by a *convex partition* of S we shall mean a disjoint family $\mathcal{P}$ of intervals of S, i.e. nonempty convex subchains of S, such that $\bigcup \mathcal{P} = $ S. In this case $\mathcal{P}$ can also be regarded as an integral chain under the ordering relation $<$ defined as follows ($P, Q \in \mathcal{P}$): $P < Q$ if and only if $x < y$ for any $x \in P$ and $y \in Q$.

Consider an *integral-indexed* sequence $(c_n)_{n \in S}$ of smooth curves in $\mathbf{R}^2$, which means that S is an integral chain. We say that (c_n) is C^∞*-contact* in case any two successive curves of (c_n) are C^∞-contact. This means that if $c_n : I_n \to \mathbf{R}^2$ and $c_{n'} : I_{n'} \to \mathbf{R}^2$ are such curves, then we have $\max I_n = \min I_{n'}$ and there is the smooth gluing $c_n \amalg c_{n'} : I_n \cup I_{n'} \to \mathbf{R}^2$. More generally, one can see that in this case there exists a unique smooth curve $c : I \to \mathbf{R}^2$ where $I = \bigcup\{I_n : n \in S\}$ such that $c|I_n = c_n$ for each $n \in S$, called the *gluing* of $(c_n)_{n \in S}$ and denoted by $\amalg_{n \in S} c_n$. If $c_1, c_2, \ldots, c_k$ is a C^∞-contact finite sequence of smooth curves in $\mathbf{R}^2$, then by $c_1 \amalg c_2 \amalg \ldots \amalg c_k$ we denote the gluing of this sequence as well. It is easily seen that the gluing of any C^∞-contact integral-indexed sequence of P-directed curves in $\mathbf{R}^2$ is again such a curve.

Let now that $(c_n)_{n \in S}$ be a C^∞-contact integral-indexed sequence of smooth curves in $\mathbf{R}^2$. If $\mathcal{P}$ is a convex partition of S, then for each $P \in \mathcal{P}$ we have defined the gluing $c_P = \amalg_{n \in P} c_n$ which is clearly a smooth curve in $\mathbf{R}^2$. In turn, since $\mathcal{P}$ is an integral chain, there is an order-preserving isomorphism from $\mathcal{P}$ onto an interval of $\mathbf{Z}$, and so, we can regard that $(c_P)_{P \in \mathcal{P}}$ is an integral-indexed sequence of smooth curves in $\mathbf{R}^2$ too. Moreover, one can see that this sequence is C^∞-contact, which implies that there exists the smooth gluing $\amalg_{P \in \mathcal{P}} c_P$. It is easy to verify the following associative law:

3.6. PROPOSITION. *Let* $(c_n)_{n\in S}$ *be a* C^∞*-contact integral-indexed sequence of smooth curves in* $\mathbf{R}^2$. *Then for any convex partition* $\mathcal{P}$ *of* S *we have*

$$\amalg_{P\in\mathcal{P}}(\amalg_{n\in P}c_n) = \amalg_{n\in S}c_n.$$

In particular, if c_1, c_2 *and* c_3 *are* C^∞*-contact smooth curves in* $\mathbf{R}^2$, *then*

$$(c_1 \amalg c_2) \amalg c_3 = c_1 \amalg (c_2 \amalg c_3) = c_1 \amalg c_2 \amalg c_3. \blacksquare$$

Recall that a *principal* K*-graph* in $\mathbf{R}^2$ is defined to be a compact connected subset of $\mathbf{R}^2$ which can be expressed as a finite union of principal closed segments (see [3], Section 2). In turn, by a *pointed principal* K*-graph* in $\mathbf{R}^2$ we shall mean a pair (G,p) where G is a principal K-graph in $\mathbf{R}^2$ and $p \in$ G. Of course, from the definition of locally P-subordinate curve in $\mathbf{R}^2$ (continuous in general) we get

3.7. PROPOSITION. *If* $c : [a;b] \to \mathbf{R}^2$ *is a locally* P*-subordinate curve where* $a,b \in \mathbf{R}$ *and* $a < b$, *then the image* $c([a;b])$ *is a principal* K*-graph.* $\blacksquare$

One can prove the following (compare [3], Lemma 2.23)

3.8. LEMMA. *If* $[a;b]$ $(a < b)$ *is an arbitrary closed interval of* $\mathbf{R}$, *then for any pointed principal* K*-graph* (G,p) *in* $\mathbf{R}^2$ *there exists a locally* P*-subordinate smooth curve* $c : [a;b] \to \mathbf{R}^2$ *such that*

$$c([a;b]) = G, \quad c(a) = c(b) = p \quad \textit{and} \quad a,b \in \operatorname{int}\operatorname{dom}_S(c)$$

where int *denotes the interior operation in* $[a;b]$. $\blacksquare$

3.9. THEOREM. *For any nonclosed interval* I *of* $\mathbf{R}$ *there exists a locally* P*-subordinate smooth curve* $c : I \to \mathbf{R}^2$ *such that the image* $c(I)$ *is dense in* $\mathbf{R}^2$.

Proof. We can regard $\mathbf{R}^2$ as a real normed space under the pointwise vector space operations and the norm $\|\cdot\|^*$ defined by $\|(x,y)\|^* = \max(|x|,|y|)$. Consider the standard net $N = \mathbf{Z}\times\mathbf{R}\cup\mathbf{R}\times\mathbf{Z}$ in $\mathbf{R}^2$. It is easily seen that for each $n \in \mathbf{N}$ the set $G_n = \{p \in 2^{-n}\cdot N : \|p\|^* \le n\}$ is a principal K-graph in $\mathbf{R}^2$ such that $o = (0,0) \in G_n$, where $2^{-n}\cdot N = \{2^{-n}q : q \in N\}$. Furthermore, note that $G_n \subseteq G_{n+1}$ for $n \in \mathbf{N}$ and the set $G = \bigcup\{G_n : n \in \mathbf{N}\}$ is a dense subset of $\mathbf{R}^2$.

By Lemma 3.8, for each $n \in \mathbf{N}$ there exists a locally P-subordinate smooth curve $c_n : [n;n+1] \to \mathbf{R}^2$ such that $c_n([n;n+1]) = G_n$, $c(n) = c(n+1) = o$ and $n, n+1 \in \operatorname{int}\operatorname{dom}_S(c_n)$. This implies that $(c_n)_{n=1}^\infty$ is a C^∞-contact sequence of locally P-subordinate smooth curves in $\mathbf{R}^2$. Thus there exists the gluing $c = \amalg_{n=1}^\infty c_n$ and from properties of all c_n it follows that c is a locally P-subordinate smooth curve in $\mathbf{R}^2$ too. Furthermore, note that $c([1;\infty)) = \bigcup\{c_n([n;n+1]) : n \in \mathbf{N}\} = G$, and so, $c([1;\infty))$ is a dense subset of $\mathbf{R}^2$.

Finally, since I is a nonclosed interval of $\mathbf{R}$, we conclude that either I is diffeomorphic to $[1;\infty)$ or I is an open interval of $\mathbf{R}$. In the first case if $\varphi : I \to [1;\infty)$ is a diffeomorphism, then the curve $c\circ\varphi : I \to \mathbf{R}^2$ satisfies the assertion of our theorem. Otherwise, for any $a \in I$ both the intervals $I^-(a)$ and $I^+(a)$ are diffeomorphic to $[1;\infty)$. Therefore, if $\varphi : I^+(a) \to [1;\infty)$ is a diffeomorphism, then the curves $c_1 : I^-(a) \to \mathbf{R}^2$ and $c_2 : I^+(a) \to \mathbf{R}^2$ defined by $c_1(x) = o$ for all $x \in I^-(a)$ and $c_2 = c\circ\varphi$ are C^∞-contact. Observe that in this case the gluing $c' = c_1 \amalg c_2$ satisfies the assertion of our theorem too. ∎

It turns out that for an arbitrary smooth curve $c : I \to \mathbf{R}^2$ the image $c(I)$ is 0-measurable (in $\mathbf{R}^2$). Indeed, we have $I = I_S \cup I_R$ where $I_S = \mathrm{dom}_S(c)$ and $I_R = \mathrm{dom}_R(c)$. Clearly, $c(I_S)$ consists of singular values of c, and so, it is 0-measurable by the Sard Theorem. On the other hand, I_R can be expressed as at most countable sum of disjoint open intervals of I. If U is such an interval, then $c(U)$ is a regular arc of c which has to be 0-measurable. Hence we conclude that $c(I_R)$ is 0-measurable too. Thus, since both $c(I_R)$ and $c(I_S)$ are 0-measurable, so is the image $c(I)$.

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INSTITUTE OF MATHEMATICS
UNIVERSITY OF ŁÓDŹ
S. Banacha 22
90-238 ŁÓDŹ, POLAND

Received February 24, 1994.