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Baotong Cui

OSCILLATION PROPERTIES OF THE SOLUTIONS OF HYPERBOLIC EQUATIONS WITH DEVIATING ARGUMENTS

1. Introduction

Nowadays one observes an expanding interest toward the study of initial value problems and oscillations for partial differential equations with deviating arguments (cf. [2]–[5]). But only a few papers have been published so far considering the oscillatory properties. The purpose of this paper is to obtain the sufficient conditions for the oscillation of solutions of following hyperbolic equation with deviating arguments

$$(1)\qquad \frac{\partial^2 u(x,t)}{\partial t^2} = a(t)\Delta u(x,t) + \sum_{i=1}^{m} a_i(t)\Delta u(x,\rho_i(t)) - q(x,t)f(u(x,\sigma(t))),\quad (x,t)\in\Omega\times[0,\infty)=G,$$

where Ω is a bounded domain in R^n, $n \geq 1$, with piecewise smooth boundary $\partial\Omega$, and Δ is the Laplacian in R^n.

Suppose that the following conditions hold:

(C_1) $a, a_i \in C([0,\infty);[0,\infty))$, $i = 1,2,\dots,m$;

(C_2) $q \in C(\overline{G};[0,\infty))$ and $q(t) = \min_{x\in\overline{\Omega}} q(x,t)$ is not identically zero on $[t_0,\infty)$ for some $t_0 > 0$;

(C_3) $\rho_i, \sigma \in C([0,\infty);R)$, $\lim_{t\to\infty}\sigma(t) = \lim_{t\to\infty}\rho_i(t) = \infty$, $i = 1,2,\dots,m$;

(C_4) $f \in C(R,R)$ is convex in $(0,\infty)$ and $uf(u) > 0$ for $u \neq 0$;

(C_5) there exists a function $F \in C(R;R)$ such that $|f(u)| \geq |F(u)|$,

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$uF(u) > 0$, $F'(u) \geq 0$ for $u \neq 0$, and

$$\int_{\varepsilon}^{\infty} \frac{du}{F(u)} < \infty, \quad \int_{-\varepsilon}^{-\infty} \frac{du}{F(u)} < \infty, \quad \text{for any } \varepsilon > 0.$$

We consider two kinds of boundary conditions:

$$(2) \qquad \frac{\partial u(x,t)}{\partial N} + \mu(x,t)u(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,\infty),$$

where N is the unit exterior normal vector to $\partial\Omega$ and μ is a nonnegative continuous function on $\partial\Omega \times [0,\infty)$, and

$$(3) \qquad u(x,t) = 0 \quad (x,t) \in \partial\Omega \times [0,\infty).$$

Our objective is to present conditions which imply that every (classical) solution $u(x,t)$ of the problem (1), (2) (or (1), (3)) is oscillatory in $\Omega \times [0\infty)$.

DEFINITION. The (classical) solution $u(x,t)$ of the problem (1), (2) (or (1), (3)) is called oscillatory in $G = \Omega \times [0,\infty)$, if $u(x,t)$ has zero in $\Omega \times [t_0,\infty)$ for each $t_0 > 0$.

2. Oscillation for the problem (1), (2)

THEOREM 1. *Let* (C_1)–(C_5) *hold and*

$$(4) \qquad \sigma(t) \leq t, \quad \sigma'(t) > 0.$$

If there exists a function $\rho \in C^2([0,\infty);[0,\infty))$ *such that*

$$(5) \qquad \int_{t_0}^{\infty} \rho(s)q(s)\,ds = \infty \quad \text{for each } t_0 > 0,$$

$$(6) \qquad \rho'(t) \geq 0 \quad \text{and } \beta(t) = \left(\frac{\rho'(t)}{\sigma'(t)}\right)' \leq 0 \quad \text{for } t \geq T \geq 0,$$

then every solution $u(x,t)$ *of the problem* (1), (2) *is oscillatory in the domain* G.

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x,t)$ of the problem (1), (2) which has no zero in $\Omega \times [t_0,\infty)$ for some $t_0 \geq 0$. Without loss of genearality we may assume that $u(x,t) > 0$ in $\Omega \times [t_0,\infty)$. From condition ($C_3$) there exists a $t_1 > t_0$ such that $u(x,t) > 0$, $u(x,\sigma(t)) > 0$ and $u(x,\rho_i(t)) > 0$, $i = 1,2,\ldots,m$, in $\Omega \times [t_1,\infty)$.

Integrating (1) with respect to x over the domain Ω, we obtain

$$(7) \quad \frac{d^2}{dt^2}\Big[\int_{\Omega} u(x,t)\,dx\Big] = a(t)\int_{\Omega} \Delta u(x,t)\,dx + \sum_{i=1}^{m} a_i(t)\int_{\Omega} \Delta u(x,\rho_i(t))\,dx - \int_{\Omega} q(x,t)f(u(x,\sigma(t)))\,dx, \quad t \geq t_1.$$

Green's formula and (2) yield

$$(8)\qquad \int_\Omega \Delta u(x,t)\,dx = \int_{\partial\Omega} \frac{\partial u}{\partial N}(x,t)\,dS = -\int_{\partial\Omega} \mu(x,t)u(x,t)\,dS,$$

$$(9)\qquad \int_\Omega \Delta u(x,\rho_i(t))\,dx = \int_{\partial\Omega} \frac{\partial u}{\partial N}u(x,\rho_i(t))\,dS$$

$$= -\int_{\partial\Omega} \mu(x,\rho_i(t))u(x,\rho_i(t))\,dS \le 0, \quad t \ge t_1,\ i = 1,2,\dots,m.$$

Moreover, from (C_4) and Jensen's inequality it follows that

$$(10)\qquad \int_\Omega q(x,t)f(u(x,\sigma(t)))\,dx \ge q(t)\int_\Omega f(u(x,\sigma(t)))\,dx$$

$$\ge |\Omega|q(t)f\Big(\frac{1}{|\Omega|}\int_\Omega u(x,\sigma(t))\,dx\Big), \quad t \ge t_1,$$

where $|\Omega| = \int_\Omega dx$. Then (7)–(10) imply

$$(11)\qquad V''(t) + q(t)f(V(\sigma(t))) \le 0, \quad t \ge t_1,$$

where $V(t) = \frac{1}{|\Omega|}\int_\Omega u(x,t)\,dx$, $t \ge t_0$. Thus, $V(t)$ is positive solution of the inequality (11). Obviously $V(t) > 0$ and $V''(t) \le 0$ for $t \ge t_1$. Hence $V'(t)$ is a decreasing function. We claim that $V'(t) > 0$ for $t \ge t_1$. If there exists a $t_2 \ge t_1$ such that $V'(t_2) \le 0$, then $V'(t) \le V'(t_2) \le 0$ for $t \ge t_2$. From (C_2) and (11) it follows that there is a $t_3 > t_2$ such that $V''(t_3) < 0$. Moreover, the inequalities $V'(t) \le V'(t_3) < V'(t_2) \le 0$ and

$$V(t) - V(t_3) = \int_{t_3}^{t} V'(s)\,ds \le \int_{t_3}^{t} V'(t_3)\,ds < 0, \quad t \ge t_3,$$

imply $\lim_{t\to\infty} V(t) = -\infty$, which contradicts the fact that $V(t) > 0$ for $t \ge t_1$. By (C_5) and (11), we obtain

$$(12)\qquad V''(t) + q(t)F(V(\sigma(t))) \le 0, \quad t \ge t_1.$$

Multiplying both sides of (12) by $\rho(t)/F(V(\sigma(t)))$ and integrating from t_1 to t, we have

$$\int_{t_1}^{t} \frac{V''(t)\rho(t)}{F(V(\sigma(t)))}\,dt + \int_{t_1}^{t} q(t)\rho(t)\,dt \le 0, \quad t \ge t_1,$$

or

$$\frac{\rho(t)V'(t)}{F(V(\sigma(t)))} \le \frac{\rho(t_1)V'(t_1)}{F(V(\sigma(t_1)))} - \int_{t_1}^{t} \rho(s)q(s)\,ds + \int_{t_1}^{t} \frac{V'(s)\rho'(s)}{F(V(\sigma(s)))}\,ds - \int_{t_1}^{t} \frac{V'(s)\rho(s)F'(V(\sigma(s)))V'(\sigma(s))\sigma'(s)}{[F(V(\sigma(s)))]^2}\,ds, \quad t \ge t_1.$$

Further, from (4) and (C_5) we have

$$\frac{\rho(t)V'(t)}{F(V(\sigma(t)))} \le C_0 - \int_{t_1}^{t} \rho(s)q(s)\,ds + \int_{t_1}^{t} \frac{V'(s)\rho'(s)}{F(V(\sigma(s)))}\,ds, \quad t \ge t_1, \tag{13}$$

where C_0 is a constant. Since $V'(t)$ is decreasing, from (13) and (4) it follows that

$$\begin{aligned} \frac{\rho(t)V'(t)}{F(V(\sigma(t)))} &\le C_0 - \int_{t_1}^{t} \rho(s)q(s)\,ds + \int_{t_1}^{t} \frac{V'(\sigma(t))\rho'(t)}{F(V(\sigma(t)))}\,dt \\ &= C_0 - \int_{t_1}^{t} \rho(s)q(s)\,ds + \int_{t_1}^{t} \frac{\rho'(t)}{\sigma'(t)} \frac{d(V(\sigma(t)))}{F(V(\sigma(t)))}\,dt \end{aligned} \tag{14}$$

$$= C_0 - \int_{t_1}^{t} \rho(s)q(s)\,ds - \frac{\rho'(t)}{\sigma'(t)}G(t) + \frac{\rho'(t_1)}{\sigma'(t_1)}G(t_1) + \int_{t_1}^{t} \beta(s)G(s)\,ds, \quad t \ge t_1,$$

where $G(t) = \int_{V(\sigma(t))}^{\infty} \frac{du}{F(u)} > 0$. Hence, using (C_5), (6) and (14), we obtain

$$\frac{\rho(t)V'(t)}{F(V(\sigma(t)))} \le C_0 - \int_{t_1}^{t} \rho(s)q(s)\,ds + C_1, \quad t \ge t_1,$$

where C_1 is a constant, too. So, by (5), we have

$$\lim_{t\to\infty} \frac{\rho(t)V'(t)}{F(V(\sigma(t)))} = -\infty.$$

From this it follows that there exists a $t_2 \ge t_1$ such that $V'(t) < 0$ for $t \ge t_2$, which leads to a contradiction.

If $u(x,t) < 0$ for $(x,t) \in \Omega \times [t_0,\infty)$, then $-u(x,t)$ is a positive solution of the problem (1), (2), and the proof is similar.

THEOREM 2. *If* (C_1)–(C_5), (4) *and* (5) *hold and*

$$\rho'(t) \ge 0, \quad \int_{t_0}^{\infty} \left| \left(\frac{\rho'(s)}{\sigma'(s)} \right)' \right| ds < \infty \quad \textit{for each } t_0 > 0, \tag{6'}$$

then every solution $u(x,t)$ *of the problem* (1), (2) *oscillates in* G.

Proof. Let $u(x,t)$ be a nonoscillatory solution of the problem (1), (2). Without loss of generality, we can assume that $u(x,t) > 0$ in $\Omega \times [t_0,\infty)$ for some $t_0 > 0$. As in the proof of Theorem 1, we can have that (7)–(13) hold and

$$\frac{\rho(t)V'(t)}{F(V(\sigma(t)))} \le C_0 - \int_{t_1}^{t} \rho(s)q(s)\,ds - \frac{\rho'(t)}{\sigma'(t)}G(t) + \frac{\rho'(t_1)}{\sigma'(t_1)}G(t_1) + \int_{t_1}^{t} \beta(t)G(t)\,dt, \quad t \ge t_1. \tag{14}$$

Now, by the condition (C_5), there exists a constant $M > 0$ such that $0 < G(t) \le M$ for any $t \ge t_0$, and from (6′) we have

$$\Big| \int_{t_1}^{t} \beta(s)G(s)\,ds \Big| \le M \int_{t_1}^{t} |\beta(s)|\,ds < \infty, \quad t \ge t_1.$$

Thus, there is a constant $K > 0$ such that

$$\frac{\rho(t)V'(t)}{F(V(\sigma(t)))} \le K - \int_{t_1}^{t} \rho(s)q(s)\,ds, \quad t \ge t_1. \tag{15}$$

Hence, from this and (5) it follows that

$$\lim_{t\to\infty} \frac{\rho(t)V'(t)}{F(V(\sigma(t)))} = -\infty,$$

which contradicts $V'(t) > 0$ for $t \ge t_1$.

THEOREM 3. *If* (C_1)–(C_5) *and* (4) *hold and*

$$\int_{t_0}^{\infty} \sigma'(s)\Big(\int_{s}^{\infty} q(\eta)\,d\eta \Big)\,ds = \infty \quad \text{for each } t_0 > 0, \tag{16}$$

then every solution $u(x,t)$ *of the problem* (1), (2) *oscillates in* G.

Proof. Let $u(x,t)$ be a positive solution of the problem (1), (2) and $u(x,t) > 0$ for $(x,t) \in \Omega \times [t_0, \infty)$ with $t_0 \ge 0$. As in the proof of Theorem 1, we can get (12). Then for $t \ge t_1$ we have the inequality

$$\int_{t_1}^{t} \frac{V''(s)}{F(V(\sigma(s)))}\,ds + \int_{t_1}^{t} q(s)\,ds \le 0$$

leading to

$$\frac{V'(t)}{F(V(\sigma(t)))} - \frac{V'(t_1)}{F(V(\sigma(t_1)))} + \int_{t_1}^{t} \frac{V'(s)V'(\sigma(s))\sigma'(s)F'(V(\sigma(s)))}{[F(V(\sigma(s)))]^2}\,ds + \int_{t_1}^{t} q(s)\,ds \le 0.$$

From this and conditions (C_5), (4) it follows that

$$(17)\qquad \frac{V'(t)}{F(V(\sigma(t)))} \le \frac{V'(t_1)}{F(V(\sigma(t_1)))} - \int_{t_1}^{t} q(s)\,ds, \quad t \ge t_1.$$

Since $V'(t) > 0$ for $t \ge t_1$ (see the proof of Theorem 1), we have $0 \le \frac{V'(t)}{F(V(\sigma(t)))} - \int_t^\infty q(s)\,ds$, $t \ge t_1$. Hence

$$\int_t^\infty q(s)\,ds \le \frac{V'(t)}{F(V(\sigma(t)))}, \quad t \ge t_1.$$

Then, from $t \ge \sigma(t)$ and $V''(t) \le 0$ for $t \ge t_1$, we obtain

$$(17')\qquad \int_t^\infty q(s)\,ds \le \frac{V'(\sigma(t))}{F(V(\sigma(t)))}, \quad t \ge t_1.$$

Multiplying both sides of (17′) by $\sigma'(t) > 0$ and integrating from t_1 to t, we get

$$\int_{t_1}^{t} \sigma'(s)\Big(\int_s^\infty q(\eta)\,d\eta\Big)\,ds \le \int_{t_1}^{t} \frac{V'(\sigma(t))\sigma'(t)}{F(V(\sigma(t)))}\,dt = \int_{V(\sigma(t_1))}^{V(\sigma(t))} \frac{du}{F(u)} < \infty,$$

which contradicts (16).

3. Oscillation for the problem (1), (3)

In the domain Ω we consider the following Dirichlet problem:

$$(18)\qquad \begin{cases} \Delta\omega(x,t) + \alpha\omega(x,t) = 0, & x \in \Omega, \\ \omega(x,t) = 0, & x \in \partial\Omega, \end{cases}$$

where α is a constant. It is well known [6], [7] that the least eigenvalue α_0 of the problem (18) is positive and the corresponding eigenfunction $\Phi(x)$ is positive on Ω.

With each solution $u(x,t)$ of the problem (1), (3) we associate a function V defined by

$$(19)\qquad V(t) = \Big(\int_\Omega \Phi(x)\,dx\Big)^{-1} \int_\Omega u(x,t)\Phi(x)\,dx, \quad t \ge 0.$$

THEOREM 4. *If all conditions of Theorem* 1 *hold, then every solution of* (1), (3) *is oscillatory in* G.

Proof. Let $u(x,t)$ be a positive solution of (1), (3) in $\Omega \times [t_0,\infty)$ for some $t_0 \ge 0$. By condition (C_3), there exists a $t_1 \ge t_0$ such that

$$u(x,\sigma(t)) > 0, \quad u(x,\rho_i(t)) > 0, \quad i = 1,2,\ldots,m,$$

for any $t \geq t_1$. Multiplying both sides of (1) by the eigenfunction $\Phi(x) > 0$ and integrating with respect to x over the domain Ω, we have

$$(20)\quad \frac{d^2}{dt^2}\Big[\int_\Omega u(x,t)\Phi(x)\,dx\Big] \leq a(t)\int_\Omega \Delta u(x,t)\Phi(x)\,dx$$
$$+\sum_{i=1}^{m} a_i(t)\int_\Omega \Delta u(x,\rho_i(t))\Phi(x)\,dx - q(t)\int_\Omega f(u(x,\sigma(t)))\Phi(x)\,dx, \quad t \geq t_1.$$

From the divergence theorem it follows that

$$(21)\quad \int_\Omega \Delta u(x,t)\Phi(x)\,dx = \int_\Omega u(x,t)\Delta\Phi(x)\,dx$$
$$= -\alpha_0 \int_\Omega u(x,t)\Phi(x)\,dx, \quad t \geq t_1,$$

$$(22)\quad \int_\Omega \Delta u(x,\rho_i(t))\Phi(x)\,dx = \int_\Omega u(x,\rho_i(t))\Delta\Phi(x)\,dx$$
$$= -\alpha_0 \int_\Omega u(x,\rho_i(t))\Phi(x)\,dx, \quad t \geq t_1,\ i = 1,2,\ldots,m,$$

where α_0 is the least eigenvalue of the problem (18). From the condition (C_4) and Jensen's inequality it follows that

$$(23)\quad \int_\Omega f(u(x,\sigma(t)))\Phi(x)\,dx$$
$$\geq \Big(\int_\Omega \Phi(x)\,dx\Big)^{-1} f\Big(\frac{\int_\Omega u(x,\sigma(t))\Phi(x)\,dx}{\int_\Omega \Phi(x)\,dx}\Big), \quad t \geq t_1.$$

Using (21)-(23) and (19), from (20) we obtain for $t \geq t_1$

$$(24)\quad V''(t) + \alpha_0 a(t)V(t) + a_0\sum_{i=1}^{m} a_1(t)V(\rho_i(t)) + q(t)f(V(\sigma(t))) \leq 0.$$

Since $V(t) > 0$, $V(\rho_i(t)) > 0$ and $V(\sigma(t)) > 0$ for any $t \geq t_1$ and all i, we have the inequality

$$(25)\quad V''(t) + q(t)f(V(\sigma(t))) \leq 0, \quad t \geq t_1.$$

The remains are similar to the proof of Theorem 1. We have also the following results.

THEOREM 5. *If all conditions of Theorem 2 hold, then every solution of the problem* (1), (3) *is oscillatory in* G.

THEOREM 6. *If all conditions of Theorem 3 hold, then every solution of the problem* (1), (3) *is oscillatory in* G.

EXAMPLE. Consider the problem

$$(26)\quad u_{tt} = 3u_{xx}(x,t) + u_{xx}(x,t-\pi) - u(x,t-\pi)\exp\{[u(x,t-\pi)]^2 - \sin^2 t\cos^2 x\}, (x,t) \in (0,\pi)\times[0,\infty),$$

$$(27)\quad u_x(0,t) = u_x(\pi,t) = 0, \quad t \geq 0.$$

Here $a(t) = 3$, $a_1(t) = 1$, $m = 1$, $\Omega = (0,\pi)$, $n = 1$, $f(u) = u\exp(u^2)$, $\sigma(t) = \rho_1(t) = t - \pi$, $q(x,t) = \exp(-\sin^2 t\cos^2 x)$ with $q(t) = \min_{x\in(0,\pi)} q(x,t) = \exp(-\sin^2 t)$. Choose $\rho(t) = t$, then

$$\rho'(t) = 1 > 0, \quad \beta(t) = \left(\frac{\rho'(t)}{\sigma'(t)}\right)' = 0,$$

$$\int_{t_0}^{\infty} \rho(s)q(s)\,ds = \int_{t_0}^{\infty} s\exp(-\sin^2 s)\,ds \geq \frac{1}{e}\int_{t_0}^{\infty} s\,ds = \infty, \quad t_0 = 0.$$

It is easy to check that all the conditions of Theorem 1 are satisfied. Therefore, every solution of the problem (26), (27) oscillates in $G = (0,\pi)\times[0,\infty)$. In fact, $u(x,t) = \sin t\cos x$ is such a solution.

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DEPARTMENT OF MATHEMATICS
BINZHOU NORMAL COLLEGE
SHANDONG 256604
PEOPLE'S REPUBLIC OF CHINA

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